Strong, weak and a posteriori error analysis of the finite element method for parabolic and hyperbolic stochastic equations

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Outline

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Co-workers

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Semigroup approach

Linear SPDEs with additive noise:

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), & t > 0 \\ X(0) = X_0 \end{cases}$$

- H, U Hilbert spaces
- $(\Omega, \mathcal{F}, \mathbf{P}, \{\mathcal{F}_t\})$, filtered probability space
- $W(t)$, Q-Wiener process on U w.r.t $\{\mathcal{F}_t\}$
- $B : U \to H$, bdd linear operator
- $X(t)$, H-valued stochastic process
- $E(t) = e^{-tA}$, C_0-semigroup of bdd linear operators on H
- X_0 is $\mathcal{F}_0$-measurable

Weak solution

A weak solution satisfies the integral equation: for all $\eta \in D(A^*)$

$$\langle X(t), \eta \rangle + \int_0^t \langle X(s), A^*\eta \rangle \, \mathrm{d}s = \langle X_0, \eta \rangle + \int_0^t \langle \eta, B \, \mathrm{d}W(s) \rangle$$

The unique weak solution is given by (mild solution)

$$X(t) = E(t)X_0 + \int_0^t E(t-s)B \, \mathrm{d}W(s)$$

Give a rigorous meaning to $W(t)$. Define the stochastic integral.

Q-Wiener process

- covariance operator $Q : U \to U$, self-adjoint, positive definite, bounded, linear operator
- $Qe_j = \gamma_j e_j, \quad \gamma_j > 0, \quad \{e_j\}_{j=1}^{\infty}$ ON basis
- $\beta_j(t)$, independent identically distributed, real-valued, Brownian motions
- $W(t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j$

Two important cases:

- $\mathrm{Tr}(Q) < \infty$. $W(t)$ converges in $L_2(\Omega, U)$:
$$\mathbf{E}\Big\| \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j \Big\|^2 = \sum_{j=1}^{\infty} \gamma_j \mathbf{E}\big(\beta_j(t)^2\big) = t \sum_{j=1}^{\infty} \gamma_j = t\,\mathrm{Tr}(Q) < \infty$$
- $Q = I$, "white noise". $W(t)$ is not U-valued, since $\mathrm{Tr}(I) = \infty$, but converges in a weaker sense.

Q-Wiener process

If $\mathrm{Tr}(Q) < \infty$:

- $W(0) = 0$
- continuous paths $t \mapsto W(t) \in U$
- independent increments
- Gaussian law $\mathbf{P} \circ (W(t) - W(s))^{-1} = N(0, (t-s)Q), \quad s \leq t$

$W(t)$ generates a filtration $\mathcal{F}_t$ so that it becomes a square integrable U-valued martingale.

We can integrate with respect to W: $$\int_0^t B(s)\,\mathrm{d}W(s).$$

The integral can be defined also when $\mathrm{Tr}(Q) = \infty$.

Stochastic integral

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s), \quad t \geq 0$$

- We can define the stochastic integral (deterministic integrand) $\int_0^t B(s)\,\mathrm{d}W(s)$ if $\int_0^t \|B(s)Q^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}s < \infty$.
- Isometry property:
$$\mathbf{E}\Big\| \int_0^t B(s)\,\mathrm{d}W(s)\Big\|^2 = \int_0^t \|B(s)Q^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}s$$

Hilbert-Schmidt operator B:

$$\|B\|_{\mathrm{HS}}^2 = \sum_{l=1}^{\infty} \|B\varphi_l\|^2 < \infty, \qquad \{\varphi_l\} \text{ arbitrary ON basis in } U$$

Da Prato and Zabczyk, *Stochastic Equations in Infinite Dimensions*
C. Prévôt and M. Röckner, *A Consise Course on Stochastic Partial Differential Equations*

Spatial approximation

Consider

$$\begin{cases} \mathrm{d}X_h(t) + A_h X_h(t)\,\mathrm{d}t = B_h\,\mathrm{d}W(t), \ t > 0, \\ X_h(0) = X_{0,h} \end{cases}$$

- $V_h \subset H$ finite dimensional
- $B_h : U \to V_h$, "approximation of B"
- $-A_h$ generates a C_0-semigroup $E_h(t) = e^{-tA_h}$ on V_h
- $X_{0,h}$ approximates X_0 in V_h

The weak solution is given by

$$X_h(t) = E_h(t)X_{0,h} + \int_0^t E_h(t-s)B_h\,\mathrm{d}W(s)$$

Strong and weak error

- Strong error:

$$\|X_h(t) - X(t)\|_{L_2(\Omega,H)} = (\mathbf{E}\|X_h(t) - X(t)\|^2)^{1/2}$$

- Weak error:

$$\mathbf{E}G(X_h(T)) - \mathbf{E}G(X(T))$$

  for $G : H \to \mathbb{R}$.

This talk focuses on the weak error.

Weak error representation: preliminaries

Consider

$$\mathrm{d}Y(t) = E(T-t)B\,\mathrm{d}W(t),\ \ t \in (0,T];\ \ Y(0) = E(T)X_0,$$

with weak solution

$$Y(t) = E(T)X_0 + \int_0^t E(T-s)B\,\mathrm{d}W(s).$$

Similarly, consider

$$\mathrm{d}Y_h(t) = E_h(T-t)B\,\mathrm{d}W(t),\ \ t \in (0,T];\ \ Y_h(0) = E_h(T)X_{0,h},$$

with weak solution

$$Y_h(t) = E_h(T)X_{0,h} + \int_0^t E_h(T-s)B_h\,\mathrm{d}W(s).$$

Notice that $X(T) = X(T)$, $X_h(T) = Y_h(T)$ and there is NO DRIFT term for Y and Y_h.

Weak error representation: preliminaries

In general, consider the auxiliary problem

$$\mathrm{d}Z(t) = E(T-t)B\,\mathrm{d}W(t),\ t \in (\tau, T];\ Z(\tau) = \xi,$$

where ξ is a $\mathcal{F}_\tau$-measurable random variable. Then the unique weak solution is given by

$$Z(t,\tau,\xi) = \xi + \int_\tau^t E(T-s)B\,\mathrm{d}W(s).$$

Define $u : H \times [0, T] \to \mathbb{R}$ by

$$u(x,t) = \mathbf{E}\left(G(Z(T,t,x))\right).$$

If $G \in C_{\mathrm{b}}^2(H,\mathbb{R})$, then it is well known that u is a solution to Kolmogorov's equation

$$\begin{aligned} &u_t(x,t) + \frac{1}{2}\operatorname{Tr}\left(u_{xx}(x,t)E(T-t)BQB^*E(T-t)^*\right) = 0, \\ &u(x,T) = G(x), \quad t \in [0,T),\ x \in H. \end{aligned}$$

Weak error representation: preliminaries

Let $\mathcal{L}_1(H)$ denote the set of nuclear operators from H to H; that is, $T \in \mathcal{L}_1(H)$ if $T \in \mathcal{B}(H)$ and there are sequences $\{a_j\}, \{b_j\} \subset H$ with $\sum_{j=1}^{\infty} \|a_j\|\|b_j\| < \infty$ and such that

$$Tx = \sum_{j=1}^{\infty} \langle x, b_j \rangle a_j \quad \forall x \in H.$$

It is a Banach space under the norm

$$\|T\|_{\mathrm{Tr}} = \inf \Big\{ \sum_{j=1}^{\infty} \|a_j\|\|b_j\| : Tx = \sum_{j=1}^{\infty} \langle x, b_j \rangle a_j \Big\}.$$

For $T \in \mathcal{L}_1(H)$ the trace of T, $\mathrm{Tr}(T)$, is well defined and is given by $\mathrm{Tr}(T) = \sum_{k=1}^{\infty} \langle Te_k, e_k \rangle$ with $\{e_k\}_{k=1}^{\infty}$ an ONB of H.

Weak error representation: preliminaries

$C_b^2(H,\mathbb{R})$ is the set of all real-valued, twice Fréchet differentiable functions G whose first and second derivatives are continuous and bounded. By the Riesz representation theorem, we may identify the first derivative $DG(x)$ at $x \in H$ with an element $G'(x) \in H$ such that

$$DG(x)y = \langle G'(x), y\rangle, \quad y \in H,$$

and the second derivative $D^2G(x)$ with a symmetric linear operator $G''(x) \in \mathcal{B}(H)$ such that

$$D^2G(x)(y,z) = \langle G''(x)y, z\rangle, \quad y,z \in H.$$

We say that $G \in C^2(H,\mathbb{R})$ if G, G', and G'' are continuous, that is, $G \in C(H,\mathbb{R})$, $G' \in C(H,H)$, and $G'' \in C(H,\mathcal{B}(H))$. We define

$$C_b^2(H) := \left\{ G \in C^2(H,\mathbb{R}) : \|G\|_{C_b^2(H)} < \infty \right\},$$

with the seminorm

$$\|G\|_{C_b^2(H)} := \sup_{x\in H} \|G'(x)\|_H + \sup_{x\in H} \|G''(x)\|_{\mathcal{B}(H)}.$$

Weak error representation

THEOREM. If

$$\operatorname{Tr}\left(\int_0^T E(t)BQ[E(t)B]^*\,\mathrm{d}t\right) < \infty$$

and $G \in C_{\mathrm{b}}^2(H, \mathbb{R})$, then the weak error $e_h(T)$ has the representation

$$\begin{aligned}
e_h(T) &= \mathbf{E}\left(u(Y_h(0),0) - u(Y(0),0)\right) \\
&\quad + \frac{1}{2}\mathbf{E}\int_0^T \operatorname{Tr}\Big(u_{xx}(Y_h(t),t) \\
&\qquad \times [E_h(T-t)B_h + E(T-t)B]Q[E_h(T-t)B_h - E(T-t)B]^*\Big)\,\mathrm{d}t \\
&= \mathbf{E}\left(u(Y_h(0),0) - u(Y(0),0)\right) \\
&\quad + \frac{1}{2}\mathbf{E}\int_0^T \operatorname{Tr}\Big(u_{xx}(Y_h(t),t) \\
&\qquad \times [E_h(T-t)B_h - E(T-t)B]Q[E_h(T-t)B_h + E(T-t)B]^*\Big)\,\mathrm{d}t.
\end{aligned}$$

Weak error representation: proof

If ξ is $\mathcal{F}_t$ measurable, then $u(\xi,t) = \mathbf{E}\Big(G(Z(T,t,\xi))\Big|\mathcal{F}_t\Big)$. Therefore, by the law of double expectation,

$$\mathbf{E}\Big(u(\xi,t)\Big) = \mathbf{E}\Big(\mathbf{E}\Big(G(Z(T,t,\xi))\Big|\mathcal{F}_t\Big)\Big) = \mathbf{E}\Big(G(Z(T,t,\xi))\Big).$$

Thus,

$$\mathbf{E}\Big(u(Y(0),0)\Big) = \mathbf{E}\Big(G(Z(T,0,Y(0))\Big) = \mathbf{E}\Big(G(Y(T))\Big) = \mathbf{E}\Big(G(X(T))\Big)$$

and with $\xi = Y_h(T)$

$$\begin{aligned}\mathbf{E}\Big(u(Y_h(T),T)\Big) &= \mathbf{E}\Big(\mathbf{E}\Big(G(Z(T,T,Y_h(T)))\Big|\mathcal{F}_T\Big)\Big) \\ &= \mathbf{E}\Big(G(Y_h(T))\Big) = \mathbf{E}\Big(G(X_h(T))\Big).\end{aligned}$$

Hence,

$$\begin{aligned}e_h(T) &= \mathbf{E}\Big(G(X_h(T)) - G(X(T))\Big) = \mathbf{E}\Big(u(Y_h(T),T) - u(Y(0),0)\Big) \\ &= \mathbf{E}\Big(u(Y_h(0),0) - u(Y(0),0)\Big) + \mathbf{E}\Big(u(Y_h(T),T) - u(Y_h(0),0)\Big).\end{aligned}$$

Weak error representation: proof

Using Itô's formula for $u(Y_h(t), t)$ and Kolmogorov's equation

$$
\begin{aligned}
&\mathbf{E}\Big(u(Y_h(T), T) - u(Y_h(0), 0)\Big) \\
&= \mathbf{E}\int_0^T u_t(Y_h(t), t) \\
&\quad + \frac{1}{2}\operatorname{Tr}\Big(u_{xx}(Y_h(t), t)[E_h(T-t)B_h]Q[E_h(T-t)B_h]^*\Big)\,\mathrm{d}t \\
&= \frac{1}{2}\mathbf{E}\int_0^t \operatorname{Tr}\Big(u_{xx}(Y_h(t), t) \\
&\qquad \times [E_h(T-t)B_h]Q[E_h(T-t)B_h]^* - [E(T-t)B]Q[E(T-t)B]^*\Big)\,\mathrm{d}t.
\end{aligned}
$$

The proof can be finished by algebraic manipulation and playing around with traces.

Applications: Heat equation

The stochastic heat equation is formally

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) = \dot{W}(x,t), & x \in D,\ t > 0 \\ u(x,t) = 0, & x \in \partial D,\ t > 0 \\ u(x,0) = u_0(x), & x \in D \end{cases}$$

where $D \subset \mathbb{R}^d$ is a bounded domain.

In the abstract framework:

- $\Lambda := -\Delta$ with $\mathcal{D}(\Lambda) = H^2(D) \cap H_0^1(D)$.
- $U = H := L_2(D)$ with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$
- $A := \Lambda$
- $B := I$

Applications: Heat equation

Introduce

$$\dot{H}^\alpha := D(\Lambda^{\alpha/2}), \quad |v|_\alpha := \|\Lambda^{\alpha/2} v\| = \Big(\sum_{j=1}^{\infty} \lambda_j^\alpha (v, \phi_j)^2 \Big)^{1/2}, \quad \alpha \in \mathbf{R}, \ v \in \dot{H}^\alpha,$$

where $\{(\lambda_j, \phi_j)\}_{j=1}^{\infty}$ are the eigenpairs of Λ with orthonormal eigenvectors.

- $\dot{H}^\alpha \subset \dot{H}^\beta$ for $\alpha \geq \beta$
- $\dot{H}^0 = L_2(\mathcal{D}),\ \dot{H}^1 = H_0^1(\mathcal{D}),\ \dot{H}^2 = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$ with equivalent norms
- $\dot{H}^{-\beta}$ can be identified with the dual space $(\dot{H}^\beta)^*$ for $\beta > 0$.

Applications: Heat equation

Let $P_h : H \to V_h \subset \dot{H}^1$ denote the orthogonal projection and let $\Lambda_h : V_h \to V_h$ be the "discrete Laplacian" defined by

$$\langle \Lambda_h \psi, \chi \rangle = \langle \nabla \psi, \nabla \chi \rangle, \quad \psi, \chi \in V_h.$$

We set $B_h := P_h$, $A_h := \Lambda_h$ and $X_{0,h} := P_h X_0$. Our basic assumption on the finite element method is that the Ritz projection $R_h : \dot{H}^1 \to V_h$ defined as

$$\langle \nabla R_h v, \nabla \chi \rangle = \langle \nabla v, \nabla \chi \rangle, \ v \in \dot{H}^1, \ \chi \in V_h,$$

satisfies the error bound

$$\|R_h v - v\| \leq C h^\beta |v|_\beta, \ v \in \dot{H}^\beta, \ 1 \leq \beta \leq r.$$

Example: D is a convex polygonal domain, $V_h := S_h$ where S_h is a family of standard finite element spaces consisting of continuous piecewise polynomials of order $r-1$ on a regular family of triangulations of D with maximum mesh-size h.

Applications: Heat equation

THEOREM. Let $g \in C_{\mathrm{b}}^2(H, \mathbb{R})$ and assume that $\|A^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathsf{HS}} < \infty$ for some $\beta \in (0, 1]$. Then, there are $C > 0$, $h_0 > 0$, depending on g, X_0, Q, β, and T but not on h, such that for $h \leq h_0$,

$$\left|\mathbf{E}\big(g(X_h(T)) - g(X(T))\big)\right| \leq C h^{2\beta} |\ln h|.$$

If, in addition $X_0 \in L_1(\Omega, \dot{H}^{2\beta})$, then C is independent of T as well.

The proof uses the error representation theorem, the basic deterministic finite element estimate

$$\|(E_h(t)P_h - E(t))v\| \leq C h^s t^{-\frac{s-\gamma}{2}} |v|_\gamma, \ 0 \leq \gamma \leq s \leq r,$$

and that

$$\sup_{(x,t) \in H \times [0,T]} \|u_{xx}(x,t)\|_{\mathcal{B}(H)} \leq \sup_{x \in H} \|g''(x)\|_{\mathcal{B}(H)}.$$

Applications: Heat equation

REMARK. This is proved under a stronger condition on g but the same condition on A and Q in [GKL'09]. It is also proved (without the logarithm) in [DP'09] under the same condition on g but under the condition that $A^{\beta-1+\gamma}Q \in \mathcal{B}(H)$ and $\mathrm{Tr}(A^{-\gamma}) < \infty$, which is a sufficient condition for $\|A^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$.

REMARK. This is twice the rate of the strong convergence proved in [Yan'04] under the condition $\|A^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$.

Special cases:

- $Q = I \Rightarrow d = 1,\ \beta < \frac{1}{2}$
- $\mathrm{Tr}\, Q < \infty \Rightarrow \beta = 1$.

Under a slightly stronger condition on A and Q the result can be extended to the case $\beta > 1$.

Applications: Heat equation

THEOREM. Let $g \in C_b^2(H, \mathbb{R})$ and assume that $\|A^{\beta-1}Q\|_{\mathrm{Tr}} < \infty$ for some $\beta \in [1, \frac{r}{2}]$. Then there are $C > 0$, $h_0 > 0$, depending on g, X_0, Q, β, and T but not on h, such that for $h \le h_0$,

$$\left|\mathbf{E}\big(g(X_h(T)) - g(X(T))\big)\right| \le Ch^{2\beta}|\ln h|.$$

If, in addition $X_0 \in L_1(\Omega, \dot{H}^{2\beta})$, then C is independent of T as well.

REMARK. We have the implications

$$A^{\beta-1+\gamma}Q \in \mathcal{B}(H) \text{ and } \mathrm{Tr}(A^{-\gamma}) < \infty \Rightarrow \|A^{\beta-1}Q\|_{\mathrm{Tr}} < \infty \Rightarrow \|A^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty.$$

If A and Q have a common basis of eigenfunctions, then

$$\|A^{\beta-1}Q\|_{\mathrm{Tr}} = \|A^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2.$$

Applications: Linear Cahn-Hilliard-Cook equation

The stochastic Cahn-Hilliard equation (Cahn-Hilliard-Cook equation) formally is

$$\begin{cases} u_t - \Delta v = \dot{W}, & x \in \mathcal{D},\ t > 0 \\ v = -\Delta u + f(u), & x \in \mathcal{D},\ t > 0 \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & x \in \partial\mathcal{D},\ t > 0 \\ u(0) = u_0 \end{cases}$$

Phenomenological model for phase separation problems.

Typically: $f(u) = u^3 - u$. In this work: $f(u) = 0$.

Applications: Linear Cahn-Hilliard-Cook equation

Abstract framework

- $U = H := \{v \in L_2(D) : \langle v, 1\rangle = 0\}$
- $B := I$
- Let $H^s = H^s(\mathcal{D})$ be the usual Sobolev space with norm $\|\cdot\|_s$. Define $\Lambda := -\Delta$ with

$$D(\Lambda) = \Big\{v \in H^2 \cap H : \frac{\partial v}{\partial n} = 0 \text{ on } \partial\mathcal{D}\Big\}.$$

  Then Λ is a selfadjoint, positive definite, densely defined operator on H. If we set $A := \Lambda^2$, then $-A$ generates an analytic semigroup on H.
- Let $V_h \subset H^1 \cap H$ and define $\Lambda_h : V_h \to V_h$ by

$$\langle \Lambda_h \chi, \eta\rangle = \langle \nabla\chi, \nabla\eta\rangle, \quad \chi, \eta \in V_h.$$

  Let $A_h := \Lambda_h^2$, $B_h := P_h : H \to V_h$ be the orthogonal projection and set $X_{0,h} := P_h X_0$.

Applications: Linear Cahn-Hilliard-Cook equation

Define $\dot{H}^s = \mathcal{D}(\Lambda^{\frac{s}{2}})$ with norms $|v|_s = \|\Lambda^{\frac{s}{2}} v\|$ for a real s. In particular, we have $\dot{H}^1 = H^1 \cap H$ and the norm $|v|_1 = \|\Lambda^{\frac{1}{2}} v\| = \|\nabla v\|$ is equivalent to $\|v\|_1$ on $\dot{H}^1$.

- As for the heat equation we assume, that the Ritz projection $R_h : \dot{H}^1 \to V_h$ defined as

  $$\langle \nabla R_h v, \nabla \chi \rangle = \langle \nabla v, \nabla \chi \rangle, \; v \in \dot{H}^1, \; \chi \in V_h,$$

  satisfies the error bound

  $$\|R_h v - v\| \leq C h^\beta |v|_\beta, \; v \in \dot{H}^\beta, \; 1 \leq \beta \leq r.$$

- We assume the inverse inequality

  $$\|\Lambda_h^{-\delta} P_h v\| \leq C \|\Lambda^{-\delta} v\|, \; v \in H, \; \delta \in [0, 2],$$

  holds. This holds if the family of triangulations is quasi-uniform.

Applications: Linear Cahn-Hilliard-Cook equation

THEOREM. Let $g \in C_{\rm b}^2(H, \mathbb{R})$ and assume that $\|A^{\frac{\beta-2}{2}} Q\|_{\rm Tr} = \|\Lambda^{\beta-2} Q\|_{\rm Tr} < \infty$ for some $0 < \beta \leq \min(2, \frac{r}{2})$. Then there are $C > 0$, $h_0 > 0$, depending on g, X_0, Q, β, and T but not on h, such that for $h \leq h_0$,

$$\left|\mathbf{E}\big(g(X_h(T)) - g(X(T))\big)\right| \leq Ch^{2\beta} |\ln h|.$$

If, in addition $X_0 \in L_1(\Omega, \dot{H}^{2\beta})$, then C is independent of T as well.

The proof uses the weak error representation theorem and the basic deterministic finite element estimate

$$\|(E_h(t)P_h - E(t))v\| \leq Ch^s t^{-\frac{s-\gamma}{4}} |v|_\gamma, \; 0 \leq \gamma \leq s \leq r.$$

Applications: Linear Cahn-Hilliard-Cook equation

REMARK. In [LM'09] the strong rate of convergence is found to be $O(h^{\beta})$ under the condition $\|\Lambda^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$. Similarly to the heat equation,

$$\Lambda^{\beta-2+\gamma} Q \in \mathcal{B}(H) \text{ and } \mathrm{Tr}(\Lambda^{-\gamma}) < \infty \Rightarrow \|\Lambda^{\beta-2} Q\|_{\mathrm{Tr}} < \infty \Rightarrow \|\Lambda^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty.$$

If Λ and Q have a common basis of eigenfunctions, then

$$\|\Lambda^{\beta-2} Q\|_{\mathrm{Tr}} = \|\Lambda^{\frac{\beta-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2.$$

In particular, if $\beta = 2$ or if $Q = I$.

Special cases: If $Q = I$, then $\beta < 2 - \frac{d}{2}$. If $\mathrm{Tr}(Q) < \infty$, then $\beta = \min(2, \frac{r}{2})$.

Applications: Wave equation

The linear stochastic wave equation formally is

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = \dot{W}(\xi, t), & \xi \in \mathcal{D} \subset \mathbf{R}^d, \ t > 0 \\ u(\xi, t) = 0, & \xi \in \partial\mathcal{D}, \ t > 0 \\ u(\xi, 0) = u_0, \ \dfrac{\partial u}{\partial t}(\xi, 0) = u_1, & \xi \in \mathcal{D} \end{cases}$$

$$\Lambda = -\Delta, \quad D(\Lambda) = \dot{H}^2 = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$$

$$\dot{H}^\beta = D(\Lambda^{\beta/2}), \quad |v|_\beta = \|\Lambda^{\beta/2} v\|, \quad \beta \in \mathbf{R}$$

$$\begin{bmatrix} \mathrm{d}u \\ \mathrm{d}u_t \end{bmatrix} + \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix} \begin{bmatrix} u \\ u_t \end{bmatrix} \mathrm{d}t = \begin{bmatrix} 0 \\ I \end{bmatrix} \mathrm{d}W, \quad X = \begin{bmatrix} u \\ u_t \end{bmatrix}, \ A = \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$H = \dot{H}^0 \times \dot{H}^{-1}, \quad D(A) = \dot{H}^1 \times \dot{H}^0, \quad U = \dot{H}^0 = L_2(\mathcal{D})$$

Applications: Wave equation

Abstract framework

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), \quad t > 0 \\ X(0) = X_0 \end{cases}$$

- $X(t)$, $H = \dot{H}^0 \times \dot{H}^{-1}$-valued stochastic process
- $W(t)$, $U = \dot{H}^0$-valued Q-Wiener process

- $E(t) = e^{-tA} = \begin{bmatrix} \cos(t\Lambda^{1/2}) & \Lambda^{-1/2}\sin(t\Lambda^{1/2}) \\ -\Lambda^{1/2}\sin(t\Lambda^{1/2}) & \cos(t\Lambda^{1/2}) \end{bmatrix}$, C_0-semigroup on H.

$$\cos(t\Lambda^{1/2})v = \sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j})(v,\phi_j)\phi_j \quad \big(\lambda_j, \phi_j \text{ are the eigenpairs of } \Lambda\big)$$

Applications: Wave equation

Approximation

$$\begin{cases} \mathrm{d}X_h(t) + A_h X_h(t)\,\mathrm{d}t = B_h\,\mathrm{d}W(t), & t > 0 \\ X_h(0) = X_{0,h} \end{cases}$$

- Let $S_h \subset \dot{H}^1$ finite dimensional subspaces and set $V_h := S_h \times S_h$
- Λ_h"discrete Laplacian" defined by

$$\langle \Lambda_h \psi, \chi \rangle = \langle \nabla \psi, \nabla \chi \rangle, \quad \psi, \chi \in S_h$$

  Set

$$A_h := \begin{bmatrix} 0 & -I \\ \Lambda_h & 0 \end{bmatrix}$$

- Let

$$B_h := \begin{bmatrix} 0 \\ P_h \end{bmatrix}$$

  where P_h is the orthogonal projection $\dot{H}^0 \to S_h$

Applications: Wave equation

- $E_h(t) = e^{-tA_h} = \begin{bmatrix} \cos(t\Lambda_h^{1/2}) & \Lambda_h^{-1/2}\sin(t\Lambda_h^{1/2}) \\ -\Lambda_h^{1/2}\sin(t\Lambda_h^{1/2}) & \cos(t\Lambda_h^{1/2}) \end{bmatrix}$,

  C_0-semigroup on V_h

- $$X_{0,h} := \begin{bmatrix} P_h X_1(0) \\ P_h X_2(0) \end{bmatrix}$$

  for $X_1(0), X_2(0) \in \dot{H}^0$.

- Basic assumption on the finite element method: the Ritz projection $R_h : \dot{H}^1 \to S_h$ defined as

  $$\langle \nabla R_h v, \nabla \chi \rangle = \langle \nabla v, \nabla \chi \rangle, \ v \in \dot{H}^1, \ \chi \in S_h,$$

  satisfies the error bound

  $$\|R_h v - v\| \le C h^{\beta} |v|_{\beta}, \ v \in \dot{H}^{\beta}, \ 1 \le \beta \le r.$$

Applications: Wave equation

THEOREM: Let $g \in C_{\mathrm{b}}^2(\dot{H}^0, \mathbb{R})$ and assume that $\|\Lambda^{\beta-\frac{1}{2}} Q \Lambda^{-\frac{1}{2}}\|_{\mathrm{Tr}} < \infty$ and that $X_0 \in L_1(\Omega, H^{2\beta})$ for some $\beta \in (0, \frac{r+1}{2}]$. Then, there are $C > 0$, $h_0 > 0$, depending on g, X_0, Q, β, and T but not on h, such that for $h \le h_0$,

$$\left|\mathbf{E}\big(g(X_{1,h}(T)) - g(X_1(T))\big)\right| \le C h^{\frac{2r}{r+1}\beta}.$$

The proof uses the weak error representation theorem with $G(X) := g(P_1 X)$, where P_1 is the canonical projection $H \to \dot{H}^0$. The relevant deterministic error estimates are

$$\|K_h(t)\| := \|\Lambda_h^{-1/2} \sin(t\Lambda_h^{1/2}) P_h v - \Lambda^{-1/2} \sin(t\Lambda^{1/2}) v\| \le C(T) h^{\frac{r}{r+1}s} |v|_{s-1}, \qquad t \in [0, T],\ s \in [0, r+1].$$

and

$$\|G_h(t)\| := \|\cos(t\Lambda_h^{1/2}) P_h - \cos(t\Lambda^{1/2}) v\| \le C(T) h^{\frac{r}{r+1}s} |v|_s, \qquad t \in [0, T],\ s \in [0, r+1].$$

Wave equation: sketch of proof

Set $G(X) := g(P_1 X)$, where P_1 is the canonical projection $H \to \dot{H}^0$. Then,

$$(u_x(Y(t),t),\Phi) = \mathbf{E}(\langle g'(P_1 Z(Y(t),t,T)), P_1\Phi\rangle | \mathcal{F}_t)$$

and

$$(u_{xx}(Y(t),t)\Phi,\Psi) = \mathbf{E}(\langle g''(P_1 Z(Y(t),t,T))P_1\Phi, P_1\Psi\rangle | \mathcal{F}_t).$$

Recall, the abstract weak error representation:

$$\begin{aligned} e_h(T) &= \mathbf{E}\left(u(Y_h(0),0) - u(Y(0),0)\right) \\ &\quad + \frac{1}{2}\mathbf{E}\int_0^T \mathrm{Tr}\Big(u_{xx}(Y_h(t),t) \\ &\qquad \times [E_h(T-t)B_h - E(T-t)B]Q[E_h(T-t)B_h + E(T-t)B]^*\Big)\,\mathrm{d}t. \end{aligned}$$

Wave equation: sketch of proof

The estimate for the first term is more or less straightforward from the deterministic error estimate and gives

$$|\mathbf{E}\left(u(Y_h(0),0)-u(Y(0),0)\right)| \le C \sup_{x\in \dot{H}^0} \|g'(x)\| C h^{\frac{2r}{r+1}\beta} \mathbf{E}|||X_0|||_{2\beta}.$$

For the second term, one can show, due to the special choice of G,

$$\begin{aligned}&\mathbf{E}(\mathrm{Tr}(u_{xx}(Y_h(t),t)(E_h(T-t)B_h+E(T-t)B)Q(E_h(T-t)B_h-E(T-t)B)^*))\\ &=\mathbf{E}\,\mathrm{Tr}(K_h(T-t)Q(\Lambda_h^{-\frac{1}{2}}S_h(T-t)P_h+\Lambda^{-\frac{1}{2}}S(T-t))g''(P_1Z(Y(t),t,T)))\end{aligned}$$

Wave equation: sketch of proof

Therefore,

$$
\begin{aligned}
&\Big|\mathbf{E}\int_0^T \mathrm{Tr}\,\big(u_{xx}(Y_h(t),t)\\
&\qquad (E_h(T-t)B_h + E(T-t)B)Q(E_h(T-t)B_h - E(T-t)B)^*\big)\,\mathrm{d}t\Big|\\
&= \Big|\mathbf{E}\int_0^T \mathrm{Tr}(K_h(T-t)Q\\
&\qquad (\Lambda_h^{-\frac{1}{2}}S_h(T-t)P_h + \Lambda^{-\frac{1}{2}}S(T-t))g''(P_1Z(Y(t),t,T)))\,\mathrm{d}t\Big|\\
&= \Big|\mathbf{E}\int_0^T \mathrm{Tr}(K_h(T-t)\Lambda^{\frac{1}{2}-\beta}\Lambda^{\beta-\frac{1}{2}}Q\Lambda^{-\frac{1}{2}}\\
&\qquad \Lambda^{\frac{1}{2}}(\Lambda_h^{-\frac{1}{2}}S_h(T-t)P_h + \Lambda^{-\frac{1}{2}}S(T-t))g''(P_1Z(Y(t),t,T)))\,\mathrm{d}t\Big|\\
&\le CT\sup_{x\in H}\|g''(x)\|_{\mathcal{B}(\dot H^0)}\|\Lambda^{\beta-\frac{1}{2}}Q\Lambda^{-\frac{1}{2}}\|_{\mathrm{Tr}}\|K_h(T-t)\Lambda^{\frac{1}{2}-\beta}\|_{\mathcal{B}(\dot H^0)}\\
&\le CTh^{\frac{2r}{r+1}\beta}\sup_{x\in H}\|g''(x)\|_{\mathcal{B}(\dot H^0)}\|\Lambda^{\beta-\frac{1}{2}}Q\Lambda^{-\frac{1}{2}}\|_{\mathrm{Tr}}.
\end{aligned}
$$

Wave equation: remark

REMARK: In [KLS'09] the strong rate $O(\frac{r}{r+1}\beta)$ is obtained under the assumption $\|\Lambda^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$. It can be shown that

$$\|\Lambda^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \leq \|\Lambda^{\beta-\frac{1}{2}}Q\Lambda^{-\frac{1}{2}}\|_{\mathrm{Tr}}$$

with equality when A and Q have a common basis of eigenfunctions, in particular, when $Q = I$. If $Q = I$, then $d = 1$ and the weak rate is $O(h^{\frac{r}{r+1}})$.

Ongoing and future work

- A posteriori error estimates
- Add time discretization for the wave equation, weak and strong error estimates
- Add semiliniarity to the wave equation and multiplicative noise
- Cahn-Hiliard-Cook equation
- Spatial wavelet approximation of the noise and SPDEs

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