An L_2 -theory of Stochastic PDEs driven by Lévy processes

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Main Contents

- Introduction of Main results
- Short review on Lévy processes and stochastic integral
- Proof of Main Theorem

Main Result

We studied

1. SPDE of divergence-type

$$du = (\frac{\partial}{\partial x^i}(a^{ij}u_{x^j} + \bar{b}^i u) + b^i u_{x^i} + cu + f)dt + \sum_{k=1}^{\infty}(\sigma^{ik}u_{x^i} + \mu^k u + g^k)dZ_t^k$$

2. SPDEs of non-divergence-type

$$du = \left(a^{ij}u_{x^i x^j} + b^i u_{x^i} + cu + f\right) dt + \sum_{k=1}^{\infty}(\sigma^{ik}u_{x^i} + \mu^k u + g^k)dZ_t^k,$$

where Z_t^k are (general) one-dimensional Lévy processes, and all the coefficients depend on (ω, t, x).

Obtained

- L_2-theory for divergence type; If $\int z^2\nu^k(dz) < \infty \Rightarrow$ unique solution in $\mathcal{H}_2^1$.
- L_p-theory for non-divergence type ; If $\int(z^2 \vee |z|^p)\nu^k(dz) < \infty \Rightarrow$ unique solution in $\mathcal{H}_p^{n+2}$, here $n \in \mathbb{R}$, $p \in [2, \infty)$.

Remark

The equations of type

$$du = (Au + f)dt + g(u)dZ_t$$

have been well studied, where A is the infinitesimal generator of some semigroup and g satisfies certain continuity condition.

Note:

- The differential operator is non-random and independent of t.
- The first derivatives of solutions do not appear in the stochastic part.

Lévy process

Definition. A stochastic process $\{Z_t : t \geq 0\}$ is a Lévy process if

• $Z_0 = 0(a.s.)$;

• (Independent increment property): For each $0 \leq t_0 < t_1 < \cdots < t_n$, random variables $Z_{t_0}, Z_{t_1} - Z_{t_0}, \cdots, Z_{t_n} - Z_{t_{n-1}}$ are independent;

• (Stationary increment property): The distribution of $Z_{s+t} - Z_s$ does not depend on s.

• Z is stochastically continuous, i.e. for any $a > 0$ and $s \geq 0$

$$\lim_{t \to s} P(|Z_t - Z_s| > a) = 0.$$

Fact. Every Lévy process has a càdlàg modification, i.e. there is a version which is right continuous with left limits.

Jump process

Define jump process ΔZ by

$$\Delta Z_t = Z_t - Z_{t-},$$

where Z_{t-} is the left limit at the point t.

Definition. If $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R} - \{0\})$,

$$\begin{aligned} N(t, A)(\omega) &= \#\{0 \leq s \leq t; \Delta Z_s(\omega) \in A\} \\ &= \sum_{0 \leq s \leq t} I_A(\Delta Z_s(\omega)). \end{aligned}$$

The set function $A \to N(t, A)(\omega)$ is a counting measure on $\mathcal{B}(\mathbb{R} - \{0\})$. Define a Borel measure

$$\mu(A) := \mathbb{E}N(1, A) \quad \text{: Lévy measure}$$

Random measure

Fact:

- $\int_{\mathbb{R}}(1 \wedge z^2)d\mu < \infty$;
- If $0 \notin \bar{A}$, then $(N(t,A), t \geq 0)$ is a Poisson process with intensity $\mu(A)$. In particular, $\mathbb{E}N(t,A) = t\mu(A)$.

Define

$$\tilde{N}(t,dz) = N(t,dz) - t\mu(dz)$$

In other word, if $0 \notin \bar{A}$, then

$$\begin{aligned}
\tilde{N}(t,A) &= N(t,A) - t\mu(A) \\
&:= \#\{0 \leq s \leq t; \Delta Z_s(\omega) \in A\} - t\mathbb{E}N(1,A)
\end{aligned}$$

This is a square integrable martingale with respect to $\mathcal{F}_t = \sigma(Z_s : s \leq t)$. Indeed,

$$\begin{aligned}
& \mathbb{E}[N(t,A) - N(s,A)|\mathcal{F}_s] \\
= \; & \mathbb{E}(N(t,A) - N(s,A)) = (t-s)\mu(A)
\end{aligned}$$

Lèvy-Itô decomposition

Let $0 \notin \bar{A}$, since $\tilde{N}(t,dz) = N(t,dz) - t\mu(dz)$,

$$\int_A z\tilde{N}(t,dz) = \int_A zN(t,dz) - t\int_A z\mu(dz) = \sum_{s\le t} \Delta Z_s I_A(\Delta Z_s) - t\int_A z\mu(dz)$$

Fact: This is a square integrable martingale and $\mathbb{E}(\int_A z\tilde{N}(t,dz))^2 = t\int_A z^2\mu(dz)$. Thus we can define the martingale

$$\int_{-\infty}^{\infty} z\tilde{N}(t,dz) := L_2 - \lim_{n\to\infty} \int_{|z|\ge\frac{1}{n}} z\tilde{N}(t,dz).$$

Theorem(The Lèvy-Itô decomposition): If $\int z^2\mu(dz) < \infty$, then $\exists b, \beta \in \mathbb{R}$ and BM B_t s.t.

$$Z_t = bt + \beta B_t + \int_{-\infty}^{\infty} z\tilde{N}(t,dz)$$

Formally, $dZ_t = bdt + \beta dB_t + z\tilde{N}(dt,dz)$. Later, we define

$$\int_0^t f dZ_s = b\int_0^t f ds + \beta\int_0^t f dB_s + \int_0^t \int_{-\infty}^{\infty} f z\tilde{N}(ds,dz).$$

Stochastic Integral

If f is predictable and $0 \notin \bar{A}$

$$\int_0^t \int_A f(s,z) N(ds,dz) = \sum_{0 \le s \le t} f(s, \Delta Z_s) \cdot \chi_A(\Delta Z_s)$$

Since $\tilde{N}(t,dz) = N(t,dz) - t\mu(dz)$, define

$$I_t(A,f) := \int_0^t \int_A f(s,z) \tilde{N}(ds,dz) = \int_0^t \int_A f(s,z) N(ds,dz) - \int_0^t \int_A f(s,z)\mu(dz)ds$$

Facts:

- The above is a square integrable martingale, $\mathbb{E}(I_t(A,f))^2 = \mathbb{E} \int_0^t \int_A f^2 \mu(dz) ds$
- Since $\mathbb{E} I_t(A,f) = 0$, by taking $f(s,z) = g^2(s) z^2$,

$$\mathbb{E} \sum_{0 \le u \le t} g^2(u)(\Delta Z_u)^2 \cdot \chi_A(\Delta Z_u) = \mathbb{E} \int_0^t g^2 ds \cdot \int_A z^2 \mu(dz)$$

- The integral is extended for any Borel set A and $f \in L_2$, which has a predictable version.

Itô's formula

Remember: For Lèvy process Z_t, $\exists b, \beta, B_t$ s.t.

$$Z_t = bt + \beta B_t + \int z\tilde{N}(t, dz)$$

Let $b = \beta = 0$, and

$$u(t) = u(0) + \int_0^t f(s)ds + \int_0^t g(s)dZ_s,$$

where $\int_0^t g dZ_s = \int_0^t \int zg\tilde{N}(ds, dz)$. Then

$$|u(t)|^2 - |u(0)|^2 = 2\int_0^t u(s)f(s)ds + 2\int_0^t u(s-)g(s)dZ_s + \sum_{0 \le s \le t} g^2(s)(\Delta Z_s)^2$$

Thus

$$\mathbb{E}|u(t)|^2 - \mathbb{E}|u(0)|^2 = 2\mathbb{E}\int_0^t ufds + \mathbb{E}\sum_{0 \le s \le t} g^2(s)(\Delta Z_s)^2 = 2\mathbb{E}\int_0^t ufds + \mathbb{E}\int_0^t g^2 ds \cdot \int_{\mathbb{R}} z^2 \mu(dz)$$

SPDE of divergence type

Consider

$$du = \left(\frac{\partial}{\partial x_i} (a^{ij} u_{x^j} + \bar{b}^i u) + b^i u_{x^i} + cu + f \right) dt + \left(\sigma^{ik} u_{x^i} + \mu^k u + g^k \right) dZ_t^k$$

Here, $Z_t^1, Z_t^2, \ldots$: independent Lèvy processes, $i, j \in \{1, 2, \cdots, d\}$ and $k \in \{1, 2, 3, \cdots\}$. For $A \in \mathcal{B}(\mathbb{R} - \{0\})$,

$$N_k(t, A) := \#\{0 \leq s \leq t;\ Z_s^k - Z_{s-}^k \in A\}, \quad \tilde{N}_k(t, A) := N_k(t, A) - t\nu_k(A)$$

where $\nu_k(A) := \mathbb{E}[N_k(1, A)]$. By Lévy-Itô decomposition

$$Z^k(t) = \alpha^k t + \beta^k B_t^k + \int z \tilde{N}_k(t, dz)$$

Assume $\alpha^k = \beta^k = 0$, then

$$\int_0^t g^k dZ_s^k = \int_0^t \int g^k(s) z \tilde{N}^k(ds, dz)$$

Assumption

- $\widehat{c}_k := \left[\int z^2 \nu_k(dz)\right]^{1/2} < \infty$.

- (Stochastic parabolicity) : $\delta I \leq (a^{ij} - \alpha^{ij}) \leq KI$
where $\alpha^{ij} := \frac{1}{2}\sum_{k=1}^{\infty} \widehat{c}_k^2 \sigma^{ik}\sigma^{jk}$.

- The coefficients are $\mathcal{P} \times \mathcal{B}(\mathbb{R})$-measurable,

$$|a^{ij}| + |\bar{b}^i| + |b^i| + |c| + \left(\sum_{k=1}^{\infty} |\widehat{c}_k \sigma^{ik}|^2 + |\widehat{c}_k \mu^k|^2\right)^{1/2} \leq K$$

Remark The first condition is used for L_2-Estimates and is dropped when we prove path-wise (H^1-valued) uniqueness and existence results.

Solution space

For $n = 0, 1, 2, \ldots,$

$$H_2^n = W_2^n = \left\{ u : u, Du, \ldots, D^n u \in L^2(\mathbb{R}^d) \right\}, \quad H_2^{-n} = (H_2^n)^*.$$

Let $\mathbb{H}_2^n(T)$ be the set of all $\mathcal{P}^{dP \times dt}$-measurable processes $u : [0,T] \times \Omega \to H_2^n$ so that

$$\|u\|^2_{\mathbb{H}_2^n(T)} := \mathbb{E} \int_0^T \|u\|^2_{H_2^n} \, dt < \infty.$$

Denote $\mathbb{L}_2(T) := \mathbb{H}_2^0(T)$, and for $g = (g^1, g^2, \ldots)$ define

$$\|g\|_{\mathbb{L}_2(T,\ell^2)} := \left(\sum_{k=1}^{\infty} \mathbb{E} \int_0^T \|\widehat{c}_k g^k\|^2_{L_2} \, dt \right)^{1/2} < \infty.$$

Finally we write $u_0 \in U_2$ if u_0 is $\mathcal{F}_0$-measurable and

$$\|u_0\|_{U_2} := (\mathbb{E}\|u_0\|^2_{L_2})^{1/2} < \infty.$$

Solution space

Definition: Write $u \in \mathcal{H}_2^1(T)$ if $u \in \mathbb{H}_2^1(T), u(0) \in U_2$, u is right continuous with left limits in L_2 $a.s.$, and for some $f \in \mathbb{H}_2^{-1}(T)$ and $g \in \mathbb{L}_2(T, \ell^2)$

$$du = f dt + g^k dZ_t^k, \quad \forall t \leq T$$

in the sense of distributions, i.e., for any $\phi \in C_c^\infty(\mathbb{R}^d)$, the equality

$$(u(t), \phi) = (u(0), \phi) + \int_0^t (f, \phi) dt + \sum_k \int_0^t (g^k, \phi) dZ_t^k$$

holds for all $t \leq T$ $a.s.$. Define

$$\|u\|_{\mathcal{H}_2^1(T)} := \|u\|_{\mathbb{H}_2^1(T)} + \|f\|_{\mathbb{H}_2^{-1}(T)} + \|g\|_{\mathbb{L}_2(T, \ell^2)} + \|u(0)\|_{U_2}.$$

Theorem. The space $\mathcal{H}_2^1(T)$ is a Banach space and

$$\mathbb{E}\left[\sup_{t \leq T} \|u(t)\|_{L_2}^2\right] \leq C(T)\|u\|_{\mathcal{H}_2^1(T)}^2.$$

Remark

For every bounded predictable H, $M_t := \int_0^t H_s dZ_s^k$ is a square integrable martingale with

$$\mathbb{E}[M_t^2] = \widehat{c}_k^2 \mathbb{E}\left[\int_0^t H_s^2 ds\right].$$

Also for $g \in \mathbb{L}_2(T, \ell_2)$ and $\phi \in C_0^\infty(\mathbb{R}^d)$,

$$\sum_{k=1}^{\infty} \int_0^T \widehat{c}_k^2 (g^k, \phi)^2 ds \leq \|\phi\|_{L_2}^2 \int_0^T \sum_{k=1}^{\infty} \|\widehat{c}_k g^k\|_{L_2}^2 ds < \infty.$$

Therefore $\sum_{k=1}^{\infty} \int_0^t (g^k, \phi) dZ_t^k$ converges uniformly in t in probability on $[0, T]$.

Thus for each $g \in \mathbb{L}(T, \ell_2)$, the equation

$$du = f dt + g^k dZ_t^k$$

makes sense.

Note: g may not be predictable.

Main Theorem

Theorem. For any $f \in \mathbb{H}_2^{-1}(T)$, $g \in \mathbb{L}_2(T, \ell^2)$ and $u_0 \in U_2$, the equation

$$du = \left(\frac{\partial}{\partial x_i} (a^{ij} u_{x^j} + \bar{b}^i u) + b^i u_{x^i} + cu + f \right) dt + \left(\sigma^{ik} u_{x^i} + \mu^k u + g^k \right) dZ_t^k$$

has a unique solution $u \in \mathcal{H}_2^1(T)$ (see below), and

$$\|u\|_{\mathcal{H}_2^1(T)} \leq C(\|f\|_{\mathbb{H}_2^{-1}(T)} + \|g\|_{\mathbb{L}_2(T,\ell^2)} + \|u_0\|_{U_2}).$$

Remark. For any $\phi \in C_c^\infty(\mathbb{R}^d)$, the equality

$$\begin{aligned}
(u(t), \phi) &= (u(0), \phi) - \int_0^t (a^{ij} u_{x^i} + \bar{b}u, \phi_{x^j}) dt + \int_0^t (b^i u_{x^i} + cu + f, \phi) ds \\
&+ \sum_k \int_0^t (\sigma^{ik} u_{x^i} + \nu^k u + g^k, \phi) dZ_t^k
\end{aligned}$$

holds for all $t \leq T$ $a.s.$.

A priori estimate

If a solution u already exists, then

$$\|u\|_{\mathcal{H}_2^1(T)} \leq C(\|f\|_{\mathbb{H}_2^{-1}(T)} + \|g\|_{\mathbb{L}_2(T,\ell^2)} + \|u_0\|_{U_2})$$

proof. Write $f = f^0 + \sum_{i=1}^d \frac{\partial}{\partial x_i} f^i$ so that

$$\sum_{i=0}^{d} \|f^i\|_{\mathbb{L}_2(T)} \leq C\|f\|_{\mathbb{H}_2^{-1}(T)}.$$

By Ito's formula, we have

$$\begin{aligned}
\mathbb{E}\|u(t)\|_{L^2}^2 &= \mathbb{E}\|u_0\|_{L^2}^2 + 2\mathbb{E}\int_0^t [-(a^{ij}u_{x^j} + \bar{b}^i u + f^i, u_{x^i}) + (b^i u_{x^i} + cu + f^0, u)]ds \\
&+ 2\mathbb{E}\sum_k \int_0^t (\sigma^{ik}u_{x^i} + \mu^k u + g^k, u(s-))dZ_s^k \\
&+ \mathbb{E}\sum_k \sum_{0<s\leq t} \|(\sigma^{ik}u_{x^i} + \mu^k u + g^k)\Delta Z_s^k\|_{L^2}^2.
\end{aligned}$$

A priori estimate

Thus

$$\mathbb{E}\|u(t)\|_{L^2}^2 + 2\mathbb{E}\int_0^t (a^{ij}u_{x^i}, u_{x^j})ds$$

$$\leq \mathbb{E}\|u_0\|_{L_2}^2 + \varepsilon\|u_x\|_{\mathbb{L}_2(t)}^2 + c\|u\|_{\mathbb{L}_2(t)}^2 + c\sum_{i=1}^d \|f^i\|_{\mathbb{L}_2(T)} + \mathbb{E}\sum_k \sum_{0<s\leq t} \|(\sigma^{ik}u_{x^i} + \mu^k u + g^k)\Delta Z_s^k\|_{L_2}^2$$

Here,

$$\mathbb{E}\left[\sum_k \sum_{0<s\leq t} \|(\sigma^{ik}u_{x^i} + \mu^k u + g^k)\Delta Z_s^k\|_{L^2}^2\right] = \sum_k \widehat{c}_k^2 \mathbb{E}\left[\int_0^t \|\sigma^{ik}u_{x^i} + \mu^k u + g^k\|_{L^2}^2 ds\right]$$

$$\leq 2\mathbb{E}\left[\int_0^t (\alpha^{ij}u_{x^i}, u_{x^j})ds\right] + \varepsilon\|Du\|_{\mathbb{L}(t)}^2 + c\|u\|_{\mathbb{L}(t)}^2 + c\|g\|_{\mathbb{L}(t,\ell_2)}^2$$

where $\alpha^{ij} := \frac{1}{2}\sum_{k=1}^\infty \widehat{c}_k^2 \sigma^{ik}\sigma^{jk}$. Finally since $\delta I \leq (a^{ij} - \sigma^{ij})$.

$$\mathbb{E}\|u(t)\|_{L_2}^2 + 2\delta\|Du\|_{\mathbb{L}_2(t)}^2 \leq \varepsilon\|Du\|_{\mathbb{L}_2(t)}^2 + c\|u\|_{\mathbb{L}_2(t)}^2 + c\|f\|_{\mathbb{H}_2^{-1}(T)}^2 + c\|g\|_{\mathbb{L}_2(t,\ell_2)}^2 + c\|u_0\|_{U_2}^2.$$

Gronwall's inequality leads to the a priori estimate.

Method of continuity

Let

$$Lu := \frac{\partial}{\partial x_i}(a^{ij}u_{x^j} + \bar{b}^i) + b^i u_{x^i} + cu,$$

$$\Lambda^k u := \sigma^{ik} u_{x^i} + \mu^k u,$$

For $\lambda \in [0,1]$, define

$$L_\lambda u := \lambda L u + (1-\lambda)\Delta u$$

$$\Lambda^k_\lambda = \lambda \Lambda^k + (1-\lambda)0$$

Note that

$$L_{\lambda_1} u - L_{\lambda_2} u = (\lambda_1 - \lambda_2)(L - \Delta)u$$

$$\Lambda_{\lambda_1} u - \Lambda_{\lambda_2} u = (\lambda_1 - \lambda_2)\Lambda u,$$

and

$$\|L_{\lambda_1} u - L_{\lambda_2} u\|_{H_2^{-1}} + \|\Lambda_{\lambda_1} u - \Lambda_{\lambda_2} u\|_{L_2} \leq c|\lambda_1 - \lambda_2| \|u\|_{H_2^1}.$$

Proof of Theorem

Let $J \subset [0,1]$ denote the set of λ, so that for any f, g, u_0, the equation

$$du = (L_\lambda u + f)dt + (\Lambda^k_\lambda u + g^k)dZ^k_t, \quad u(0) = u_0$$

has a solution $u \in \mathcal{H}^1_2(T)$. Then J is not empty since $0 \in J$. Now assume $\lambda_0 \in J$, and note that u is a solution of the above equation if and only if

$$du = (L_{\lambda_0} u + (L_\lambda u - L_{\lambda_0} u + f))dt + (\Lambda_{\lambda_0} u + (\Lambda^k_\lambda u - \Lambda^k_{\lambda_0} u + g^k))dZ^k_t.$$

Recall $\lambda_0 \in J$. Denote $u^0 = u_0$ and for $n = 1, 2, \cdots$ we define $u^{n+1} \in \mathcal{H}^1_2(T)$ as the solution of the equation

$$du^{n+1} = (L_{\lambda_0} u^{n+1} + f_n)dt + (\Lambda_{\lambda_0} u^{n+1} + g^k_n)dZ^k_t$$

where $f_n := L_\lambda u^n - L_{\lambda_0} u^n + f, \quad g^k_n := \Lambda^k_\lambda u^n - \Lambda^k_{\lambda_0} u^n + g^k$. Then for $v^n := u^{n+1} - u^n$

$$dv^n = \left(L_{\lambda_0} v^n + (\lambda - \lambda_0)(L - \Delta)v^{n-1}\right) dt + \left(\Lambda^k_{\lambda_0} v^n + (\lambda - \lambda_0)\Lambda^k v^{n-1}\right) dZ^k_t.$$

Proof of Theorem

Then by the a priori estimate

$$\|u^{n+1} - u^n\|_{\mathcal{H}_2^1(T)} \le c\|(\lambda - \lambda_0)(L - \Delta)(u^n - u^{n-1})\|_{\mathbb{H}_2^{-1}(T)}$$

$$+c\|(\lambda - \lambda_0)\Lambda^k(u^n - u^{n-1})\|_{\mathbb{L}_2(T,\ell_2)}$$

$$\le c|\lambda - \lambda_0|\|u^n - u^{n-1}\|_{\mathcal{H}_2^1(T)}.$$

If $|\lambda - \lambda_0| \le c/2$, then $\|u^{n+1} - u^n\|_{\mathcal{H}_2^1(T)} \le \frac{1}{2}\|u^n - u^{n-1}\|_{\mathcal{H}_2^1(T)}$ for every $n \ge 1$. The map is a contraction and has a unique fixed point. Hence $J = [0, 1]$ and the theorem is proved.

Thank you.