Pointwise Approximation of a Stochastic Heat Equation with Additive Space-Time White Noise

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Contents

Stochastic Heat Equation

$$
\begin{aligned}
dX(t) &= \Delta X(t)\,dt + B(t)\,dW(t), \quad t \in [0, T], \\
X(0) &= x_0,
\end{aligned}
$$

with

- $x_0 \in H = L_2((0,1))$,
- Dirichlet boundary conditions,
- multiplication operator B, i.e.
  $(B(t)h)(u) = g(t,u) \cdot h(u)$
  for $u \in [0,1]$, $h \in H$ and $g \in C^{(1,1)}([0,T] \times [0,1])$,
- cylindrical Brownian motion $W = (\langle W(t), h \rangle)_{t \geq 0, h \in H}$.

Computational Problem

Task

Approximate $X(T)$ based on evaluations of finitely many scalar Brownian motions β_i's at a finite number of points, where

$$\beta_i(\cdot) = \langle W(\cdot), h_i \rangle$$

with $h_i(u) = \sqrt{2}\sin(i\pi u)$ and $i \in \mathbb{N}$.

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Error and cost of approximation $\widehat{X}(T)$

$$\begin{aligned} e\left(\widehat{X}(T)\right) &= \left(\mathrm{E}\|X(T) - \widehat{X}(T)\|_H^2\right)^{1/2}, \\ \mathrm{cost}\left(\widehat{X}(T)\right) &= \text{total number of evaluations of the } \beta_i\text{'s}. \end{aligned}$$

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Goal

Approximation with optimal relation between error and cost.

Classes of Approximations and Minimal Errors

Class $\mathfrak{X}_{\text{uni}}$ of algorithms with **uniform** time discretization:

- choose finite set $\mathcal{I} \subset \mathbb{N}$: only evaluate β_i with $i \in \mathcal{I}$.
- choose $n \in \mathbb{N}$: number of evaluations for β_i with $i \in \mathcal{I}$.
- choose nodes $t_k = \frac{k}{n} T$, $k = 1, \dots, n$, for β_i with $i \in \mathcal{I}$.
- choose $\phi : \mathbb{R}^N \to H$ measurable with $N = |\mathcal{I}| \cdot n$.

Approximation:

$$\widehat{X}(T) = \phi(\beta_{i_1}(t_1), \dots, \beta_{i_1}(t_n), \dots, \beta_{i_\ell}(t_1), \dots, \beta_{i_\ell}(t_n))$$

for $\mathcal{I} = \{i_1, \dots, i_\ell\}$.

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for $\mathcal{I} = \{i_1, \dots, i_\ell\}$.
Clearly

$$\text{cost}\left(\widehat{X}(T)\right) = N.$$

N-th minimal error

$$\text{e}_{\text{uni}}(N) = \inf\left\{\text{e}\left(\widehat{X}(T)\right); \text{cost}\left(\widehat{X}(T)\right) \le N \text{ and } \widehat{X}(T) \in \mathfrak{X}_{\text{uni}}\right\}.$$

Classes of Approximations and Minimal Errors

Class $\mathfrak{X}_{\text{equi}}$ of algorithms with **equidistant** time discretization:

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- choose nodes $t_{k,i} = \frac{k}{n_i} T$, $k = 1, \dots, n_i$, for β_i with $i \in \mathcal{I}$.
- choose $\phi : \mathbb{R}^N \to H$ measurable with $N = \sum_{i \in \mathcal{I}} n_i$.

Approximation:

$$\widehat{X}(T) = \phi(\beta_{i_1}(t_{1,i_1}), \dots, \beta_{i_1}(t_{n_{i_1},i_1}), \dots, \beta_{i_\ell}(t_{1,i_\ell}), \dots, \beta_{i_\ell}(t_{n_{i_\ell},i_\ell}))$$

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- choose nodes $0 < t_{1,i} < \ldots < t_{n_i,i} \leq T$ for β_i with $i \in \mathcal{I}$.
- choose $\phi : \mathbb{R}^N \to H$ measurable with $N = \sum_{i \in \mathcal{I}} n_i$.

Approximation:

$$\widehat{X}(T) = \phi(\beta_{i_1}(t_{1,i_1}), \ldots, \beta_{i_1}(t_{n_{i_1},i_1}), \ldots, \beta_{i_\ell}(t_{1,i_\ell}), \ldots, \beta_{i_\ell}(t_{n_{i_\ell},i_\ell}))$$

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Classes of Approximations and Minimal Errors

N-th minimal errors

$$\begin{aligned} \mathrm{e}_{\mathrm{uni}}(N) &= \inf\left\{\mathrm{e}\left(\widehat{X}(T)\right); \mathrm{cost}\left(\widehat{X}(T)\right) \leq N \text{ and } \widehat{X}(T) \in \mathfrak{X}_{\mathrm{uni}}\right\}, \\ \mathrm{e}_{\mathrm{equi}}(N) &= \inf\left\{\mathrm{e}\left(\widehat{X}(T)\right); \mathrm{cost}\left(\widehat{X}(T)\right) \leq N \text{ and } \widehat{X}(T) \in \mathfrak{X}_{\mathrm{equi}}\right\}, \\ \mathrm{e}(N) &= \inf\left\{\mathrm{e}\left(\widehat{X}(T)\right); \mathrm{cost}\left(\widehat{X}(T)\right) \leq N \text{ and } \widehat{X}(T) \in \mathfrak{X}\right\}. \end{aligned}$$

Clearly

$$\mathrm{e}(N) \leq \mathrm{e}_{\mathrm{equi}}(N) \leq \mathrm{e}_{\mathrm{uni}}(N).$$

But are these minimal errors basically equal?

Results for Additive Noise

Theorem *Henkel* (2009)
Assume that $x_0 = 0$ and

$$\langle g(t) \cdot h_i, h_j \rangle^2 \asymp \frac{1}{|i-j|^\alpha + 1}$$

with $\alpha > 1$. Then

$$\mathrm{e}_{\mathrm{equi}}(N) \asymp \mathrm{e}_{\mathrm{uni}}(N) \asymp N^{-1/6}$$

and

$$N^{-1/2} \preceq \mathrm{e}(N) \preceq \begin{cases} N^{-1/6} & \text{if } 1 < \alpha \leq \frac{5}{3}, \\ N^{-(\alpha-1)/4} & \text{if } \frac{5}{3} < \alpha < 2, \\ N^{-1/4} & \text{if } 2 \leq \alpha < \infty. \end{cases}$$

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Remark Limiting case $g = 1$, i.e. $\alpha \to \infty$:

$$e_{\text{uni}}(N) \asymp e_{\text{equi}}(N) \asymp N^{-1/6}$$

and

$$e(N) \asymp N^{-1/2},$$

see *Müller-Gronbach, Ritter, Wagner* (2007).

Results for Additive Noise

Remark

- Suboptimality of $\mathfrak{X}_{\text{equi}} \supset \mathfrak{X}_{\text{uni}}$ (at least), if $\alpha > \frac{5}{3}$.
- Optimal time discretizations are quantiles of the density $\exp(-\frac{\mu_j}{3}(T-t))$, i.e.

$$\int_0^{s_{k,j}} \exp\left(-\frac{\mu_j}{3}(T-t)\right)\, dt = \frac{k}{\nu_j} \int_0^T \exp\left(-\frac{\mu_j}{3}(T-t)\right)\, dt$$

for $j \in \mathcal{J} \subset \mathbb{N}$, $\mu_j = \pi^2 j^2$, $\nu_j \in \mathbb{N}$, $k = 1, \ldots, \nu_j$ and $\{t_{1,i}, \ldots, t_{n,i}\} = \bigcup_{j \in \mathcal{J}} \{s_{1,j}, \ldots, s_{\nu_j,j}\}$ for every β_i.

Related Results

Equation

$$\begin{aligned}
\frac{\partial}{\partial t}X(t,u) &= \frac{\partial^2}{\partial u^2}X(t,u) + a(t,u,X(t,u)) \\
&\quad + \sigma(t,u,X(t,u))\frac{\partial^2}{\partial t \partial u}\mathcal{W}(t,u), \\
X(t,0) &= X(t,1) = 0, \quad t \in [0,T], \\
X(0,u) &= x_0(u), \quad u \in [0,1],
\end{aligned}$$

with regularity conditions for a, σ and x_0.

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Theorem *Gyöngy* (1999)

With $\widehat{X}_N$, based on an equidistant N-point grid in $[0,T] \times [0,1]$ to evaluate the Brownian sheet $\mathcal{W}$,

$$\sup_{u \in [0,1]} \left(\mathrm{E}|X(t,u) - \widehat{X}_N(t,u)|^2 \right)^{1/2} \preceq N^{-1/6}$$

for all $p \geq 1$ and $t \in (0,T]$.

Related Results

Equation

$$\begin{aligned} \frac{\partial}{\partial t} X(t,u) &= \frac{\partial^2}{\partial u^2} X(t,u) + X(t,u) \frac{\partial^2}{\partial t \partial u} \mathcal{W}(t,u), \\ X(t,0) &= X(t,1) = 0, \quad t \in [0,T], \\ X(0,u) &= 1, \quad u \in [0,1]. \end{aligned}$$

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\end{aligned}
$$

Theorem *Davie, Gaines* (2001)

$$\mathrm{e}(N) \succeq N^{-1/6}.$$

Related Results

Errorcriterion

$$\epsilon(\widehat{X}) = \left(\mathrm{E} \int_0^T \|X(t) - \widehat{X}(t)\|_H^2 \, dt \right)^{1/2},$$

i.e. approximation of $X(t)$ for every $t \in [0, T]$, see the following talk.

Summary and Outlook

Summary

At a fixed time point, for additive noise with decay condition

$$e_{\text{equi}}(N) \asymp e_{\text{uni}}(N) \asymp N^{-1/6}$$

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Outlook

- Multiplicative noise.
- Sharp upper terms for $e(N)$.