Regularity of solution to stochastic linear equations

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We are concerned with the stochastic evolution equation

$$dX = AXdt + dZ, \qquad X(0) = 0,$$

where $(A, D(A))$ generates a C_0-semigroup on a Hilbert space H, Z is a Lévy process in a Hilbert space U, $H \hookrightarrow U$.

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$$X(t) = \int_0^t e^{A(t-s)} dZ(s),$$

we assume that X is mean square integrable in H, $\mathbb{E}\|X(t)\|_H^2 < \infty$, in particular $e^{At} \colon U \to H$ for $t > 0$.

There are plenty of continuous deterministic $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$\int_0^t f(t-s)dB(s), \qquad t \geq 0,$$

does not have a modification which is with positive probability bounded on any bounded interval.

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Z. Brzezniak, S. Peszat, J. Zabczyk, Continuity of stochastic convolutions, Czechoslovak Math. J. 51 (2001), 679–684.

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$$\int_0^t e^{A(t-s)} dZ(s) = e^{At} \int_0^t e^{-As} dZ(s),$$

and càdlàg (continuity) follows.

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$$X(t) = PU(t) \int_0^t U(-s) dZ(s).$$

If Z is a Wiener then one can apply Da Prato, Kwapien, Zabczyk factorization, or Kolmogorov test.

Da Prato–Kwapień–Zabczyk Factorization

$$\int_0^t S(t-s)dM(s) = \pi I_{A,\alpha}(Y_\alpha)(t),$$

where $I_{A,\alpha}$ is the Liouville–Riemann operator

$$I_{A,\alpha}\psi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}S(t-s)\psi(s)ds,$$

and

$$Y_\alpha(t); = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}S(t-s)dM(s).$$

Then $I_{\alpha,A}$ maps $L^q(0,T;H)$ into $C([0,T[;H)$ if $1/q<\alpha$. It is enough to show that Y_α has trajectories in $L^q(0,T;H)$.

There are counterexamples, see
Iscoe, Marcus, Mc Donald, Talagrand, Zinn 1990 (Ann. Probab.)
Millet and Smoleński 1992 (PTRF).

Assume that $h = l^2$, $A = (\lambda_k \delta_{k,j})$, $Z = (Z_k)$, Z_k are independent. Let μ_k be the Lévy measure of Z_k. Assume that

$$m_j(k) := \int_{\mathbb{R}} |x|^j \mu_j(dx) < \infty, \qquad j = 1, 2, 3, 4.$$

Theorem

If

$$\sup_k \left(m_2(k) + m_4(k)\right) < \infty, \quad \sum \frac{1}{\lambda_k} < \infty,$$

then X is weakly càdlàg in l^2.

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$$(m_2(k) + m_4(k)) \to 0, \quad \sum \frac{1}{\lambda_k} < \infty,$$

then X is càdlàg in l^2.

Theorem

Assume that ξ is a separable process in a metic spce $(\mathcal{X}, \rho)$. Assume that there are $p \geq 1$, $r, K > 0$ such that for $t \in [0, T]$, and $h > 0$,

$$\mathbb{E}\left[\rho(\xi(t), \xi(t-h))\rho(\xi(t), \xi(t+h))\right]^p \leq Kh^{1+r}.$$

Then ξ does not have discontinuities of the second type, and

$$\mathbb{E}\rho(\xi(t), \xi(s))^q \leq C\left(\mathbb{E}\rho(\xi(T), \xi(0))^q + 1\right),$$

$1 \leq q < 2p$ and $C = C(K, r, T, p, q)$.