A Concise Course on Stochastic Partial Differential Equations

Michael Röckner

Reference:

C. Prevot, M. Röckner: Springer LN in Math. 1905, Berlin (2007)

And see the references therein for the original literature!

Table of content

1 The Stochastic Integral in General Hilbert Spaces (w.r.t. Brownian Motion)

For a topological space E we denote its Borel σ-algebra by $\mathcal{B}(E)$. For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$(i.e. Ω is an arbitrary set, $\mathcal{F}$ a σ-algebra of subsets of Ω and $\mathbb{P} : \mathcal{F} \to [0,1]$ a probability measure) and an $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}$ we set

$$\mathbb{E}(X) := \int_\Omega X(\omega)\mathbb{P}(\,d\omega)$$

provided $\int_\Omega |X(\omega)|\mathbb{P}(\,d\omega) < \infty$.
We fix two Hilbert spaces $(U, \langle\ ,\ \rangle_U)$ and $(H, \langle\ ,\ \rangle_H)$.

Definition (1.1.1)

A probability measure μ on $\big(U, \mathcal{B}(U)\big)$ is called *Gaussian* if

$$\hat{\mu}(u) := \int_U e^{i\langle u,v\rangle_U}\ \mu(\mathrm{d}v) = e^{i\langle m,u\rangle_U - \frac{1}{2}\langle Qu,u\rangle_U}, \quad u \in U,$$

where $m \in U$ and $Q \in L(U)$ is nonnegative, symmetric. (Here $L(U)$ denotes the set of all bounded linear operators on U). In this case Q necessarily has finite trace and μ will be denoted by $N(m, Q)$ where m is called mean and Q is called covariance (operator). Furthermore, for all $h, g \in U$

$$\int \langle x, h\rangle_U\ \mu(\mathrm{d}x) = \langle m, h\rangle_U,$$

$$\int \big(\langle x, h\rangle_U - \langle m, h\rangle_U\big)\big(\langle x, g\rangle_U - \langle m, g\rangle_U\big)\ \mu(\mathrm{d}x) = \langle Qh, g\rangle_U.$$

Theorem (1.1.2)

Let Q be a nonnegative and symmetric operator in $L(U)$ with finite trace and let $m \in U$. Then there exists a Gaussian measure $\mu = N(m, Q)$ on $\big(U, \mathcal{B}(U)\big)$.

Now we can give the definition of a standard Q-Wiener process. To this end we fix an element $Q \in L(U)$, nonnegative, symmetric, with finite trace and a positive real number T.

Definition (1.1.3)

A U-valued stochastic process $W(t)$, $t \in [0,T]$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a (standard) Q-Wiener process if:

- $W(0) = 0$,
- W has $\mathbb{P}$-a.s. continuous trajectories,
- the increments of W are independent, i.e. the random variables

$$W(t_1),\, W(t_2) - W(t_1), \dots,\, W(t_n) - W(t_{n-1})$$

  are independent for all $0 \leqslant t_1 < \cdots < t_n \leqslant T$, $n \in \mathbb{N}$,
- the increments have the following Gaussian laws:

$$\mathbb{P}\circ\big(W(t)-W(s)\big)^{-1} = N\big(0,(t-s)Q\big) \quad \text{for all } 0 \leqslant s \leqslant t \leqslant T.$$

Proposition (1.1.4 Representation of the Q-Wiener process)

Let e_k, $k \in \mathbb{N}$, be an orthonormal basis of U consisting of eigenvectors of Q with corresponding eigenvalues λ_k, $k \in \mathbb{N}$. Then a U-valued stochastic process $W(t)$, $t \in [0,T]$, is a Q-Wiener process if and only if

$$W(t) = \sum_{k\in\mathbb{N}} \sqrt{\lambda_k}\beta_k(t)e_k, \quad t \in [0,T], \tag{1.1}$$

where β_k, $k \in \{n \in \mathbb{N} \mid \lambda_n > 0\}$, are independent real-valued Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The series even converges in $L^2\big(\Omega, \mathcal{F}, \mathbb{P}; C([0,T],U)\big)$, and thus always has a $\mathbb{P}$-a.s. continuous modification. (Here the space $C\big([0,T],U\big)$ is equipped with the sup norm.) In particular, for any Q as above there exists a Q-Wiener process on U.

Definition (1.1.5 Normal filtration)

A filtration $\mathcal{F}_t$, $t \in [0,T]$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (i.e., each $\mathcal{F}_t$ is a σ-field and $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}, s \leqslant t$) is called normal if:

- $\mathcal{F}_0$ contains all elements $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$ and
- $\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$ for all $t \in [0,T]$.

Definition (1.1.6 Q-Wiener process with respect to a filtration)

A Q-Wiener process $W(t)$, $t \in [0,T]$, is called a Q-Wiener process with respect to a filtration $\mathcal{F}_t$, $t \in [0,T]$, if:

- $W(t)$, $t \in [0,T]$, is adapted to $\mathcal{F}_t$, $t \in [0,T]$, and
- $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $0 \leqslant s \leqslant t \leqslant T$.

We define

$$\mathcal{N} := \left\{A \in \mathcal{F} \mid \mathbb{P}(A) = 0\right\}, \quad \tilde{\mathcal{F}}_t := \sigma\big(W(s) \mid s \leqslant t\big)$$

and $\tilde{\mathcal{F}}_t^0 := \sigma(\tilde{\mathcal{F}}_t \cup \mathcal{N})$.

Then it is clear that

$$\mathcal{F}_t := \bigcap_{s>t} \tilde{\mathcal{F}}_s^0, \quad t \in [0,T], \tag{1.2}$$

is a normal filtration and we get:

Proposition (1.1.7)

Let $W(t)$, $t \in [0, T]$, be an arbitrary U-valued Q-Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then it is a Q-Wiener process with respect to the normal filtration $\mathcal{F}_t$, $t \in [0, T]$, given by (1.2).

Analogously to the real-valued case it is possible to define the conditional expectation of any Bochner integrable random variable with values in an arbitrary separable Banach space $(E, \| \ \|)$. This result is formulated in the following proposition.

Proposition (1.2.1 Existence of the conditional expectation)

Assume that E is a separable real Banach space and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X : \Omega \to E$ be $\mathcal{F}$-measurable and Bochner integrable. Let $\mathcal{G}$ be a σ-field contained in $\mathcal{F}$. Then there exists a unique, up to a set of $\mathbb{P}$-probability zero, Bochner integrable E-valued map $Z : \Omega \to E$, measurable with respect to $\mathcal{G}$ such that

$$\int_A X \, \mathrm{d}\mathbb{P} = \int_A Z \, \mathrm{d}\mathbb{P} \quad \text{for all } A \in \mathcal{G}. \tag{1.3}$$

Z is denoted by $\mathbb{E}(X \mid \mathcal{G})$ and is called the conditional expectation of X given $\mathcal{G}$. Furthermore,

$$\big\|\mathbb{E}(X \mid \mathcal{G})\big\| \leqslant \mathbb{E}\big(\|X\| \mid \mathcal{G}\big).$$

Definition (1.2.2)

Let $M(t)$, $t \geqslant 0$, be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a separable Banach space E, and let $\mathcal{F}_t$, $t \geqslant 0$, be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$.

The process M is called an *$(\mathcal{F}_t)$-martingale*, if:

- $\mathbb{E}\big(\|M(t)\|\big) < \infty$ for all $t \geqslant 0$,
- $M(t)$ is $\mathcal{F}_t$-measurable for all $t \geqslant 0$,
- $\mathbb{E}\big(M(t) \bigm| \mathcal{F}_s\big) = M(s)$ $\mathbb{P}$-a.s. for all $0 \leqslant s \leqslant t < \infty$.

Remark (1.2.3)

Let M be as above such that $\mathbb{E}(\|M(t)\|) < \infty$ for all $t \in [0, T]$. Then M is an ($\mathbb{R}$-valued) $(\mathcal{F}_t)$-martingale if and only if $l(M)$ is an $(\mathcal{F}_t)$-martingale for all $l \in E^*$. In particular, results like optional stopping etc. extend to E-valued martingales.

Theorem (1.2.4 Maximal inequality)

Let $p > 1$ and let E be a separable Banach space. If $M(t)$, $t \in [0, T]$, is a right-continuous E-valued $(\mathcal{F}_t)$-martingale, then

$$\left(\mathbb{E}\left(\sup_{t\in[0,T]} \|M(t)\|^p\right)\right)^{\frac{1}{p}} \leqslant \frac{p}{p-1} \sup_{t\in[0,T]} \left(\mathbb{E}\left(\|M(t)\|^p\right)\right)^{\frac{1}{p}}$$
$$= \frac{p}{p-1} \left(\mathbb{E}\left(\|M(T)\|^p\right)\right)^{\frac{1}{p}}.$$

Remark (1.2.5)

We note that in the inequality in Theorem 1.11 the first norm is the standard norm on $L^p\big(\Omega, \mathcal{F}, \mathbb{P}; C([0,T]; E)\big)$, *whereas the second is the standard norm on* $C\big([0,T]; L^p(\Omega, \mathcal{F}, \mathbb{P}; E)\big)$. *So, for right-continuous E-valued $(\mathcal{F}_t)$-martingales these two norms are equivalent.*

Now we fix $0 < T < \infty$ and denote by $\mathcal{M}_T^2(E)$ the space of all E-valued continuous, square integrable martingales $M(t)$, $t \in [0, T]$.

Proposition (1.2.6)

The space $\mathcal{M}_T^2(E)$ equipped with the norm

$$\|M\|_{\mathcal{M}_T^2} := \sup_{t\in[0,T]} \Big(\mathbb{E}\big(\|M(t)\|^2\big)\Big)^{\frac{1}{2}} = \Big(\mathbb{E}\big(\|M(T)\|^2\big)\Big)^{\frac{1}{2}}$$

$$\leqslant \Big(\mathbb{E}\big(\sup_{t\in[0,T]} \|M(t)\|^2\big)\Big)^{\frac{1}{2}} \leqslant 2 \cdot \mathbb{E}\big(\|M(T)\|^2\big)^{\frac{1}{2}}.$$

is a Banach space.

Proposition (1.2.7)

Let $T > 0$ and $W(t)$, $t \in [0, T]$, be a U-valued Q-Wiener process with respect to a normal filtration $\mathcal{F}_t$, $t \in [0, T]$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then $W(t)$, $t \in [0, T]$, is a continuous square integrable $\mathcal{F}_t$-martingale, i.e. $W \in \mathcal{M}_T^2(U)$.

For the whole section we fix a positive real number T and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we define $\Omega_T := [0, T] \times \Omega$ and $P_T := \mathrm{d}x \otimes \mathbb{P}$ where $\mathrm{d}x$ is the Lebesgue measure.
Moreover, let $Q \in L(U)$ be symmetric, nonnegative and with finite trace and we consider a Q-Wiener process $W(t)$, $t \in [0, T]$, with respect to a normal filtration $\mathcal{F}_t$, $t \in [0, T]$.

1.3.1 Scheme of the construction of the stochastic integral

Step 1: First we consider a certain class $\mathcal{E}$ of elementary $L(U, H)$-valued processes and define the mapping

$$\begin{aligned} \text{Int} : \quad \mathcal{E} \quad &\rightarrow \quad \mathcal{M}_T^2(H) =: \mathcal{M}_T^2 \\ \Phi \quad &\mapsto \quad \int_0^t \Phi(s)\, \mathrm{d}W(s), \quad t \in [0, T]. \end{aligned}$$

Step 2: We prove that there is a certain norm on $\mathcal{E}$ such that

$$\text{Int} : \mathcal{E} \to \mathcal{M}_T^2$$

is a linear isometry. Since $\mathcal{M}_T^2$ is a Banach space, this implies that Int can be extended to the abstract completion $\bar{\mathcal{E}}$ of $\mathcal{E}$. This extension remains isometric and it is unique.

Step 3: We give an explicit representation of $\bar{\mathcal{E}}$.

Step 4: We show how the definition of the stochastic integral can be extended by localization.

Step 1: First we define the class $\mathcal{E}$ of all elementary processes as follows.

Definition (1.3.1 Elementary process)

An $L = L(U, H)$-valued process $\Phi(t)$, $t \in [0, T]$, on $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $\mathcal{F}_t$, $t \in [0, T]$, is said to be *elementary* if there exist $0 = t_0 < \cdots < t_k = T$, $k \in \mathbb{N}$, such that

$$\Phi(t) = \sum_{m=0}^{k-1} \Phi_m 1_{]t_m, t_{m+1}]}(t), \quad t \in [0, T],$$

where:

- $\Phi_m : \Omega \to L(U, H)$ is $\mathcal{F}_{t_m}$-measurable, w.r.t. strong Borel σ-algebra on $L(U, H)$, $0 \leqslant m \leqslant k - 1$,
- Φ_m takes only a finite number of values in $L(U, H)$, $1 \leqslant m \leqslant k - 1$.

Define the linear map $\mathrm{Int}: \mathcal{E} \to M_T^2$ by

$$\mathrm{Int}(\Phi)(t) := \int_0^t \Phi(s)\,\mathrm{d}W(s) := \sum_{m=0}^{k-1} \Phi_m \big(W(t_{m+1} \wedge t) - W(t_m \wedge t)\big), \qquad t \in [0,T]$$

(this is obviously independent of the representation) for all $\Phi \in \mathcal{E}$.

Proposition (1.3.2)

Let $\Phi \in \mathcal{E}$. Then the stochastic integral $\int_0^t \Phi(s)\,\mathrm{d}W(s)$, $t \in [0,T]$, *defined above, is a continuous square integrable martingale with respect to* $\mathcal{F}_t$, $t \in [0,T]$, *i.e.*

$$\mathrm{Int} : \mathcal{E} \to \mathcal{M}_T^2.$$

Step 2: To verify the assertion that there is a norm on $\mathcal{E}$ such that $\mathrm{Int} : \mathcal{E} \to \mathcal{M}_T^2$ is an isometry, we have to introduce the following notion.

Definition (1.3.3 Hilbert–Schmidt operator)

Let e_k, $k \in \mathbb{N}$, be an orthonormal basis of U. An operator $A \in L(U, H)$ is called Hilbert-Schmidt if

$$\sum_{k \in \mathbb{N}} \langle Ae_k, Ae_k \rangle < \infty.$$

Moreover, the space $L_2 := L_2(U, H)$ of all Hilbert–Schmidt operators from U to H equipped with the inner product

$$\langle A, B\rangle_{L_2} := \sum_{k\in\mathbb{N}} \langle Ae_k, Be_k\rangle \text{ (independent of ONB!)}$$

is a separable Hilbert space.

Proposition (1.3.4 "Itô-isometry")

If $\Phi = \sum_{m=0}^{k-1} \Phi_m 1_{]t_m, t_{m+1}]}$ *is an elementary* $L(U,H)$*-valued process then*

$$\left\| \int_0^{\cdot} \Phi(s) \, \mathrm{d}W(s) \right\|_{\mathcal{M}_T^2}^2 = \mathbb{E}\left(\int_0^T \left\| \Phi(s) \circ Q^{\frac{1}{2}} \right\|_{L_2}^2 \mathrm{d}s \right) =: \|\Phi\|_T^2$$

Step 3: To give an explicit representation of $\bar{\mathcal{E}}$ it is useful, to introduce the subspace $U_0 := Q^{\frac{1}{2}}(U)$ with the inner product given by

$$\langle u_0, v_0 \rangle_0 := \left\langle Q^{-\frac{1}{2}} u_0, Q^{-\frac{1}{2}} v_0 \right\rangle_U,$$

$u_0, v_0 \in U_0$, where $Q^{-\frac{1}{2}}$ is the pseudo inverse of $Q^{\frac{1}{2}}$ in the case that Q is not one-to-one. Then $(U_0, \langle\ ,\ \rangle_0)$ is again a separable Hilbert space. We have

$$\|L\|_{L_2^0} = \left\| L \circ Q^{\frac{1}{2}} \right\|_{L_2} \quad \text{for each } L \in L_2^0 := L_2(U_0, H).$$

Define $L(U,H)_0 := \{T|_{U_0} \mid T \in L(U,H)\}$. Since $Q^{\frac{1}{2}} \in L_2(U)$ it is clear that $L(U,H)_0 \subset L_2^0$ and that the $\| \, \|_T$-norm of $\Phi \in \mathcal{E}$ can be written as

$$\|\Phi\|_T = \left(\mathbb{E}\left(\int_0^T \|\Phi(s)\|_{L_2^0}^2 \, \mathrm{d}s\right)\right)^{\frac{1}{2}}.$$

We need the following σ-field:

$$\begin{aligned}
&\mathcal{P}_T : \\
&= \sigma\Big(\big\{]s,t] \times F_s \mid 0 \leqslant s < t \leqslant T,\, F_s \in \mathcal{F}_s\big\} \cup \big\{\{0\} \times F_0 \mid F_0 \in \mathcal{F}_0\big\}\Big) \\
&= \sigma\big(Y : \Omega_T \to \mathbb{R} \mid Y \text{ is left-continuous and adapted to } \\
&\qquad\qquad \mathcal{F}_t,\, t \in [0,T]\big).
\end{aligned}$$

We are now able to characterize $\bar{\mathcal{E}}$.

Proposition (1.3.5)

$$\begin{aligned}\bar{\mathcal{E}} :&= \left\{\Phi : [0,T]\times\Omega \to L_2^0 \;\middle|\; \Phi \textit{ is predictable and } \|\Phi\|_T < \infty\right\}\\ &= L^2\left([0,T]\times\Omega, \mathcal{P}_T, dt\otimes\mathbb{P}; L_2^0\right) =: \mathcal{N}_W^2(0,T;H).\end{aligned}$$

For simplicity we also write $\mathcal{N}_W^2(0,T)$ or $\mathcal{N}_W^2$ instead of $\mathcal{N}_W^2(0,T;H)$. Since $L(U,H)_0 \subset L_2^0$ and since any $\Phi \in \mathcal{E}$ is predictable by construction we indeed have that $\mathcal{E} \subset \mathcal{N}_W^2$.

Step 4: Finally by the so-called localization procedure we extend the stochastic integral even to the linear space

$$\mathcal{N}_W(0,T;H) := \left\{ \Phi : \Omega_T \to L_2^0 \,\middle|\, \Phi \text{ is predictable with } \mathbb{P}\left(\int_0^T \|\Phi(s)\|_{L_2^0}^2 \, \mathrm{d}s < \infty \right) = 1 \right\}.$$

For simplicity we also write $\mathcal{N}_W(0,T)$ or $\mathcal{N}_W$ instead of $\mathcal{N}_W(0,T;H)$ and $\mathcal{N}_W$ is called the class of *stochastically integrable* processes on $[0,T]$.
The extension is done in the following way:
For $\Phi \in \mathcal{N}_W$ we define

$$\tau_n := \inf\left\{ t \in [0,T] \,\middle|\, \int_0^t \|\Phi(s)\|_{L_2^0}^2 \, \mathrm{d}s > n \right\} \wedge T. \tag{1.4}$$

Then τ_n, $n \in \mathbb{N}$, is an increasing sequence of stopping times with respect to $\mathcal{F}_t$, $t \in [0,T]$ (i.e., $\{\tau_n \leqslant t\} \in \mathcal{F}_t \quad \forall t \in [0,T]$), such that

$$\mathbb{E}\left(\int_0^T \|1_{]0,\tau_n]}(s)\Phi(s)\|_{L_2^0}^2 \, \mathrm{d}s\right) \leqslant n < \infty.$$

Thus the stochastic integrals

$$\int_0^t 1_{]0,\tau_n]}(s)\Phi(s)\,\mathrm{d}W(s), \quad t \in [0,T],$$

are well-defined for all $n \in \mathbb{N}$. For arbitrary $t \in [0,T]$ we set

$$\int_0^t \Phi(s)\,\mathrm{d}W(s) := \int_0^t 1_{]0,\tau_n]}(s)\Phi(s)\,\mathrm{d}W(s), \tag{1.5}$$

where n is an arbitrary natural number such that $\tau_n \geqslant t$. This definition is consistent.

Lemma (1.3.6)

Let $\Phi \in \mathcal{N}_W(0,T)$ *and* f *an* $(\mathcal{F}_t)$*-adapted continuous* H*-valued process. Set*

$$\int_0^T \langle f(t), \Phi(t)\, \mathrm{d}W(t)\rangle := \int_0^T \tilde{\Phi}_f(t)\, \mathrm{d}W(t) \qquad (1.6)$$

with

$$\tilde{\Phi}_f(t)(u) := \langle f(t), \Phi(t)u\rangle,\ u \in U_0.$$

Then $\tilde{\Phi}_t \in \mathcal{N}_W(0,T;\ \mathbb{R})$ *and the stochastic integral is well-defined as a continuous* $\mathbb{R}$*-valued stochastic process.*

Until now we have considered the case that $W(t)$, $t \in [0, T]$, was a standard Q-Wiener process where $Q \in L(U)$ was nonnegative, symmetric and with finite trace. We could integrate processes in

$$\mathcal{N}_W := \Big\{ \Phi : \Omega_T \to L_2(Q^{\frac{1}{2}}(U), H) \mid \Phi \text{ is predictable and } \mathbb{P}\Big(\int_0^T \|\Phi(s)\|^2_{L_2^0} \, ds < \infty \Big) = 1 \Big\}.$$

1.4 The stochastic integral for cylindrical Wiener processes

In fact it is possible to extend the definition of the stochastic integral to the case that Q is not necessarily of finite trace. To this end we first have to introduce the concept of cylindrical Wiener processes.

Let $Q \in L(U)$ be nonnegative definite and symmetric.
Remember that in the case that Q is of finite trace the Q-Wiener process has the following representation:

$$W(t) = \sum_{k \in \mathbb{N}} \beta_k(t) e_k, \quad t \in [0, T],$$

where e_k, $k \in \mathbb{N}$, is an orthonormal basis of $Q^{\frac{1}{2}}(U) = U_0$ and β_k, $k \in \mathbb{N}$, is a family of independent real-valued Brownian motions.

The series converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; U)$, because the inclusion $U_0 \subset U$ defines a Hilbert–Schmidt embedding from $(U_0, \langle\ ,\ \rangle_0)$ to $(U, \langle\ ,\ \rangle)$. In the case that Q is no longer of finite trace one looses this convergence. Nevertheless, it is possible to define the Wiener process. For simplicity let $Q = I$, thus $U_0 = U$.

To this end we need a further Hilbert space $(U_1, \langle\ ,\ \rangle_1)$ and a Hilbert–Schmidt embedding

$$J : (U, \langle\ ,\ \rangle) \to (U_1, \langle\ ,\ \rangle_1).$$

Remark (1.4.1)

$(U_1, \langle\ ,\ \rangle_1))$ and J as above always exist; e.g. choose $U_1 := U$ and $\alpha_k \in]0, \infty[$, $k \in \mathbb{N}$, such that $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$. Define $J : U_0 \to U$ by

$$J(u) := \sum_{k=1}^{\infty} \alpha_k \langle u, e_k \rangle_0 \, e_k, \quad u \in U_0.$$

Then J is one-to-one and Hilbert–Schmidt.

Then the process given by the following proposition is called a *cylindrical Wiener process* in U.

Proposition (1.4.2)

Let e_k, $k \in \mathbb{N}$ be an orthonormal basis of U and β_k, $k \in \mathbb{N}$, a family of independent real-valued Brownian motions. Define $Q_1 := JJ^*$. Then $Q_1 \in L(U_1)$, Q_1 is nonnegative definite and symmetric with finite trace and the series

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) J e_k, \quad t \in [0, T], \tag{1.7}$$

converges in $\mathcal{M}_T^2(U_1)$ and defines a Q_1-Wiener process on U_1.

1.4.2 The definition of the stochastic integral for cylindrical Wiener processes

Basically we integrate with respect to the standard U_1-valued Q_1-Wiener process given by Proposition 1.4.2. In this sense we get that a process $\Phi(t)$, $t \in [0, T]$, is integrable with respect to $W(t)$, $t \in [0, T]$, if it takes values in $L_2(Q_1^{\frac{1}{2}}(U_1), H)$, is predictable and if

$$\mathbb{P}\left(\int_0^T \|\Phi(s)\|^2_{L_2(Q_1^{\frac{1}{2}}(U_1),H)}\, ds < \infty\right) = 1.$$

It is easy to check that

$$\Phi \in L_2(U,H) \Longleftrightarrow \Phi \circ J^{-1} \in L_2(Q_1^{\frac{1}{2}}(U_1),H)$$

Now we define

$$\int_0^t \Phi(s)\ \mathrm{d}W(s) := \int_0^t \Phi(s)\circ J^{-1}\ \mathrm{d}W(s), \quad t \in [0,T]. \quad (1.8)$$

where $\Phi \in \mathcal{N}_W :=$

$$\Big\{\Phi : \Omega_T \to L_2(U,H) \mid \Phi \text{ predictable}, \mathbb{P}\Big(\int_0^T \|\Phi(s)\|^2_{L_2^0}\, ds < \infty\Big) = 1\Big\}.$$

Chapter 2: Stochastic Differential Equations on Hilbert spaces

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathcal{F}_t$, $t \in [0, \infty[$, a normal filtration. Let $(W_t)_{t \geqslant 0}$ be a standard Wiener process on $\mathbb{R}^{d_1}$, $d_1 \in \mathbb{N}$, with respect to $\mathcal{F}_t$, $t \in [0, \infty[$. So, in the terminology of the previous section $U := \mathbb{R}^{d_1}$, $Q := \mathrm{I}$, and $H := \mathbb{R}^d$.

2.1 The finite dimensional case

Let $M(d \times d_1, \mathbb{R})$ denote the set of all real $d \times d_1$-matrices. Let the following maps $\sigma = \sigma(t, x, \omega)$, $b = b(t, x, \omega)$ be given:

$$\sigma : [0, \infty[\times \mathbb{R}^d \times \Omega \to M(d \times d_1, \mathbb{R})$$

$$b : [0, \infty[\times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$$

such that both are continuous in $x \in \mathbb{R}^d$ for each fixed $t \in [0, \infty[$, $w \in \Omega$, and progressively measurable, i.e. both σ and b restricted to $[0, t] \times \mathbb{R}^d \times \Omega$ are $\mathcal{B}([0, t]) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_t$-measurable for every $t \in [0, \infty[$.

We also assume that

$$\int_0^T \sup_{|x|\leqslant R} \{\|\sigma(t,x)\|^2 + |b(t,x)|\}\, \mathrm{d}t < \infty \text{ on } \Omega, \tag{2.1}$$

for all $T, R \in [0,\infty[$. Here $|\cdot|$ denotes the Euclidean distance on $\mathbb{R}^d$ and

$$\|\sigma\|^2 := \sum_{i=1}^{d} \sum_{j=1}^{d_1} |\sigma_{ij}|^2 (= \|\sigma\|^2_{L_2(\mathbb{R}^{d_1},\mathbb{R}^d)}). \tag{2.2}$$

$\langle\, ,\, \rangle$ below denotes the Euclidean inner product on $\mathbb{R}^d$.

Theorem (2.1.1)

Let b, σ be as above satisfying (2.1). Assume that on Ω for all $t, R \in [0, \infty[$, $x, y \in \mathbb{R}^d$, $|x|, |y| \leqslant R$

$$\begin{aligned} &2\langle x - y, b(t,x) - b(t,y)\rangle + \|\sigma(t,x) - \sigma(t,y)\|^2 \\ &\leqslant K_t(R)|x-y|^2 \quad \textit{(local weak monotonicity)} \end{aligned} \tag{2.3}$$

and

$$2\langle x, b(t,x)\rangle + \|\sigma(t,x)\|^2 \leqslant K_t(1)(1+|x|^2), \quad \textit{(weak coercivity)} \tag{2.4}$$

Theorem (2.1.1)

where for $R \in [0, \infty[$, $K_t(R)$ is an $\mathbb{R}_+$-valued $(\mathcal{F}_t)$-adapted process satisfying on Ω for all $R, T \in [0, \infty[$

$$\alpha_T(R) := \int_0^T K_t(R)\,\mathrm{d}t < \infty. \tag{2.5}$$

Then for any $\mathcal{F}_0$-measurable map $X_0 : \Omega \to \mathbb{R}^d$ there exists a unique solution to the stochastic differential equation

$$\mathrm{d}X(t) = b(t, X(t))\,\mathrm{d}t + \sigma(t, X(t))\,\mathrm{d}W(t). \tag{2.6}$$

Theorem (2.1.1)

Here solution means that $(X(t))_{t\geqslant 0}$ is a $\mathbb{P}$-a.s. continuous $\mathbb{R}^d$-valued $(\mathcal{F}_t)$-adapted process such that $\mathbb{P}$-a.s. for all $t \in [0, \infty[$

$$X(t) = X_0 + \int_0^t b(s, X(s))\,\mathrm{d}s + \int_0^t \sigma(s, X(s))\,\mathrm{d}W(s). \quad (2.7)$$

Furthermore, for all $t \in [0, \infty[$

$$\mathbb{E}(|X(t)|^2 e^{-\alpha_t(1)}) \leqslant \mathbb{E}(|X_0|^2) + 1. \quad (2.8)$$

Remark (2.1.2)

We note that by (2.1) the integrals on the right-hand side of (2.7) are well-defined.

Proof of Theorem 2.1.1.

Stopping and Euler approximation. □

Proposition (2.1.3)

Let the assumptions of Theorem 2.1.1 apart from (2.4) be satisfied. Let $X_0, X_0^{(n)} : \Omega \to \mathbb{R}^d$, $n \in \mathbb{N}$, *be* $\mathcal{F}_0$*-measurable such that*

$$\mathbb{P} - \lim_{n\to\infty} X_0^{(n)} = X_0.$$

Let $T \in [0, \infty[$ *and assume that* $X(t), X^{(n)}(t)$, $t \in [0, T]$, $n \in \mathbb{N}$, *are solutions of* (2.6) *(up to time* T*) such that* $X(0) = X_0$ *and* $X^{(n)}(0) = X_0^{(n)}$ $\mathbb{P}$*-a.e. for all* $n \in \mathbb{N}$. *Then*

$$\mathbb{P} - \lim_{n\to\infty} \sup_{t\in[0,T]} |X^{(n)}(t) - X(t)| = 0. \tag{2.9}$$

2.2 Gelfand triples, conditions on the coefficients and examples

Let H be a separable Hilbert space with inner product $\langle \ , \ \rangle_H$ and H^* its dual. Let V be a reflexive Banach space, such that $V \subset H$ continuously and densely. Then for its dual space V^* it follows that $H^* \subset V^*$ continuously and densely. Identifying H with H^* via the Riesz isomorphism we have that

$$V \subset H \subset V^* \tag{2.10}$$

continuously and densely and if ${}_{V^*}\langle \ , \ \rangle_V$ denotes the dualization between V^* and V (i.e. ${}_{V^*}\langle z, v\rangle_V := z(v)$ for $z \in V^*, v \in V$), it follows that

$${}_{V^*}\langle z, v\rangle_V = \langle z, v\rangle_H \quad \text{for all } z \in H, v \in V. \tag{2.11}$$

(V, H, V^*) is called a *Gelfand triple.*

Note that since $H \subset V^*$ continuously and densely, also V^* is separable, hence so is V. Furthermore, $\mathcal{B}(V)$ is generated by V^* and $\mathcal{B}(H)$ by H^*. We also have by Kuratowski's theorem that $V \in \mathcal{B}(H)$, $H \in \mathcal{B}(V^*)$ and $\mathcal{B}(V) = \mathcal{B}(H) \cap V$, $\mathcal{B}(H) = \mathcal{B}(V^*) \cap H$. Below we want to study stochastic differential equations on H of type

$$\mathrm{d}X(t) = A(t, X(t))dt + B(t, X(t))\,\mathrm{d}W(t) \tag{2.12}$$

with $W(t)$, $t \in [0, T]$ a cylindrical Q-Wiener process with $Q = I$ on another separable Hilbert space $(U, \langle\ ,\ \rangle_U)$ and with B taking values in $L_2(U, H)$ as in Chapter 1, but with A taking values in the larger space V^*.

The solution X will, however, take values in H again. In this section we give precise conditions on A and B.
Let $T \in [0, \infty[$ be fixed and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with normal filtration $\mathcal{F}_t$, $t \in [0, \infty[$. Let

$$A : [0, T] \times V \times \Omega \to V^*, \; B : [0, T] \times V \times \Omega \to L_2(U, H)$$

be *progressively measurable*, i.e. for every $t \in [0, T]$, these maps restricted to $[0, t] \times V \times \Omega$ are $\mathcal{B}([0, t]) \otimes \mathcal{B}(V) \otimes \mathcal{F}_t$-measurable.

As usual by writing $A(t, v)$ we mean the map $\omega \mapsto A(t, v, \omega)$. Analogously for $B(t, v)$. We impose the following conditions on A and B:

(H1) (*Hemicontinuity*) For all $u, v, x \in V$, $\omega \in \Omega$ and $t \in [0, T]$ the map

$$\mathbb{R} \ni \lambda \mapsto {}_{V^*}\langle A(t, u + \lambda v, \omega), x\rangle_V$$

is continuous.

(H2) (*Weak monotonicity*) There exists $c \in \mathbb{R}$ such that for all $u, v \in V$

$$2\,{}_{V^*}\langle A(\cdot,u) - A(\cdot,v), u - v\rangle_V + \|B(\cdot,u) - B(\cdot,v)\|^2_{L_2(U,H)}$$
$$\leqslant c\|u-v\|_H^2 \text{ on } [0,T]\times\Omega.$$

(H3) (*Coercivity*) There exist $\alpha \in]1, \infty[$, $c_1 \in \mathbb{R}$, $c_2 \in]0, \infty[$ and an $(\mathcal{F}_t)$-adapted process $f \in L^1([0, T] \times \Omega,\ \mathrm{d}t \otimes \mathbb{P})$ such that for all $v \in V, t \in [0, T]$

$$\begin{aligned} &2\,{}_{V^*}\langle A(t,v), v\rangle_V + \|B(t,v)\|^2_{L_2(U,H)} \\ \leqslant &c_1\|v\|_H^2 - c_2\|v\|_V^\alpha + f(t) \quad \textit{on } \Omega. \end{aligned}$$

(H4) (*Boundedness*) There exist $c_3 \in [0, \infty[$ and an $(\mathcal{F}_t)$-adapted $g \in L^{\frac{\alpha}{\alpha-1}}([0,T] \times \Omega,\ \mathrm{d}t \otimes \mathbb{P})$ s.th. for all $v \in V$, $t \in [0,T]$

$$\|A(t,v)\|_{V^*} \leqslant g(t) + c_3 \|v\|_V^{\alpha-1} \quad \text{on} \ \ \Omega,$$

where α is as in (H3).

Remark (2.2.1)

1. *By (H3) and (H4) it follows that for all* $v \in V$, $t \in [0,T]$

$$\begin{aligned} &\|B(t,v)\|^2_{L_2(U,H)} \\ &\quad \leqslant c_1\|v\|^2_H + f(t) + 2\|v\|_V\, g(t) + 2c_3\|v\|^\alpha_V \quad \text{on } \Omega. \end{aligned}$$

2. *We stress that we shall never need any explicit representation of* V^*. V^* *is only used as an auxiliary space!*

Remark (2.2.1)

3. *Let* $\omega \in \Omega$, $t \in [0,T]$. *(H1) and (H2) imply that* $A(t,\cdot,\omega)$ *is demicontinuous, i.e.*

$$u_n \to u \text{ as } n \to \infty \text{ (strongly) in } V$$

implies

$$A(t,u_n,\omega) \to A(t,u,\omega) \text{ as } n \to \infty \text{ weakly in } V^*,$$

In particular if $H = \mathbb{R}^d$, $d \in \mathbb{N}$, *hence* $V = V^* = \mathbb{R}^d$, *then (H1) and (H2) imply that* $u \mapsto A(t,u,\omega)$ *is continuous from* $\mathbb{R}^d$ *to* $\mathbb{R}^d$.

Consider the general situation described at the beginning of the previous section. So, we have a Gelfand triple

$$V \subset H \subset V^*$$

and maps

$$A : [0, T] \times V \times \Omega \to V^*, \quad B : [0, T] \times V \times \Omega \to L_2(U, H)$$

as specified there, satisfying (H1)–(H4), and consider the stochastic differential equation

$$\mathrm{d}X(t) = A(t, X(t))\,\mathrm{d}t + B(t, X(t))\,\mathrm{d}W(t) \tag{2.13}$$

on H with $W(t)$, $t \in [0, T]$, a cylindrical Q-Wiener process with $Q := I$ taking values in another separable Hilbert space $(U, \langle\ ,\ \rangle_U)$ and being defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $\mathcal{F}_t$, $t \in [0, T]$.

Definition (2.3.1)

A continuous H-valued $(\mathcal{F}_t)$-adapted process $(X(t))_{t\in[0,T]}$ is called a *solution of* (2.13), if $X \in L^\alpha([0,T]\times\Omega,\ \mathrm{d}t\otimes\mathbb{P};V)\cap L^2([0,T]\times\Omega,\ \mathrm{d}t\otimes\mathbb{P};H)$ with α as in (H3) and $\mathbb{P}$-a.s.

$$X(t) = X(0) + \int_0^t A(s,\bar{X}(s))ds + \int_0^t B(s,\bar{X}(s))\,\mathrm{d}W(s), \quad t\in[0,T], \tag{2.14}$$

where $\bar{X}$ is any V-valued progressively measurable $\mathrm{d}t\otimes\mathbb{P}$-version of X.

Theorem (2.3.2 Krylov-Rosowskii)

Let A, B above satisfy (H1)–(H4) and let $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$. Then there exists a unique solution X to (2.13) in the sense of Definition 2.3.1. Moreover,

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \|X(t)\|_H^2\Big) < \infty. \tag{2.15}$$

Proof.

By finite dimensional (Galerkin) approximation and the following Itô-formula. □

Theorem (2.3.3 Krylov-Rosowskii)

Let $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H), \alpha > 1,$ *and*
$Y \in L^{\frac{\alpha}{\alpha-1}}([0,T] \times \Omega,\ \mathrm{d}t \otimes \mathbb{P}; V^*),$
$Z \in L^2([0,T] \times \Omega,\ \mathrm{d}t \otimes \mathbb{P}; L_2(U,H)),$ *both progressively measurable.*
Define the continuous V^**-valued process*

$$X(t) := X_0 + \int_0^t Y(s)ds + \int_0^t Z(s)\,\mathrm{d}W(s),\ t \in [0,T].$$

Theorem (2.3.3 Krylov-Rosowskii)

If $X \in L^\alpha([0,T] \times \Omega,\ \mathrm{d}t \otimes \mathbb{P}, V)$, *then* X *is an* H*-valued continuous* $(\mathcal{F}_t)$*-adapted process,*

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|X(t)\|_H^2\right) < \infty$$

and the following Itô-formula holds for the square of its H*-norm* $\mathbb{P}$*-a.s.*

Theorem (2.3.3 Krylov-Rosowskii)

$$\|X(t)\|_H^2 = \|X_0\|_H^2 + \int_0^t \Big(2\,{}_{V^*}\langle Y(s), \bar{X}(s)\rangle_V + \|Z(s)\|_{L_2(U,H)}^2\Big)\,\mathrm{d}s + 2\int_0^t \langle X(s), Z(s)\,\mathrm{d}W(s)\rangle_H \quad \text{for all } t \in [0,T] \tag{2.16}$$

for any V-valued progressively measurable $\mathrm{d}t \otimes \mathbb{P}$-version $\bar{X}$ of X.

Proposition (2.3.4)

Consider the situation of Theorem 2.3.2 and let X, Y be two solutions. Then for $c \in \mathbb{R}$ as in (H2)

$$\mathbb{E}(\|X(t) - Y(t)\|_H^2) \leqslant e^{ct}\, \mathbb{E}(\|X(0) - Y(0)\|_H^2) \text{ for all } t \in [0, T]. \tag{2.17}$$

Below we shall give examples of operators $A : V \to V^*$, independent of t and ω, satisfying (H1)–(H4). If B is as in Chapter 2 satisfying (H2) with $A \equiv 0$, then the pair (A, B) satisfies (H1)–(H4). In all examples below V and H will be spaces of (possibly generalized) functions.

Example (3.1 $L^p \subset L^2 \subset L^{p/(p-1)}$ and $A(u) := -u|u|^{p-2}$)

Let $p \in [2, \infty[$, $\Lambda \subset \mathbb{R}^d$, Λ open. Let

$$V := L^p(\Lambda) := L^p(\Lambda,\ \mathrm{d}\xi),$$

equipped with its usual norm $\|\cdot\|_p$, and

$$H := L^2(\Lambda) := L^2(\Lambda,\ \mathrm{d}\xi),$$

where $\mathrm{d}\xi$ denotes Lebesgue measure on Λ. Then

$$V^* = L^{p/(p-1)}(\Lambda).$$

Example (3.1 $L^p \subset L^2 \subset L^{p/(p-1)}$ and $A(u) := -u|u|^{p-2}$)

If $p > 2$ we assume that

$$|\Lambda| := \int_{\mathbb{R}^d} \mathbb{I}_\Lambda(\xi)\, \mathrm{d}\xi < \infty. \tag{3.1}$$

Then

$$V \subset H \subset V^*,$$

or concretely

$$L^p(\Lambda) \subset L^2(\Lambda) \subset L^{p/(p-1)}(\Lambda)$$

continuously and densely. Recall that since $p > 1$, $L^p(\Lambda)$ is reflexive.

Example (3.1 $L^p \subset L^2 \subset L^{p/(p-1)}$ and $A(u) := -u|u|^{p-2}$)

Define $A : V \to V^*$ by

$$Au := -u|u|^{p-2},\ u \in V = L^p(\Lambda).$$

Indeed, A takes values in $V^* = L^{p/(p-1)}(\Lambda)$, since

$$\int |Au(\xi)|^{p/(p-1)}\,\mathrm{d}\xi = \int |u(\xi)|^p\,\mathrm{d}\xi < \infty$$

for all $u \in L^p(\Lambda)$.

Then A satisfies (H1)–(H4).

Example (3.1 $L^p \subset L^2 \subset L^{p/(p-1)}$ and $A(u) := -u|u|^{p-2}$)

E.g. (H2):

$$
\begin{aligned}
&{}_{V^*}\langle A(u) - A(v), u - v\rangle_V \\
&= \int (v(\xi)|v(\xi)|^{p-2} - u(\xi)|u(\xi)|^{p-2})(u(\xi) - v(\xi))\,\mathrm{d}\xi \leqslant 0,
\end{aligned}
$$

since the map $s \mapsto s|s|^{p-2}$ is increasing on $\mathbb{R}$. Thus (H2) holds, with $c := 0$.

The corresponding SDE (2.12) then reads

$$
\mathrm{d}X(t) = -X(t)|X(t)|^{p-2}\,\mathrm{d}t + B(t, X(t))\,\mathrm{d}W(t)
$$

Now we turn to cases where A is given by a (possibly nonlinear) partial differential operator. We shall start with the linear case; more concretely, A will be given by the classical Laplace operator

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial \xi_i^2}.$$

Again let $\Lambda \subset \mathbb{R}^d$, Λ open, and let $C_0^\infty(\Lambda)$ denote the set of all infinitely differentiable real-valued functions on Λ with compact support. Let $p \in [1, \infty[$ and for $u \in C_0^\infty(\Lambda)$ define

$$\|u\|_{1,p} := \left(\int (|u(\xi)|^p + |\nabla u(\xi)|^p) \, \mathrm{d}\xi \right)^{1/p} \tag{3.2}$$

and

$$H_0^{1,p}(\Lambda) := \text{ completion of } C_0^\infty(\Lambda) \text{ with respect to } \|\cdot\|_{1,p} \ (\hookrightarrow L^p(\Lambda)!!!). \tag{3.3}$$

$H_0^{1,p}(\Lambda)$ is called the *Sobolev space* of order 1 in $L^p(\Lambda)$ with *Dirichlet boundary conditions*.

Example (3.2 $H_0^{1,2} \subset L^2 \subset (H_0^{1,2})^*$, $A = \Delta$)

Though later we shall see that to have (H3) we have to take $p = 2$, we shall first take any $p \in [2, \infty[$ and define

$$V := H_0^{1,p}(\Lambda), H := L^2(\Lambda),$$

so

$$V^* := H_0^{1,p}(\Lambda)^*.$$

Example (3.2 $H_0^{1,2} \subset L^2 \subset (H_0^{1,2})^*$, $A = \Delta$)

Again we assume (3.1) to hold if $p > 2$. Since then $V \subset L^p(\Lambda) \subset H$, continuously and densely, identifying H with its dual we obtain the continuous and dense embeddings

$$V \subset H \subset V^*$$

or concretely

$$H_0^{1,p}(\Lambda) \subset L^2(\Lambda) \subset H_0^{1,p}(\Lambda)^*. \tag{3.4}$$

Example (3.2 $H_0^{1,2} \subset L^2 \subset (H_0^{1,2})^*$, $A = \Delta$)

Then

$$\Delta : C_0^\infty(\Lambda) \to C_0^\infty(\Lambda) \subset L^2(\Lambda) \subset V^*.$$

extends to a bounded linear operator

$$A(:= \Delta) : H_0^{1,p}(\Lambda) \to H_0^{1,p}(\Lambda)^*.$$

which satisfies (H1),(H2),(H4) and provided $p = 2$, also (H3).

Example (3.2 $H_0^{1,2} \subset L^2 \subset (H_0^{1,2})^*$, $A = \Delta$)

E.g. (H2): for $u_n, v_n \in C_0^\infty(\nabla)$ with $u_n \to u,\ v_n \to v$ in V,

$$
\begin{aligned}
{}_{V^*}\langle A(u) - A(v), u - v\rangle_V &= \lim_{n\to\infty} {}_{V^*}\langle \Delta u_n - \Delta v_n, u_n - v_n\rangle_V \\
&= \lim_{n\to\infty} \langle \Delta(u_n - v_n), u_n - v_n\rangle_H \\
&= \lim_{n\to\infty} -\int |\nabla(u_n - v_n)(\xi)|^2 \, d\xi \leqslant 0.
\end{aligned}
$$

The corresponding SDE (2.12) then reads

$$\mathrm{d}X(t) = \Delta X(t)\,\mathrm{d}t + B(t, X(t))\,\mathrm{d}W(t).$$

If $B \equiv 0$, this is just the classical *heat equation*. If $B \not\equiv 0$, but constant, the solution is an *Ornstein–Uhlenbeck process* on H.

Example (3.3 $H_0^{1,p} \subset L^2 \subset (H_0^{1,p})^*$, $A = p$-Laplacian)

Again we take $p \in [2, \infty[$, $\Lambda \in \mathbb{R}^d$, Λ open and bounded, and $V := H_0^{1,p}(\Lambda)$, $H := L^2(\Lambda)$, so $V^* = (H_0^{1,p}(\Lambda))^*$. Define $A : H_0^{1,p}(\Lambda) \to H_0^{1,p}(\Lambda)^*$ by

$$A(u) := \mathrm{div}(|\nabla u|^{p-2}\nabla u), \; u \in H_0^{1,p}(\Lambda);$$

more precisely, given $u \in H_0^{1,p}(\Lambda)$ for all $v \in H_0^{1,p}(\Lambda)$

$${}_{V^*}\langle A(u), v\rangle_V := -\int |\nabla u(\xi)|^{p-2}\langle \nabla u(\xi), \nabla v(\xi)\rangle \, \mathrm{d}\xi. \tag{3.5}$$

Example (3.3 $H_0^{1,p} \subset L^2 \subset (H_0^{1,p})^*$, $A = p$-Laplacian)

A is called the *p-Laplacian*, also denoted by Δ_p. Note that $\Delta_2 = \Delta$. (To show that $A : V \to V^*$ is well-defined we have to show that the right-hand side of (3.5) defines a linear functional in $v \in V$ which is continuous with respect to $\|\cdot\|_V = \|\cdot\|_{1,p}$.) Then A satisfies (H1)–(H4).

E.g.(H2): Let $u, v \in H_0^{1,p}(\Lambda)$. Then by (3.5)

$$- {}_{V^*}\langle A(u) - A(v), u - v \rangle_V$$

Example (3.3 $H_0^{1,p} \subset L^2 \subset (H_0^{1,p})^*$, $A = p$-Laplacian)

$$
\begin{aligned}
&= \int \langle |\nabla u(\xi)|^{p-2} \nabla u(\xi) - |\nabla v(\xi)|^{p-2} \nabla v(\xi), \nabla u(\xi) - \nabla v(\xi) \rangle \, \mathrm{d}\xi \\
&= \int (|\nabla u(\xi)|^p + |\nabla v(\xi)|^p - |\nabla u(\xi)|^{p-2} \langle \nabla u(\xi), \nabla v(\xi) \rangle \\
&\qquad - |\nabla v(\xi)|^{p-2} \langle \nabla u(\xi), \nabla v(\xi) \rangle) \, \mathrm{d}\xi \\
&\geqslant \int (|\nabla u(\xi)|^p + |\nabla v(\xi)|^p - |\nabla u(\xi)|^{p-1} |\nabla v(\xi)| \\
&\qquad - |\nabla v(\xi)|^{p-1} |\nabla u(\xi)|) \, \mathrm{d}\xi \\
&= \int (|\nabla u(\xi)|^{p-1} - |\nabla v(\xi)|^{p-1})(|\nabla u(\xi)| - |\nabla v(\xi)|) \, \mathrm{d}\xi \geqslant 0.
\end{aligned}
$$

The corresponding SDE (2.12) then reads

$$\mathrm{d}X(t) = div(|\nabla X(t)|^{p-2}\nabla X(t))\,\mathrm{d}t + B(t, X(t))\,\mathrm{d}W(t).$$

Example (3.4 $L^p \subset (H_0^{1,2})^* \subset (L^p)^*$, $A =$ porous medium operator)

Let $\Lambda \subset \mathbb{R}^d$, Λ open and bounded, $p \in [2, \infty[$ and

$$V := L^p(\Lambda), H := (H_0^{1,2}(\Lambda))^*.$$

Since Λ is bounded we have by Poincaré's inequality, that for some constant $c > 0$

$$\begin{aligned} \|u\|_{1,2} \geqslant \|u\|_{H_0^{1,2}} := &\left(\int |\nabla u(\xi)|^2 \, \mathrm{d}\xi \right)^{\frac{1}{2}} \\ \geqslant &c\|u\|_{1,2} \quad \text{for all } u \in H_0^{1,2}(\Lambda). \end{aligned} \tag{3.6}$$

Example (3.4 $L^p \subset (H_0^{1,2})^* \subset (L^p)^*$, $A =$ porous med. op.)

So, we can (and will do so below) consider $H_0^{1,2}(\Lambda)$ with norm $\|\cdot\|_{H_0^{1,2}}$ and corresponding scalar product

$$\langle u, v\rangle_{H_0^{1,2}} := \int \langle \nabla u(\xi), \nabla v(\xi)\rangle \, \mathrm{d}\xi, \; u, v \in H_0^{1,2}(\Lambda).$$

Since $H_0^{1,2}(\Lambda) \subset L^2(\Lambda)$ continuously and densely, so is

$$H_0^{1,2}(\Lambda) \subset L^{\frac{p}{p-1}}(\Lambda).$$

Hence

$$L^p(\Lambda) \equiv \left(L^{\frac{p}{p-1}}(\Lambda)\right)^* \subset (H_0^{1,2}(\Lambda))^* = H,$$

continuously and densely.

Example (3.4 $L^p \subset (H_0^{1,2})^* \subset (L^p)^*$, $A =$ porous med. op.)

Now we identify H with its dual H^* by the Riesz map $(-\Delta)^{-1} : H \to H^*$, so $H \equiv H^*$ in this sense, hence

$$V = L^p(\Lambda) \subset H \subset (L^p(\Lambda))^* = V^* \tag{3.7}$$

continuously and densely.

Example (3.4 $L^p \subset (H_0^{1,2})^* \subset (L^p)^*$, $A =$ porous med. op.)

Exercise: *The map*

$$\Delta : H_0^{1,2}(\Lambda) \to (L^p(\Lambda))^*$$

extends to a linear isometry

$$\Delta : L^{\frac{p}{p-1}}(\Lambda) \to (L^p(\Lambda))^* = V^*$$

and for all $u \in L^{\frac{p}{p-1}}(\Lambda)$, $v \in L^p(\Lambda)$

$$_{V^*}\langle -\Delta u, v \rangle_V = {}_{L^{\frac{p}{p-1}}}\langle u, v \rangle_{L^p} = \int u(\xi) v(\xi) \, \mathrm{d}\xi. \qquad (3.8)$$

This isometry is in fact surjective, hence

$$(L^p(\Lambda))^* = \Delta(L^{\frac{p}{p-1}}) \neq L^{\frac{p}{p-1}}.$$

Example (3.4 $L^p \subset (H_0^{1,2})^* \subset (L^p)^*$, $A =$ porous med. op.)

Now we want to define the "porous medium operator A". So, let $\Psi : \mathbb{R} \to \mathbb{R}$ be a continuous **increasing** function having the following properties:

(Ψ1) There exist $p \in [2, \infty[$, $a \in]0, \infty[$, $c \in [0, \infty[$ such that for all $s \in \mathbb{R}$

$$s\Psi(s) \geqslant a|s|^p - c.$$

(Ψ2) There exist $c_3, c_4 \in]0, \infty[$ such that for all $s \in \mathbb{R}$

$$|\Psi(s)| \leqslant c_4 + c_3|s|^{p-1},$$

where p is as in (Ψ1).

Example (3.4 $L^p \subset (H_0^{1,2})^* \subset (L^p)^*$, $A =$ porous med. op.)

We note that (Ψ4) implies that

$$\Psi(v) \in L^{\frac{p}{p-1}}(\Lambda) \quad \text{for all } v \in L^p(\Lambda). \tag{3.9}$$

Now we can define the *porous medium operator*
$A : L^p(\Lambda) = V \to V^* = (L^p(\Lambda))^*$ by

$$A(u) := \Delta\Psi(u), \quad u \in L^p(\Lambda). \tag{3.10}$$

Then (H1)–(H4) hold.

Example (3.4 $L^p \subset (H_0^{1,2})^* \subset (L^p)^*$, $A =$ porous med. op.)

E.g.(H2): Let $u, v \in V = L^p(\Lambda)$. Then by (3.8)

$$
\begin{aligned}
&{}_{V^*}\langle A(u) - A(v), u - v)\rangle_V \\
&= {}_{V*}\langle \Delta(\Psi(u) - \Psi(v)), u - v\rangle_V \\
&= -\int [\Psi(u(\xi)) - \Psi(v(\xi))](u(\xi) - v(\xi))\, \mathrm{d}\xi \\
&\leqslant 0,
\end{aligned}
$$

where we used that Ψ is increasing in the last step.