Weak approximations of Burgers like SPDEs

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Stochastic Burger's equation wit additive noise

We consider a real valued stochastic process $X = X(t,\xi)$ solution of

$$\begin{cases} X_t - X_{\xi\xi} + \left(\frac{X^2}{2}\right)_\xi = \dot{\eta}, \quad \xi \in (0,1), t > 0, \\ X(t,0) = X(t,1), \; t > 0, \\ X(0,\xi) = x(\xi), \; \xi \in (0,1). \end{cases}$$

where $\dot{\eta} := \dfrac{\partial \eta}{\partial t}$ with $\eta = \eta(t,\xi)$ is a real valued Gaussian process with zero mean value.

Abstract framework

- $H = L^2(0,1)$ Hilbert space (inner product $(\cdot,\cdot)$, norm $|\cdot|$).
- $A = -(\cdot)_{\xi\xi}$ unbounded linear operator on H, $A : D(A) \to H$ with domain $D(A) = H_0^1(0,1) \cap H^2(0,1)$.
- $D(A) \subset\subset H$.
- There exists $\{e_n\}_{n\geq 1}$ Hilbertian basis of H, $Ae_n = \lambda_n e_n$,

$$0 < \lambda_1 < \lambda_2 < \lambda_n \cdots$$

- For real s, $A^s X := \sum_{n\geq 1} \lambda_n^s (X, e_n) e_n$.
- For $s \geq 0$, $D(A^s) = \{X \in H \text{ s.t. } \sum_{n\geq 1} \lambda_n^{2s} (X_n, e_n)^2 < +\infty\}$ Hilbert space endowed by the norm

$$X \mapsto |X|_s := (|X|^2 + |A^s X|^2)^{1/2}.$$

- $s > 0$, $D(A^{-s})$ defined as the completion of H with respect to the norm $|\cdot|_{-s}$ defined as above. One denotes that $D(A^{-s}) = D(A^s)^\star$.
- $s_1 < s < s_2$, $s = \theta s_2 + (1-\theta)s_1$,

$$|X|_s \leq |X|_{s_1}^{1-\theta} |X|_{s_2}^{\theta}, \quad \text{for any } X \in D(A^{s_1}).$$

Spatial regularity of the nonlinearity

For any test function φ, one has

$$\begin{aligned}
\left|\int (X^2)_\xi \varphi\right| &\leq \left|\int X^2 \varphi_\xi\right| \\
&\leq \sup_{(0,1)} |\varphi_\xi||X|^2 \\
&\leq C\|\varphi\|_{H^{3/2+\varepsilon}(0,1)}|X|^2 \quad \text{(Sobolev embedding)} \\
&\leq C|\varphi|_{3/4+\varepsilon'}|X|^2,
\end{aligned}$$

We define $f(X)(\xi) := (X^2(\xi)/2)_\xi$. Then

$$(f(X), \varphi) \leq C|\varphi|_{3/4+\varepsilon'}|X|^2.$$

Then $f : H \to D(A^{-s})$ and $|f(X)|_{-s} \leq C|X|^2$ for any $3/4 < s < 1$.

Example of noise

Let W be a cylindrical Wiener process on $H = L^2(0,1)$ ($= \sum_n \beta_n e_n$ e.g.). We consider noise of the form $\eta = Q^{1/2} W$ where Q is some symmetric non negative bounded linear operator on H.

Example : given a function q on $(0,1)$, we take

$$\eta = \int_0^1 q(x,y)\, W(y,t)$$

then $E\eta(t,x)\eta(s,y) = c(x,y)(t \wedge s)$ where
$c(x,y) = \int_0^1 q(x,z)q(y,z)dz$.
In this case $(Qf)(x) = \int_0^1 c(x,y)f(y)dy$.

Covariance operator $Q = Q^{1/2}(Q^{1/2})^\star$:

$$\mathbb{E}(\eta(t), u)(\eta(s), v) = (t \wedge s)(Qu, v), \quad \text{for any } u, v \in H.$$

Parabolic semi-linear stochastic PDE

$$dX + (AX + f(X))dt = Q^{1/2}dW, \quad X(0) = x \in H,$$

where

- W is a cylindrical Wiener process on H,
- $\mathrm{Tr}(A^{-\alpha}) < \infty$ for some positive $\alpha > 0$,
- $f : H \to D(A^{-s})$ and
- $Q \in \mathcal{L}(H, D(A^{\beta}))$ for some real β s.t. $1 - \alpha + \beta > 0$.

Examples : reaction-diffusion equations, Navier-Stokes, Ginzburg-Landau, Cahn-Hilliard, . . .

Main assumption on the noise

- In this talk, Q is not required to be nuclear or to commute with A.
- In the case of the heat equation in dimension $d \in \{1, 2, 3\}$ (e.g. $A = -\Delta$ on a bounded domain and $f = 0$), we have

$$\mathrm{Tr}(A^{-\alpha}) = \sum_{n \geq 1} \lambda_n^{-\alpha} < \infty \quad \text{iff} \quad \alpha > \frac{d}{2}.$$

Then $1 - \alpha + \beta > 0$ iff $\beta > \frac{d}{2} - 1$, i.e.

$$\beta > -1/2, \quad d = 1; \quad \beta > 0 \quad d = 2; \quad \beta > 1/2, \quad d = 3, \cdots$$

Full discretization

We discretize the former SPDEs with the Euler scheme

- $\Delta t > 0$, $t_k = k\Delta t$.
- Time discretization

$$X(t_{k+1}) - X(t_k) + \int_{t_k}^{t_{k+1}} (AX(t) + f(X(t)))dt + \int_{t_k}^{t_{k+1}} Q^{1/2} dW(t)$$

  which gives raise to (at least) two numerically schemes

$$\begin{array}{lll} \stackrel{1}{\rightarrow} & X_{k+1} - X_k + \Delta t(AX_{k+1} + f(X_k)) & = \sqrt{\Delta t} Q^{1/2} \chi_{k+1}, \\ \stackrel{2}{\rightarrow} & X_{k+1} - X_k + \Delta t(AX_{k+1} + f(X_{k+1})) & = \sqrt{\Delta t} Q^{1/2} \chi_{k+1} \end{array}$$

  where $\sqrt{\Delta t}\chi_{k+1} = W(t_{k+1}) - W(t_k)$.
- Full discretization Let $\{V_h\}_{h>0}$, $V_h \subset V := D(A^{1/2})$ be finite dimensional spaces and consider the scheme

$$(X^h_{k+1} - X^h_k, Y^h) + \Delta(A^{1/2} X^h_{k+1}, A^{1/2} Y^h) + \Delta t(f(X^h_k), Y^h) = \sqrt{\Delta t}(Q^{1/2}\chi_{k+1}, Y^h),$$

  for any $Y^h \in V_h$.

Strong error

We investigate the error committed when approximating $X(t_k)$ par X_k^h at time $t_k = k\Delta t$.

Strong error : $\mathbb{E}|X_k - X(t_k)|$ (time error) or $\mathbb{E}|X_k^h - X(t_k)|$ (Full error)

Other type of strong errors : pathwise error, in the supremum in time, in probability , . . .

(Gyöngy, Nualart; 1997), (Gyöngy; 1999), (Shardlow; 1999), (Gaines, Davies; 2001), (P; 2001), (Hausenblas, 2002-2003), (Yan; 2004), (Millet, Morien; 2005), (Gyöngy, Millet; 2005), (Walsh; 2005), (Debussche, de Bouard; 2004-2006), . . .

Typical result : $A = -\Delta$, $H = L^2(0,1)$, $f \in \mathcal{C}_b^1(H)$ then

$$\begin{aligned} \mathbb{E}|X(t_k) - X_k| &\leq C\Delta t^\gamma, \quad \text{for any } \gamma < 1/4, \\ \mathbb{E}|X(t_k) - X_k^h| &\leq C(\Delta t^\gamma + h^\delta), \quad \text{for any } \alpha < 1/4,\ \delta < 1/2. \end{aligned}$$

Strong error for the Burger's equation

- Lipschitz case $(A^{-s}f)$ (P; 2001)

  $\mathbb{E}|X(t_k) - X_k| \leq C\Delta t^{\gamma}$, for any $\gamma < \min(1-s, (1-\alpha+\beta)/2)$.

- Burger's equation, non additive case : order $1/4^{-}$ in probability, i.e.

$$\lim_{C\to+\infty} \mathbb{P}(\max_{0\leq k\leq n} |X(t_k) - X_k| \geq C\Delta t^{\gamma}) = 0,$$

  uniformly in Δt, for any $\gamma < 1/4$.

- Burger's equation, additive case : pathwise order $1/4^{-}$, i.e.

$$\max_{0\leq k\leq n} |X(t_k) - X_k| \leq C\Delta t^{\gamma},\ a.s.,$$

  for any $\gamma < 1/4$.

Some commentaries

- In general, we strongly use the fact that the equation is semi-linear so that we can write it in an integral form together with its discrete counterpart.In this way, the minimal regularity assumptions are required.
- The Euler scheme has a strong order depending on the regularity of the noise.
  e.g. $1/4$ with white noise, $1/2$ (finite dimensional case) for a smooth noise.
- The strong error has the same ordre than the local one, i.e. *the time regularity of the continuous solution*. Because these results use regularizing properties of the discrete semi-group.

Weak error

Estimate of

$$\mathbb{E}(\varphi(X(T)) - \varphi(X_n)) = \int_H \varphi d\mu_{X(T)} - \int_H \varphi d\mu_{X_n}, \quad \varphi : H \to \mathbb{R}.$$

Finite dimensional case (SDE): Talay, Tubaro, Bally, Kohatsu-Higa, Milstein, Tretyakov, Kloeden, Platten

Inifnite dimensional case (SPDE) : Buckwar, Shardlow, Lmaberton, Clément, Kohatsu-Higa, . . .

The finite dimensional case : the PDE method

We consider $X(t,x)$ the solution at time t of the SDE

$$dX = f(X)dt + \sigma(X)dW, \qquad X(0) = x,$$

where f and σ are smooth real valued functions and W is the standart Brownian motion.

We also consider its Euler approximation

$$X_{k+1} = X_k + \Delta t\, f(X_k) + \sqrt{\Delta t}\sigma(X_k)\chi_{k+1}, \qquad X_0 = x,$$

with the same notations as above.

We set $u(t,x) = \mathbb{E}\varphi(X(t,x))$. We known that u is the solution of the finite dimensional Kolomogorov equation

$$\frac{\mathrm{d}u}{\mathrm{d}t}(t,x) = (Du(t,x), f(x)) + \frac{1}{2}\mathrm{Tr}(\sigma(x)\sigma^\star(x)D^2 u(t,x)), \qquad u(0) = \varphi.$$

Hence the weak error can be written as

$$\begin{aligned}\mathbb{E}\varphi(X_n) - \mathbb{E}\varphi(X(T,x)) &= \mathbb{E}(u(0,X_n) - u(T,X_0)) \\ &= \sum_{k=0}^{n-1} \mathbb{E}(u(T-t_{k+1},X_{k+1}) - u(T-t_k,X_k))\end{aligned}$$

- Interpolation of the discrete solution : we interpolate X_k by a continuous adpated process $\widetilde{X}(t)$ at time t_k :

$$\widetilde{X}(t) = X_k + \int_{t_k}^{t} f(X_k)\, ds + \int_{t_k}^{t} \sigma(X_k)\, dW(s), \quad t \in [t_k, t_{k+1}).$$

  or on $[t_k, t_{k+1})$, we set $d\widetilde{X}(t) = f(X_k)\, dt + \sigma(X_k)\, dW(t)$, $\widetilde{X}(t_k) = X_k$,

- We apply Itô formula at $t \mapsto u(T-t, \widetilde{X}(t))$ between t_k and t_{k+1} :

$$\mathbb{E}(u(T-t_{k+1}, X_{k+1}) - u(T-t_k, X_k)) = \mathbb{E}\int_{t_k}^{t_{k+1}} du(T-t, \widetilde{X}(t)),$$

  where

$$\begin{aligned} du(T-t,\widetilde{X}(t)) \quad = \quad & -\frac{\partial u}{\partial t}(T-t,\widetilde{X}(t))dt + (Du(T-t,\widetilde{X}(t)), f(X_k))\, dt \\ & +(Du(T-t,\widetilde{X}(t)), \sigma(X_k)\, dW(t)) \\ & +\frac{1}{2}\mathrm{Tr}(\sigma(X_k)\sigma(X_k)^\star D^2 u(T-t,\widetilde{X}(t)))\, dt \end{aligned}$$

Using the Kolomogorov equation, we obtain

$$\mathbb{E}(u(T-t_{k+1},\widetilde{X}(t_{k+1}))-u(T-t_k,\widetilde{X}(t_k)))$$

$$=\mathbb{E}\Big(\int_{t_k}^{t_{k+1}}(Lu(T-t,\widetilde{X}(t))-L_{k,\Delta t}u(T-t,\widetilde{X}(t)))\,dt\Big),$$

where

$$L\psi(x)=(D\psi(t,x),f(x))+\frac{1}{2}\mathrm{Tr}(\sigma(x)\sigma^\star(x)D^2\psi(x)),$$

and

$$L_{k,\Delta t}\psi(x)=(D\psi(t,x),f(X_k))+\frac{1}{2}\mathrm{Tr}(\sigma(X_k)\sigma^\star(X_k)D^2\psi(x)),$$

Let us see the first term of the error

$$\mathbb{E}(f(\widetilde{X}(t))-f(X_k),Du(T-t,\widetilde{X}(t)))=\mathbb{E}\left(\int_{t_k}^{t}df(\widetilde{X}(r)),Du(T-t,\widetilde{X}(t))\right)$$

$$=\mathbb{E}\left(\left(\int_{t_k}^{t}L_{k,\Delta t}f(\widetilde{X}(r))\,dr,Du(T-t,\widetilde{X}(t))\right)\right)$$

$$+\mathbb{E}\left(\left(\int_{t_k}^{t}\Big(Df(\widetilde{X}(r)),\sigma(X_k)dW(r)\Big),Du(T-t,\widetilde{X}(t))\right)\right)$$

$$
\begin{aligned}
&\mathbb{E}(u(T-t_{k+1},\widetilde{X}(t_{k+1}))-u(T-t_k,\widetilde{X}(t_k)))\\
&=\mathbb{E}\left(\int_{t_k}^{t_{k+1}}(Lu(T-t,\widetilde{X}(t))-L_{k,\Delta t}u(T-t,\widetilde{X}(t)))\,dt\right)\\
&=\mathcal{O}(\Delta t^2)
\end{aligned}
$$

Hence the weak error is

$$
\begin{aligned}
\mathbb{E}\varphi(X_n)-\mathbb{E}\varphi(X(T,x)) &= \mathbb{E}(u(0,X_n)-u(T,X_0))\\
&= \sum_{k=0}^{n-1}\mathbb{E}(u(T-t_{k+1},X_{k+1})-u(T-t_k,X_k))\\
&= \mathcal{O}(\Delta t)
\end{aligned}
$$

The SPDE case

$$dX + (AX + f(X))dt = Q^{1/2}dW, \quad X(0) = x$$

Scheme 1

$$X_{k+1} - X_k + \Delta t(AX_{k+1} + f(X_k)) = \sqrt{\Delta t}Q^{1/2}\chi_{k+1}$$

Interpolation of the discrete solution

$$\begin{aligned} X_{k+1} &= S_{\Delta t}X_k - \Delta t S_{\Delta t}f(X_k) + \sqrt{\Delta t}S_{\Delta t}Q^{1/2}\chi_{k+1} \\ &= X_k - \Delta t A_{\Delta t}X_k - \Delta t S_{\Delta t}f(X_k) + \sqrt{\Delta t}S_{\Delta t}Q^{1/2}\chi_{k+1} \end{aligned}$$

$\rightarrow d\widetilde{X}(t) + (A_\Delta X_k + S_{\Delta t}f(X_k))\,dt = S_{\Delta t}Q^{1/2}dW(t), \quad t \in [t_k, t_{k+1}),$

where $S_{\Delta t} = (I + \Delta t A)^{-1}$ and $A_{\Delta t} = \frac{1}{\Delta t}(I - S_{\Delta t})$.

The weak error

$$\mathbb{E}\varphi(X_n) - \mathbb{E}\varphi(X(T,x)) = \mathbb{E}(u(0,X_n) - u(T,X_0))$$

$$= \sum_{k=0}^{n-1} \mathbb{E}(u(T - t_{k+1}, \widetilde{X}(t_{k+1})) - u(T - t_k, \widetilde{X}(t_k)))$$

$$= \sum_{k=0}^{n-1} \mathbb{E}\left(\int_{t_k}^{t_{k+1}} (Lu(T-t, \widetilde{X}(t)) - L_{k,\Delta t} u(T-t, \widetilde{X}(t)))\, dt \right),$$

where

$$L\psi(x) = -(D\psi(x), Ax + f(x)) + \frac{1}{2}\mathrm{Tr}(Q^{1/2}(Q^{1/2})^\star D^2\psi(x)),$$

and

$$L_{k,\Delta t}\psi(x) = -(D\psi(x), A_{\Delta t}X_k + S_{\Delta t} f(X_k)) + \frac{1}{2}\mathrm{Tr}((S_{\Delta t}Q^{1/2})(S_{\Delta t}Q^{1/2})^\star D^2\psi(x)),$$

First term

$$\Big(Du(T-t,\widetilde{X}(t)),A_{\Delta t}X_k - A\widetilde{X}(t)\Big)$$
$$= \Big(Du(T-t,\widetilde{X}(t)),(A_{\Delta t}-A)X_k\Big)$$
$$+\Big(Du(T-t,\widetilde{X}(t)),A(X_k-\widetilde{X}(t))\Big).$$

Let us recall that $A - A_{\Delta t} = \Delta t S_{\Delta t} A^2$ (In fact $A_{\Delta t} = S_{\Delta t} A$).

First idea : remove A – Linear case $f = 0$

- Change of unknown $Y(t, e^{-TA}x) = e^{-(T-t)A}X(t,x)$:

$$dY = e^{-(T-t)A}Q^{1/2}dW, \quad Y(0) = e^{-TA}x.$$

- Discrete counterpart $Y_k = S_{\Delta t}^{N-k}X_k$.
- Interpolation of the unknowns

$$\widetilde{Y}(t) = Y_k + \int_{t_k}^{t} S_{\Delta t}^{N-k}Q^{1/2}dW(s), \quad t \in [t_k, t_{k+1})$$

- Weak error

$$\mathbb{E}\varphi(X(T,x)) - \mathbb{E}\varphi(X_n) = \mathbb{E}v(T, e^{-TA}x) - \mathbb{E}v(0, Y_n)$$

$$= \mathbb{E}v(T, e^{-TA}x) - \mathbb{E}v(T, S_{\Delta t}^{n}x) \quad \Big\} \textit{deterministic}\, \mathcal{O}(\Delta t)$$

$$+ \mathbb{E}v(T, Y_0) - \mathbb{E}v(0, Y_n)$$

- Apply Itô formula to $t \mapsto v(T-t, \widetilde{Y}_t)$.

$$\mathbb{E}(v(T-t_{k+1},Y_{k+1})-v(T-t_k,Y_k))$$

$$=\frac{1}{2}\mathbb{E}\int_{t_k}^{t_{k+1}}\mathrm{Tr}\left[(S_{\Delta t}^{n-k-1}Q^{1/2})(S_{\Delta t}^{n-k-1}Q^{1/2})^\star-(e^{-(T-t)A}Q^{1/2})(e^{-(T-t)A}Q^{1/2})^\star\right]dt,$$

We revover twice the strong order.

Theorem (Debussche, Printems ; 2009)

Let $\varphi\in\mathcal{C}_b^2(H)$. *For any* $\gamma<1-\alpha+\beta$ *there exists a constant* $C(T,\varphi,|x|,\gamma)>0$ *independant from* h *and* n *s.t.*

$$|\mathbb{E}\varphi(X_k^h)-\varphi(X(t_k))|\le C(T,\varphi,|x|,\gamma)\left(\Delta t^\gamma+h^{2\gamma}\right).$$

(See also Geissert, Larsson, Kovàcs, 2009)

The nonlinear case

Such a technique can not be generalized to the nonlinear case, even for a smooth f : since the semi-group is not invertible, the new unknown $Y(t)$ is no more Markovian.

We go back to the first method. We recall that the first term was

$$\Big(Du(T-t,\widetilde{X}(t)),\Delta t S_{\Delta t}A^2X_k\Big)+\Big(Du(T-t,\widetilde{X}(t)),A(X_k-\widetilde{X}(t))\Big).$$

Some remarks : in dimension 1 for the nonlinear heat equation when f is Lipschitz, we know that (Debussche, 2009)

- $|A^\gamma Du(t,x)|\leq ct^{-\gamma}$, for any $\gamma<1/2$.
- $X_k\in D(A^{1/4-\varepsilon})$.
- $|A^\delta S_{\Delta t}|\leq c\Delta t^{-\gamma}$.

$$\rightarrow \Delta t\Big(Du(T-t,\widetilde{X}(t)),S_{\Delta t}A^2X_k\Big)\leq c\Delta t^{-1/4-\varepsilon}$$

The lack of regularity of X_k is due to the stochastic integral (discrete) :

$$X_k = S_{\Delta t}^k x - \Delta t \sum_{\ell=0}^{k-1} S_{\Delta t}^{k-\ell} f(X_\ell) + \sqrt{\Delta t} \sum_{\ell=0}^{k-1} S_{\Delta t}^{k-\ell} Q^{1/2} \chi_{\ell+1}.$$

We integrate by part this term this stochastic integral thanks to Malliavin calculus.

$\to$ we need good estimates on $D^2 u(t,x)$ as

$$|A^{1/2-\varepsilon} D^2 u(t,x) A^{1/2-\varepsilon}|_{\mathcal{L}(H)} \leq c t^{-1/2+\varepsilon}.$$

Theorem (Debussche, 2009)

- *In dimension 1, for f and in* $C_b^3(H)$ *then* $|\mathbb{E}\varphi(X(T)) - \mathbb{E}\varphi(X_n) \leq C\Delta t^{\alpha}$ *for any* $\alpha < 1/2$.
- *It is also true in the multiplicative case depending on some restricition on* $\sigma(X) := Q^{1/2}(X)$ *:* $\sigma \in C_b^3(H)$ *and* $D\sigma$ *Lipschitz from* $D(A^{-1/4})$ *into* H.

Locally Lipschitz case

We consider the scheme 1 truncated by a cut-off function. Let $R > 0$ and

$$X_{k+1}^R - X_k^R + \Delta t(AX_{k+1}^R + f_R(X_k^R)) = \sqrt{\Delta t} Q^{1/2} \chi_{k+1},$$

where $f_R(X) = \theta(|X|/R) f(X)$ for some compactly supported function θ s.t. $\theta \equiv 1$ on $[0, 1]$.

We recover the Lipschitz case.

$$\begin{aligned} |\mathbb{E}\varphi(X_n^R) - \mathbb{E}\varphi(X(T))| &\leq |\mathbb{E}\varphi(X_n^R - \mathbb{E}\varphi(X^R(T))| + |\mathbb{E}\varphi(X^R(T)) - \mathbb{E}\varphi(X(T))| \\ &\leq CR^k \Delta t^{1/2-\varepsilon} + CR^{-p} \end{aligned}$$

for any p.

Theorem (Debussche, Printems, 2009)

- *For $f : H \to D(A^{-s})$ Lipchitz, for any*

$$\gamma < \min(2-2s, 1-\alpha+\beta),$$

there exists a constant C independant of n s.t.

$$|\mathbb{E}\varphi(X_n) - \mathbb{E}\varphi(X(T,x))| \leq C\Delta t^{\gamma}.$$

- *In the case of the stochastic additive Burgers equation, we have*

$$|\mathbb{E}\varphi(X_n^R) - \mathbb{E}\varphi X(T,x)| \leq c\Delta t^{\delta},$$

for any $\delta < 1/2$, where X_n is given by the scheme 1.

Scheme 2 ? Work in progress.