Random dynamical systems for SPDE driven by an FBM

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1. RDS

Investigations of the qualitative behavior of de's under the influence of noise

qualitative behavior ≙ stability behavior

Noise: Metric dynamical system

$(\Omega, \mathcal{F}, \mathbb{P}, \theta)$

$(\Omega, \mathcal{F}, \mathbb{P})$ probability space (canonical)

θ flow: $\theta : \mathbb{R} \times \Omega \to \Omega$, $\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F}$ measur.

$\theta_t \circ \theta_\tau = \theta_{t+\tau} \quad \theta_0 = id_\Omega$

$\mathbb{P}$ is θ invariant (θ ergodic)

Examples: $(\Omega, \mathcal{F}, \mathbb{P}, \theta) = (C_0^H, \mathcal{B}(C_0^H), \mathbb{P}_W, \theta^W)$

$\theta_t^W \omega = \omega(\cdot + t) - \omega(t)$

$\mathbb{P}_W$ Wiener measure ($\mathbb{P}_W$ is θ^W ergodic)

Ornstein-Uhlenbeck noise

$(\Omega, \mathcal{F}, \mathbb{P}_{OU}, \theta^S) = (C^H, \mathcal{B}(C^H), \mathbb{P}_{OU}, \theta^S)$

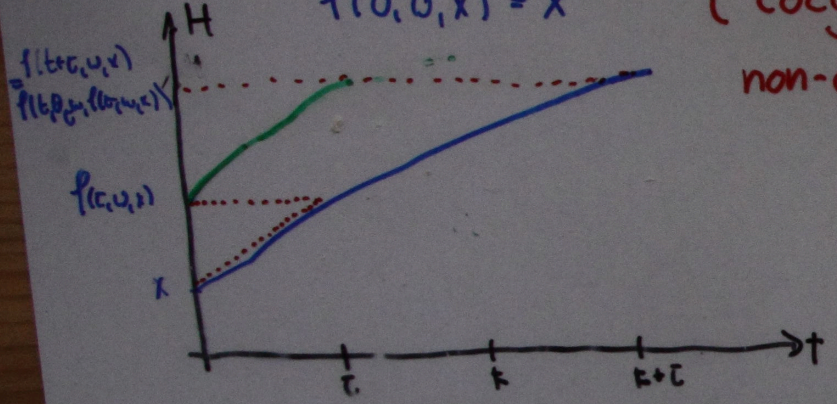
$\theta_t^S \omega(\cdot) = \omega(\cdot + t)$

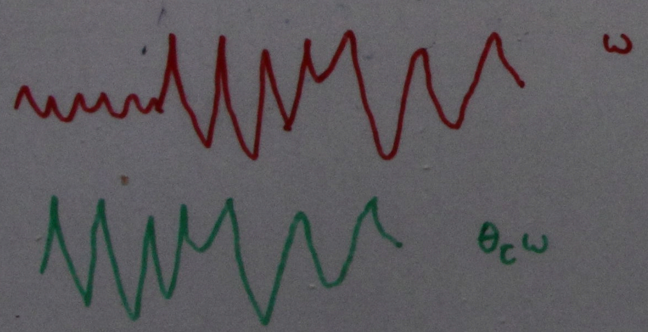
Dynamics

$\varphi: \mathbb{R}^+ \times \Omega \times H \to H$, measurable

$\forall t, \tau \in \mathbb{R}^+, x \in H$ $\varphi(t+\tau, \omega, x) = \varphi(t, \theta_\tau \omega, \cdot) \circ \varphi(\tau, \omega, x)$
$\forall \omega \in \Omega$

$\varphi(0, \omega, x) = x$ (cocycle property: non-autonomous d.s)





nonautonomous de generate cocycles
finite dim SDE generate RDS
unknown in general for infinite dim SDE (SPDE)

Attractors:

general concept of DS, attractors contain the essential dynamics of an DS

Pullback attractor $\mathcal{D}$ set system $D \in \mathcal{D}$ $(\mathcal{D} \ni D = (D(\omega))_{\omega\in\Omega})$

A set $A \in \mathcal{D}$ $A(\omega)$ compact, $\neq \emptyset$ such that

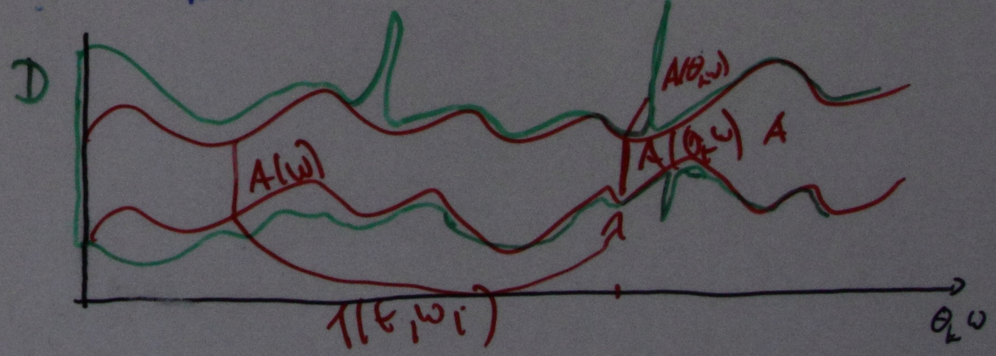
$$\varphi(t, \omega, A(\omega)) = A(\theta_t \omega) \qquad \omega \in \Omega$$

$$\lim_{t\to\infty} dist_H(\varphi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), A(\omega)) = 0 \quad \forall D \in \mathcal{D}, \ \omega \in \Omega$$

is called $\mathcal{D}$-pullback attractor

If A is a random set then A is called random attractor:

$$\mathbb{P}\lim_{t\to\infty} dist_H(\varphi(t, \omega, D(\omega)), A(\theta_t \omega)) = 0 \quad \forall D \in \mathcal{D}$$



Existence of pullback attractors

Assume
- $x \mapsto f(t,\omega,x)$ cont.
- absorbing set $B \in \mathcal{D}$, $B(\omega)$ comp.

  $\exists\, t_0(\omega,D)$: $f(t,\theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega)$ $t \geq t_0$

Then there exists a unique $\mathcal{D}$ pullback attractor

2. RDS and FBM

FBM A centered Gauß-process B^h with

covariance $R(s,t) = \frac{1}{2}\left(|t|^{2h} + |s|^{2h} - |t-s|^{2h}\right)$

is called FBM $h \in (0,1)$ $h = \frac{1}{2} \Rightarrow$ BM

infinite dim FBM: $(B_i^h)_{i \in \mathbb{N}}$ iid FBM

$(e_i)_{i \in \mathbb{N}}$ complete base in sep. HS H

$$B^H = \sum_i \lambda_i B_i^h e_i \qquad \lambda_i \geq 0 \qquad \sum \lambda_i < \infty.$$

There exists an h'-Hölder continuous version $(h' < h)$

Noise $(\Omega, \mathcal{F}, \mathbb{P}, \theta) = (C_0^H, \mathcal{B}(C_0^H), \mathbb{P}_{FBM}^H, \theta^H)$

defines a metric dynamical system

$\mathbb{P}_{FBM}^H$ is ergodic

Stochastic integrals and FBM

$$h \in \left(\tfrac{1}{2}, 1\right)$$

$$\int_0^t f\,d\omega := c \int_0^t (D_{0+}^{\alpha} f)(s)(D_{t-}^{1-\alpha}\omega_{t-})(s)\,ds \qquad (\text{Zähle} \ldots)$$

$$\left.\begin{aligned} D_{0+}^{\alpha} f[t] &:= \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(t)}{t^{\alpha}} + \alpha \int_0^t \frac{f(t)-f(q)}{(t-q)^{1+\alpha}}\,dq\right) \\ D_{t-}^{1-\alpha}\omega_{t-}[t] &:= \ldots \end{aligned}\right\} \text{ fract. der.}$$

$\alpha + h > 1$

This is an ω-wise definition of a stochastic integral

in particular

$$\int_T^{t+T} f\,d\omega = \int_0^t f(\cdot + T)\,d\theta_T\omega \quad (\text{cocycle property})$$

Estimate

$$\left|\int_0^t f\,d\omega\right| \le \Lambda_{\alpha}^{0,t}(\omega)\,|f|_{\alpha}$$

$$\Lambda_{\alpha}^{0,t}(\omega) = c \sup_{0 \le t_1 < t_2 \le t}\left(\frac{|\omega(t_1)-\omega(t_2)|}{|t_1-t_2|^{1-\alpha}} + \alpha \int_{t_2}^{t_1} \frac{|\omega(q)-\omega(t_2)|}{|q-t_2|^{2-\alpha}}\,dq\right)$$

$$|f|_{\alpha} = \int_0^t \left(\frac{|f(s)|}{s^{\alpha}} + \int_0^s \frac{|f(s)-f(q)|}{(s-q)^{1+\alpha}}\,dq\right) ds$$

3 SPDE and FBM

$$du = Au\,dt + F(u)\,dt + D(u)\,d\tilde{\omega} = Au\,dt + G(u)\,d\omega \quad *$$

$u(0) = u_0 \in H$ — infinite FBM — $\begin{pmatrix} dt \\ d\tilde{\omega}\end{pmatrix}$

A is generator of an exp decreasing analytic semigroup S, A symmetric

G is bounded, continuously differentiable

DG is bounded, Lipschitz-cont.

Existence and uniqueness (Maslowski, Nualart)

For every $u_0 \in H$ there exists a unique global mild solution for all $\omega \in \Omega_0 \subset \Omega$ independent of u_0

Idea of proof: Banach FPT with norm

$$W^{\alpha,\infty} \quad \|u\| = \sup_{t \in [0,T]}\left(\|u(t)\| + \int_0^t \frac{\|u(t)-u(q)\|}{(t-q)^{1+\alpha}}\,dq\right)$$

$\Rightarrow$ $*$ generates an RDS $\varphi(t,\omega,u_0)$

$\varphi(t,\omega,\cdot)$ is continuous in H

$\varphi(\cdot,\omega,u)$ is continuous : $\mathbb{R}^+ \to H$

Apriori estimates, absorbing sets

$$\Delta T(\omega) = \inf\{T > 0 : \Lambda^{0,T}(\omega) = \lambda\} \qquad \lambda \in (0,1)$$

$$\Delta \bar{T}(\omega) = \sup\{T < 0 : \Lambda^{T,0}(\omega) = \lambda\}$$

$$T_0(\omega) = 0, \quad T_1(\omega) = \Delta T(\omega) = T_2(\omega) = T_1(\omega) + T_1(\theta_{T_1(\omega)}\omega), \dots$$

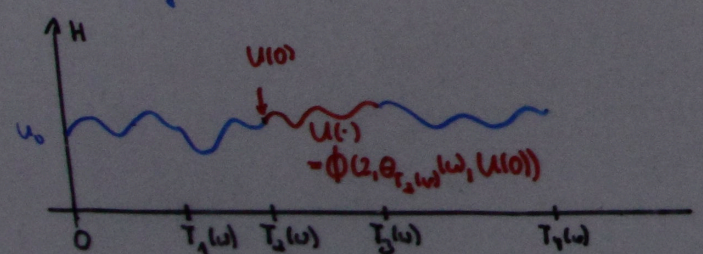
$T_i(\omega), i \in \mathbb{Z}$ forms an additive cocycle (helix)

$$\mathcal{X}_k(\omega) = \{u \in W^{1,p}(0, \overbrace{T_{k+1}(\omega) - T_k(\omega)}^{= \Delta T(\theta_{T_k(\omega)}\omega)}) \; u \text{ is solution of on } [0, T_{k+1}(\omega) - T_k(\omega)]\} \text{ for some } u_0 \in H\}$$

$\mathcal{X}_k(\omega)$ forms a complete space w.r.t. the $W^{1,p}$ norm

another cocycle

$$\Phi(i, \theta_{T_j(\omega)}\omega, U)(s) : \; f(s + T_j(\omega), \theta_{T_j(\omega)}\omega, U(0)) = U(s)$$



Absorbing set

$$\phi(n, \omega, u_0)(t) = S(t + T_n)u_0 + \int_0^{T_n + t} S(T_n + t - \tau)\, G(U(\tau))\, d\omega$$

$$= \dots + \int_{T_n}^{T_n + t} S(T_n + t - \tau)\, G(\phi(n, \omega, u_0)(\tau - T_n))\, d\omega$$

$$+ \int_{T_{n-1}}^{T_n} \dots \quad + \dots, \qquad t \in [0, T_{n+1}(\omega) - T_n(\omega)]$$

$$\|\phi(n, \omega, u_0)\|_{\mathcal{X}_n(\omega)} \leq \|S(\cdot + T_n)u_0\| + \Lambda^{0, T_{n+1} - T_n}(\theta_{T_n}\omega)(M_2 \|\phi(n, \omega, u_0)\|_{\mathcal{X}_n} + M_1)$$

$$+ \dots$$

Stopping $\Delta T(\theta_{T_n}(\omega))$:

$$\Lambda^{0, T_{n+1} - T_n}(\theta_{T_n}\omega) M_2 = \Lambda^{0, \Delta T(\theta_{T_n}\omega)}(\theta_{T_n}\omega) M_2 = \lambda M_2 = \tilde{k}_1 < 1$$

$$\|\phi(n, \omega, u)\|_{\mathcal{X}_n(\omega)} \leq M_3 e^{-aT_n} \|u(0)\|$$

$$+ \lambda M_1 \sum_{i=0}^{n} e^{-a(T_n - T_{n-1})} \|\phi(n, \omega, u)\|_{\mathcal{X}_i(\omega)}$$

$$+ \lambda M_1 \sum_{i=0}^{n} e^{-a(T_n(\omega) - T_{n-1}(\omega))}$$

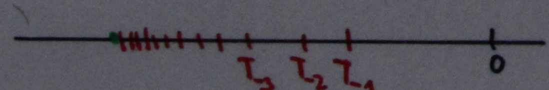
discrete Gronwall lemma, pullback absorbing set ($\omega \to \theta_{T_n(\omega)}\omega$)

There exists a $k_1 = k_1(\lambda_1, M_2)$

$$\|\phi(n,\omega,u)\|_{\gamma_n(\omega)} \le (1+k_1)^n \left(c_1 \|u(0)\| e^{-aT_n(\omega)}\right.$$
$$\left. + k_2 \sum_{i=0}^{n} (1+k_1)^{n-i} e^{-a(T_n(\omega)-T_i(\omega))}\right)$$

$\omega \to \theta_{T_{-n}(\omega)}\omega$, then

$$R^2(\omega) = k_2 \sum_{i=-\infty}^{0} e^{aT_i(\omega)} (1+k_1)^{-i}$$

- convergence is related to the growth of $T_i(\omega)$ $i \to -\infty$
- covariance of ω small, then $|T_i(\omega)| \underset{(<1)}{\approx} 1$
- $T_i(\omega)$ is not a Markov stopping time
  $\theta_{T_i(\omega)}\omega$ is not an FBM.
- 

Using ergodic theory:

$$\liminf_{i\to-\infty} \frac{|T_i(\omega)|}{\omega} = d$$

$d \underset{(<)}{\approx} 1$ if cov of ω is small

$\Rightarrow R^2(\omega)$ is finite

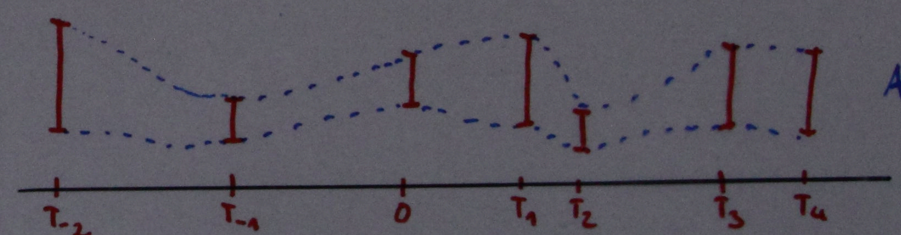
$\Rightarrow$ • The cocycle φ restricted to $(T_i(\omega))_{i\in\mathbb{Z}}$ has an absorbing set $(C(\theta_{T_i(\omega)}\omega))_{i\in\mathbb{Z}}$

• Smoothing property of $S(t)$, $t>0$:
φ restricted to $(T_i(\omega))_{i\in\mathbb{Z}}$ has a compact absorbing set $(B(\theta_{T_i(\omega)}\omega))_{i\in\mathbb{Z}}$

• $u_0 \to \varphi(t,\omega,u_0)$ is continuous on H.

$\Rightarrow$

φ restricted to $(T_i(\omega))_{i\in\mathbb{Z}}$ has a pullback attractor



Main result:

Attractor A can be extended to the measurability of φ

The RDS φ generated by the SPDE driven by an FBB has a random attractor.

Outlook

- stable / unstable manifolds for spde driven by FBM $(h \in (\frac{1}{2}, 1])$
- $h \in (\frac{1}{3}, \frac{1}{2}]$ ODE Lyons..., and Nualart and Hu
- attractors ?!