SPDE09

Stochastic Partial Differential Equations: Modelling, Analysis, and Approximation

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On Numerical Solutions of Stochastic PDEs

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I. Stochastic evolution equations and their approximations

II. Splitting up approximations

III. Accelerated numerical schemes

Abstract.

In part I stochastic evolution equations with monotone operators are considered. A general framework to investigate various numerical approximations is introduced that can be used to study finite difference, wavelets and finite elements space discretizations. Rate of convergence estimates are obtained. The results can be applied to a class of quasilinear stochastic PDEs of parabolic type. In particular, rate of convergence estimates for finite difference schemes to linear SPDEs are deduced from the abstract setting.

In part II splitting-up approximations for linear SPDEs are studied and moment estimates for the error are obtained. We also show that our estimates are sharp.

In part III accelerated numerical schemes are presented.

The first part is based on a recent joint paper with Annie Millet. The second and third parts are based on joint results with Nicolai Krylov.

I. Stochastic evolution equations

Notation: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$,

$W = \{W^k(t) : t \geq 0, k = 1, \dots\}$,
cylindrical $\mathcal{F}_t$-Wiener martingale,

$\mathcal{P}$ predictable sigma-algebra,

$\mathcal{B}(V)$ Borel sigma-algebra generated on V, *Normal triple:*

$$V \hookrightarrow H \hookrightarrow V^*,$$

V is a separable reflexive Banach space, embedded continuously and densely into a Hilbert space H, identified with its dual H^* by the inner product $(\,,\,)$ in H. The embedding $H \hookrightarrow V^*$ is the adjoint of $V \hookrightarrow H$. Thus

$$\langle v, h \rangle = (v, h), \quad \forall v \in V,\, h \in H^* = H,$$

where $\langle \cdot, \cdot \rangle$ is the duality between V and V^*.

Consider

$$(E)\begin{cases} du(t) = A(t,u(t))\,dt + B(t,u(t))\,dW(t),\ t \in [0,T], \\ u(0) = u_0 \end{cases}$$

where u_0 is $\mathcal{F}_0$-measurable r.v. in H,

$$A : [0,\infty) \times \Omega \times V \to V^*,$$

$$B = (B_k) : [0,\infty) \times \Omega \times V \to l_2(H),$$

$\mathcal{P} \otimes \mathcal{B}(V)$-measurable, satisfying

(i) *Monotonicity of* (A, B)

$$2\langle u - v, A(u) - A(v)\rangle + |B(u) - B(v)|^2_{l_2(H)} \leq K|u - v|^2_H,$$

(ii) *Coercivity of* (A, B)

$$2\langle u, A(u)\rangle + |B(u)|^2_{l_2(H)} + \lambda\|u\|^2_V \leq K|u|^2_H + \xi,$$

(iii) *Growth condition on* A *and* B

$$|A(u)|^2_{V^*} \leq K_1\|u\|^2_V + \xi, \quad |B(u)|^2_{l_2(H)} \leq K_2\|u\|^2_V + \xi,$$

(iv) Hemicontinuity of A

$$\lim_{\varepsilon \to 0} \langle w, A(u + \varepsilon v) \rangle = \langle w, A(u) \rangle,$$

$\forall \omega \in \Omega$, $t \in [0, T]$, $u, v, w \in V$.

Here and below $\lambda > 0$, $K \geq 0$, $K_1 \geq 0$ $K_2 \geq 0$ are some constants, ξ is a process, such that $\xi \in L_1([0,T], \mathbb{R}_+)$ (a.s). We use the notation

$$l_2(H) = \{(h_k)_{k=1}^{\infty} : h_k \in H, \, |h|^2_{l_2(H)} = \sum_k |h_k|^2_H < \infty\}.$$

Definition 1. An adapted H-valued continuous process $u = (u(t))_{t\in[0,T]}$ is a solution if almost surely $u(t) \in V$ for a.e. t,

$$\int_0^T \|u(t)\|_V^2 \, dt < \infty \,, \quad \text{and}$$

$$(u(t), v) = (u_0, v) + \int_0^t \langle A(s, u(s)), v\rangle \, ds$$

$$+ \int_0^t (B_k(s, u(s)), v) \, dW^k(s) \quad \forall t \in [0, T], \, v \in V.$$

(Unless otherwise indicated, we use the summation convention w.r. to repeated indices in these lectures.)

Theorem 1. *Assume (i)-(iv). Then there is a unique solution* u*, and there is a constant* $C = C(T, \lambda, K, K_2)$ *such that*

$$E \sup_{t\in[0,T]} |u(t)|_H^2 + E\int_0^T \|u(s)\|_V^2\, ds$$

$$\leq C(E|u_0|_H^2 + E\int_0^T \xi(t)\, dt).$$

Proof: This is a classical result from [P] and [KR]. Its proof can be found also, e.g., in [R] and [PR]. □

Remark. A generalization of this theorem to the case when the driving processes are random measures and semimartingales which may have jumps (e.g., they can be Lévy processes), is given in [G1] (see also [G2]).

Under the above conditions one can prove also the convergence of various numerical schemes (see [GM1]). To estimate the rate of convergence of these schemes we introduce stronger conditions below (cf. [GM2]).

Example 1. *Quasilinear equations*

$$du(t,x) = (Lu(t,x) + F(t,x,\nabla u(t,x), u(t,x))\,dt$$

$$+\,(M_k u(t,x) + g_k(t,x))\,dW^k(t), \qquad u(0,x) = u_0(x)$$

for $t \in (0,T]$, $x \in \mathbb{R}^d$, where $F = F(\omega,t,x,w,r)$, $g_k = g_k(\omega,t,x)$, $w \in \mathbb{R}^d$, $r \in \mathbb{R}$;

$$Lv = D_p(a^{pq}D_q v), \qquad M_k v = b_k^p D_p v,$$

$$D_i v := \frac{\partial}{\partial x_i} v, \qquad i = 1,...,d, \qquad D_0 v := v,$$

with bounded $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions

$$a^{pq} : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}, \quad p,q := 0,1,...,d$$

$$b^p = (b^p_k) : \Omega \times [0,T] \times \mathbb{R}^d \to l_2(\mathbb{R}), \quad p := 0,...,d$$

Assumption (A1) (Stochastic λ-parabolicity)

$$(a^{ij} - \tfrac{1}{2} b^i_k b^j_k) z_i z_j \geq \lambda \sum_{i=1}^{d} z_i^2 \quad \forall (z_1,...,z_d) \in \mathbb{R}^d$$

Assumption (A2)

$$|\nabla_p F(t,x,p,r)| + |\tfrac{\partial}{\partial r} F(t,x,p,r)| \leq K$$

Assumption (A3)

u_0 is an L_2-valued $\mathcal{F}_0$-measurable r.v., and

$$\mathcal{K}_T^2 := \int_0^T \int_{\mathbb{R}^d} |F(t,x,0,0)|^2 + \sum_k |g_k(t,x)|^2 \, dx \, dt < \infty.$$

Let $W_2^m = W_2^m(\mathbb{R}^d)$ denote for integers $m \geq 0$ the Sobolev space defined as the closure of $C_0^\infty(\mathbb{R}^d)$ in the norm $|\cdot|_m$ given by

$$|\varphi|_m^2 := \sum_{|\alpha| \leq m} \int_{\mathbb{R}^d} |D^\alpha \varphi(x)|^2 \, dx,$$

where $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} ... D_d^{\alpha_d}$ for multi-indices $\alpha = (\alpha_1, ..., \alpha_d) \in \{0, 1, ...\}^d$, $|\alpha| := \sum_{i=1}^d \alpha_i$.

Definition. An $L_2(\mathbb{R}^d)$-valued adapted continuous process u is a solution if almost surely $u(t) \in W_2^1(\mathbb{R}^d)$ for a.e. $t \in [0, T]$,

$$\int_0^T |u(t)|_1^2 \, dt < \infty,$$

$$(u(t), \varphi) = (u_0, \varphi) - \int_0^t (a^{ij} D_j u(s), D_i \varphi) \, ds$$

$$+ \int_0^t (F(s, \cdot, \nabla u(s), u(s)), \varphi) \, ds$$

$$\int_0^t (b_k^r D_r u(s) + g_k(s), \varphi) \, dW^k(s)$$

for all $t \in [0, T]$, $\varphi \in C_0^\infty(\mathbb{R}^d)$.

Theorem. *Let (A1)-(A3) hold. Then there is a unique solution u, and for a constant C*

$$E \sup_{t\in[0,T]} |u(t)|_0^2 + E\int_0^T |u(s)|_1^2\,ds \le CE(|u_0|_0^2 + \mathcal{K}_T^2).$$

Proof: Set $V = W_2^1(\mathbb{R}^d)$, $H = L_2(\mathbb{R}^d)$, define $A(t,\cdot) : V \to V^*$ and $(B_k)(t,\cdot) : V \to l_2(H)$ by

$$\langle A(t,v),\varphi\rangle = (a^{pq}(t)D_q v, D_p^*\varphi) + (F(t,\nabla v, v),\varphi)$$

$$B_k(t,v) = b_k^p(s)D_p v + g_k(s),$$

with $D_i^*\varphi := -D_i\varphi$, $i = 1,...,d$, $D_0^*\varphi := \varphi$, and apply Theorem 1.

Example 2. Linear equations

$$du(t,x) = \Big(D_p(a^{pq}D_q u(t,x)) + f(t,x)\Big)\,dt$$

$$+\Big(b^p_k D_p u(t,x) + g_k(t,x)\Big)\,dW^k(t), \qquad u(0,x) = u_0(x)$$

for $t \in (0,T]$, $x \in \mathbb{R}^d$, where $p,q = 0,1,\ldots,d$,

$$a^{pq}, f : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R},$$

$$b^p = (b^p_k), g = (g_k) : \Omega \times [0,T] \times \mathbb{R}^d \to l_2(\mathbb{R}),$$

u_0 is an L_2-valued $\mathcal{F}_0$-measurable r. v., and D_0 is the unit operator.

Smoothness assumption: For an integer $l \geq 0$

- $|D^\alpha a^{pq}| \leq K$, $|D^\alpha b^p|_{l_2(\mathbb{R})} \leq K$ for $|\alpha| \leq l$, $p, q = 0, \ldots, d$

- $\mathcal{K}_T^2 := \int_0^T |f(t)|_l^2 + \sum_k |g_k|_l^2 \, dt < \infty$ (a.s.)

- $u_0 \in W_2^l$

Theorem. *Let (A1) and the above 'smoothness assumption' hold. Then there is a unique solution* u. *Moreover,* u *is a* W_2^l*-valued continuous process, and with a constant* C

$$E \sup_{t\in[0,T]} |u(t)|_l^2 + E\int_0^T |u(t)|_{l+1}^2\, dt \le CE(|u_0|_l^2 + \mathcal{K}_T^2).$$

Proof: Apply Theorem 1 with appropriate normal triples.

Numerical Schemes

Our aim is to present a general framework to study numerical schemes for SPDEs of parabolic type. Our approach uses the L_2-theory of SPDEs and evolves around the basic observation in numerical analysis of linear equations that suitable *regularity* properties of the approximated equation and *stability* of *consistent* numerical schemes ensure the convergence of the approximations.

First spatial discretizations for (E) are considered and then space-time discretizations are studied. To get estimates for the accuracy of the schemes, regularity conditions (R...) for equation (E), together with stability conditions (S...) and consistency conditions (C..) for the schemes are imposed. For detailed proofs and examples we refer to [GM3], where some of the results are inspired by [Y].

1. Space discretization

Let $V_n \hookrightarrow H_n \hookrightarrow V_n^*$ be a sequence of normal triples and let Π_n be a bounded linear operator from V to V_n such that for a constant μ

$$|\Pi_n v|_{V_n} \leq \mu |v|_V \quad \forall v \in V, \quad n \geq 1.$$

We call them *approximation triples* for the triple $V \hookrightarrow H \hookrightarrow V^*$.

Example 3. Consider for integers $k \geq 1$ the normal triples

$$W^k_{h,2} \hookrightarrow W^{k-1}_{h,2} \hookrightarrow (W^k_{h,2})^* \qquad (A)$$

of *discret Sobolev spaces* $W^m_{h,2}$, defined for integers $m \geq 0$ and real numbers $h > 0$ as the space of real functions φ on the grid

$$\mathbb{G} := h\mathbb{Z}^d = \{hk : k = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d\},$$

such that

$$|\varphi|^2_{h,m} := \sum_{|\alpha| \leq m} \sum_{z \in \mathbb{G}} |\delta^+_\alpha \varphi(z)|^2 < \infty,$$

where $\delta^+_\alpha := \delta^+_{\alpha_1} \dots \delta^+_{\alpha_m}$, $|\alpha| = m$ for sequences $\alpha = \alpha_1 ... \alpha_m$, $\alpha_j \in \{0, 1, ..., d\}$, and

$$\delta^\pm_i \varphi(z) := \pm\tfrac{1}{h}(\varphi(z \pm he_i) - v(z)), \quad \delta^\pm_0 \varphi = \varphi$$

for the standard basis $(e_i)^d_{i=1}$ in $\mathbb{R}^d$.

The space $W^m_{h,2}$ with $|\cdot|_{h,m}$ is a Hilbert space.

For integers $l > d/2$ and $k \geq 0$ the linear operator Π defined by $\Pi\varphi = \bar{\varphi}\big|_{\mathbb{G}}, \quad \varphi \in W^{l+k}_2$, maps W^{l+k}_2 into $W^k_{h,2}$, where $\bar{\varphi}\big|_{\mathbb{G}}$ is the restriction onto $\mathbb{G}$ of the continuous modification $\bar{\varphi}$ of φ, which exists by the Sobolev embedding theorem of W_2^l in $\mathcal{C}$. By Sobolev's theorem one can also see that for $l > d/2$ and $k \geq 0$

$$|\Pi\varphi|_{h,k} \leq C|\varphi|_{W_2^{k+l}} \quad \text{for all } \varphi \in W_2^{l+k},$$

with a constant C independent of h. Thus (A) with $h_n \in (0,1]$, $n = 1, 2, \ldots$, gives approximation triples for

$$W_2^{k+l} \hookrightarrow W_2^{k-1+l} \hookrightarrow (W_2^{k+l})^* \quad l > d/2,\ k \geq 1.$$

Other examples include appropriate wavelet and finite element spaces.

Consider

$$(E^n)\begin{cases} du^n(t) = A^n(t, u^n(t))\,dt + B^n(t, u^n(t))\,dW(t) \\ u^n(0) = u_0^n, \end{cases}$$

where u_0 is $\mathcal{F}_0$-measurable r.v. in H,

$$A^n : [0, T] \times \Omega \times V_n \to V_n^*,$$

$$B^n = (B_k^n) : [0, T] \times \Omega \times V_n \to l_2(H_n),$$

$\mathcal{P} \otimes \mathcal{B}(V_n)$-measurable, satisfying

(S1) *Strong monotonicity of* (A^n, B^n)

$$2\langle u-v, A^n(u)-A^n(v)\rangle_n + \sum_k |B^n_k(u)-B^n_k(v)|^2_{H_n}$$
$$+\lambda\|u-v\|^2_{V_n} \le K|u-v|^2_{H_n} \quad \text{for } t\in[0,T],\ u,v\in V_n.$$

(Here and later on we use $\langle\cdot,\cdot\rangle_n$ for the duality product between V_n and V_n^*.)

(S2) Almost surely $\forall\ t\in[0,T]$, $v\in V_n$, $n\ge 1$,

$$|A^n(v)|^2_{V_n^*} \le K_1\|v\|^2_{V_n} + f^n,$$
$$|B^n(v)|^2_{H_n} \le K_2\|v\|^2_{V_n} + g^n$$

with constants K_1, K_2, independent of n, processes $f^n \geq 0$, $g^n \geq 0$ such that

$$\sup_n E \int_0^T f^n(t)\,dt < \infty, \quad \sup_n E \int_0^T g^n(t)\,dt < \infty.$$

(S3) *Hemicontinuity of A^n:* For $v, u, w \in V_n$

$$\lim_{\varepsilon \to 0} \langle A^n(v + \varepsilon u)\,, w \rangle_n = \langle A^n(v)\,, w \rangle_n \quad \forall t, \omega.$$

(S4) *Lipschitz condition on B^n*

$$\sum_k |B_k^n(u) - B_k^n(v)|^2_{H_n} \leq L_B \, \|u - v\|^2_{V_n}$$

for all $u, v \in V_n$ and t, ω with a constant L_B.

By Thm.1 **(S1)–(S3)** imply the existence of a unique solution u^n, and

$$E \sup_{0\le t\le T} |u^n(t)|^2_{H_n} + E\int_0^T \|u^n(t)\|^2_{V_n}\,dt$$
$$\le C\left(E|u^n_0|^2_{H_n} + E\int_0^T f^n(t) + g^n(t)\,dt\right).$$

with a constant $C = C(T, L_B, \lambda, K_2)$.

We want to approximate $\Pi_n u$ by u^n. To estimate the accuracy we relate (A, B) to (A_n, B_n) via *regularity* and *consistency* conditions.

To formulate them assume there exist Hilbert spaces $\mathcal{V}$, $\mathcal{H}$ such that

$$\mathcal{V} \to \mathcal{H} \to V \quad \text{with continuous embeddings}$$

and impose the following conditions.

(R1) There is a unique solution u of (E), it takes values in $\mathcal{V}$ for $dt \times P$-almost every $(t, \omega) \in [0, T] \times \Omega$. Moreover, $u_0 \in V$ and

$$E\|u_0\|_V^2 < \infty, \quad E\int_0^T |u(t)|_{\mathcal{V}}^2 \, dt < \infty.$$

(R2) Almost surely $A(v) \in V$, $B_k(u) \in V$ and

$$\|A(v)\|_V^2 \leq K|v|_{\mathcal{V}}^2 + \zeta, \quad \sum_k \|B_k(u)\|_V^2 \leq K|u|_{\mathcal{H}}^2 + \eta$$

$\forall t \in [0, T]$, $v \in \mathcal{V}$ and $u \in \mathcal{H}$, with processes $\zeta \geq 0$, $\eta \geq 0$, $E\int_0^T \zeta(t)\,dt + \sup_{t\in[0,T]} E\eta(t) < \infty$.

Consistency (C1) For a sequence $\varepsilon_n \downarrow 0$

$$|\Pi_n A(v) - A^n(\Pi_n v)|_{V_n^*}^2 + \sum_k |\Pi_n B_k(v) - B_k^n(\Pi_n v)|_{H_n}^2$$

$$\leq \varepsilon_n^2 (|v|_{\mathcal{V}}^2 + \xi^n), \quad \forall v \in \mathcal{V}, t \in [0, T], \omega \in \Omega.$$

Theorem 2. *Let (S1)-(S4), (R1)-(R2) and (C1) hold. Then for* $e^n := \Pi_n u - u^n$

$$E \sup_{t\le T} |e^n(t)|^2_{H_n} + E\int_0^T \|e^n(t)\|^2_{V_n} dt \le C(E|e^n(0)|^2_{H_n} + \varepsilon_n^2)$$

with a constant C *independent of* n.

Proof: Step1. Prove

$$\sup_{t\le T} E|e^n(t)|^2_n + E\int_0^T \|e^n(t)\|^2_n dt \le C(E|e^n(0)|^2_n + \varepsilon_n^2),$$

where $\|\cdot\|_n := \|\cdot\|_{V_n}$ and $|\cdot|_n := |\cdot|_{H_n}$. Using

$$d\Pi_n u(t) = \Pi_n A\big(t, u(t)\big)\, dt + \Pi_n B_k\big(t, u(t)\big) dW^k(t),$$

by Itô's formula we have

$$|e^n(t)|_n^2 = |e^n(0)|_n^2 + \sum_{i\le 3} I_i(t),$$

where

$$\begin{aligned}
I_1(t) &= 2\int_0^t \langle e^n(s)\,, \Pi_n A(u(s)) - A^n(u^n(s))\rangle_n ds\,,\\
I_2(t) &= 2\int_0^t (e^n(s)\,, \Pi_n B_k(u(s)) - B_k^n(u^n(s)))_n dW^k(s),\\
I_3(t) &= \sum_k \int_0^t \Big|\Pi_n B_k(u(s)) - B_k^n(u^n(s))\Big|_n^2 ds\,.
\end{aligned}$$

By (S1) we get $I_1(t) + I_3(t)$

$$\leq -\lambda \int_0^t \|e^n(s)\|_n^2 \, ds + L \int_0^t |e^n(s)|_n^2 \, ds + \sum_{i=1,2} R_i(t),$$

where

$$\begin{aligned} R_1(t) &= \int_0^t 2\langle e^n(s) \, , \, \Pi_n A(s, u(s)) - A^n(\Pi_n u(s))\rangle_n \, ds, \\ R_2(t) &= \sum_k \int_0^t \Big[|\Pi_n B_k(u(s)) - B_k^n(u^n(s))|_n^2 \\ &\quad -|B_k^n(\Pi_n u(s)) - B_k^n(u^n(s))|_n^2 \Big] \, ds. \end{aligned}$$

By (C) we have $|R_1(t)| \leq \frac{\lambda}{3} \int_0^t \|e^n(s)\|_n^2 \, ds$

$$+\frac{3}{\lambda} \varepsilon_n^2 \int_0^t \big(|u(s)|_{\mathcal{V}}^2 + \xi^n(s) \big) \, ds.$$

By (C) and (S4)

$$|R_2(t)| \le C\,\varepsilon_n^2 \int_0^t \Big(|u(s)|_{\mathcal{V}}^2 + \xi^n(s)\Big)\,ds + \frac{\lambda}{3}\int_0^t \|e^n(s)\|_n^2\,ds.$$

Hence $E|e^n(t)|_n^2 + \frac{\lambda}{3}E\int_0^t \|e^n(s)\|_n^2\,ds$

$$\le LE\int_0^t |e^n(s)|_n^2 ds + E|e^n(0)|_n^2 + C\varepsilon_n^2,$$

and Gronwall's lemma yields

$$\sup_{0\le t\le T} E|e^n(t)|_n^2 \le e^{LT}(C\varepsilon_n^2 + E|e^n(0)|_n^2).$$

Step2. Clearly,

$$E\sup_{t\le T}|e^n(t)|_n \le E\sup_{0\le t\le T}\{I_1(t)+I_3(t)\} + E\sup_{t\le T} I_2(t).$$

By (S1) and Step1

$$E \sup_{0\le t\le T} \Big(I_1(t) + I_3(t)\Big) \le C(E|e^n(0)|_n^2 + \varepsilon_n^2).$$

Using Davis' inequality, by (S4), (C) and Step1

$$E \sup_{t\le T} |I_2(t)| \le \tfrac{1}{2} E \sup_{0\le t\le T} |e^n(t)|_n^2 + C\varepsilon_n^2.$$

Hence

$$E \sup_{t\le T} |e^n(t)|_n \le \tfrac{1}{2} E \sup_{t\le T} |e^n(t)|_n^2 + C(E|e^n(0)|_n^2 + \varepsilon_n^2).$$

□

Examples. Approximations by *wavelets*, *Galerkin method*, *finite elements*, and *finite differences*.

- **Finite difference schemes for SPDEs**.

Consider the problem from Example 1,

$$du(t) = (Lu(t) + F(t, \nabla u(t), u(t))\, dt$$

$$+\Big(M_k u(t) + g_k(t))\, dW^k(t), \quad u(0,x) = u_0(x)$$

for $t \in (0, T]$, $x \in \mathbb{R}^d$.

For $h \in (0,1)$ approximate it by

$$\begin{aligned} dv(t) &= \Big(L_h(t)v(t) + F_h(t, \nabla_h v(t), v(t))\Big)\,dt \\ &\quad + \Big(M_{k,h}(t)v(t) + g_{k,h}(t)\Big)\,dW^k(t), \\ v(0) &= (u_0(z))_{z\in\mathbb{G}}, \end{aligned}$$

$t \in [0,T]$, for $v(t) = (v(t,z))_{z\in\mathbb{G})}$ on the grid

$$\mathbb{G} = h\mathbb{Z}^d = \{hk : k = (k_1, k_2, \dots, k_d) \in \mathbb{Z}^d\},$$

where $g_{k,h}(t) = (g_k(t,z))_{z\in\mathbb{G}}$,

$$F_h(t,w,r) = (F(t,z,w,r))_{z\in\mathbb{G}},$$

$$L_h\varphi = \delta_p^-(a^{pq}\delta_q^+\varphi), \qquad M_{k,h}\varphi = b_k^p\delta_p^+\varphi,$$
$$\delta_i^{\pm}\varphi(z) = \pm\tfrac{1}{h}(\varphi(z \pm he_i) - v(z)), \qquad \delta_0^{\pm}\varphi = \varphi$$
$$\nabla_h\varphi = (\delta_1^+\varphi, \ldots, \delta_d^+\varphi),$$

for functions φ defined on $\mathbb{G}$ and for the standard basis $(e_i)_{i=1}^d$ in $\mathbb{R}^d$.

Theorem. Let (A1)-(A3) hold. Then there is a unique continuous $W_{h,2}^0$-valued solution v.

Proof: Let $H := W^0_{h,2}$ and $V := W^1_{h,2}$. Define $A(t,\cdot) : V \to V^*$, $(B_k)(t,\cdot) : V \to l_2(H)$,

$$\langle A(t,v),\varphi\rangle = -(a^{ij}(t)\delta^+_j v, \delta^-_i \varphi) + (F(t,\nabla_h v, v),\varphi),$$

$$B_k(t,v) = b^r_k(s)\delta^+_r v + g_k(s)$$

and apply Theorem 1. □

To estimate the accuracy of the approximation we ensure the regularity conditions (R1)-(R2) by

(A4) $|D^\gamma a^{pq}| \le K$, $|D^\gamma b^p|_{l_2} \le K$ for $|\gamma| \le 2$

(A5) $|\nabla_x F(t,x,w,r)| \le L(|w| + |r|) + \eta$,

$$\sum_k |g_k(t)|_2^2 \le \eta \quad E|u_0|_2^2 < \infty$$

for all t, x, ω, w, r, with r.v. η, $E\eta < \infty$.

Theorem. *Let* $d = 1$ *and (A1)-(A5) hold. Then*

$$E \sup_{t \le T} |u(t) - v(t)|^2_{h,0} + E \int_0^T |u(t) - v(t)|^2_{h,1}\, dt \le Ch^2$$

with a constant C independent of h.

Proof: Set $\mathcal{V} = \mathcal{H} := W^3_2$, $V := W^2_2$, $H := W^1_2$. Using Thm 1 with $\mathcal{V} \hookrightarrow V \hookrightarrow \mathcal{V}^*$, show that u has a W^2_2-valued continuous modification, denoted also by u, such that

$$E \sup_{t \le T} |u(t)|^2_2 + E \int_0^T |u(t)|^2_3\, dt < \infty.$$

Hence (R1) holds and (R2) can easily be verified. By Remark 3, $V_n \hookrightarrow H_n \hookrightarrow V_n^*$ with $V_n := W^1_{h_n,2}$ and $H_n := W^0_{h_n,2}$ are approximation triples for V and H for any $h_n \in (0,1)$. Verify

$$
\begin{aligned}
|D_p(a^{pq}D_q\varphi) - \delta_p^-(a^{pq}\delta_q^+\varphi)|_{h,0} &\leq Ch|\varphi|_{W_2^3}, \\
\sum_k |b_k^p D_p\varphi - b_k^p(t)\delta_p^+\varphi|^2_{h,0} &\leq C^2h^2|\varphi|^2_{W_2^2}, \\
|F_h(t, D\varphi, \varphi) - F_h(t, \delta\varphi, \varphi)|_{h,0} &\leq Ch|\varphi|_{W_2^2}
\end{aligned}
$$

for all $\varphi \in W_2^3(\mathbb{R})$ and $h \in (0,1)$, i.e. (C) holds with $\varepsilon_n = h$. Apply Thm 2. $\square$

2. **Space-time discretizations:**

(a) Semi-implicit scheme. Set $\tau := T/m$ for a fixed integer $m \geq 1$, and $t_i = i\tau$ for $i = 0, \cdots, m$. Let $V_n \hookrightarrow H_n \hookrightarrow V_n^*$ with Π_n be approximations for $V \hookrightarrow H \hookrightarrow V^*$.

Consider for each n the system of equations

$$u_{i+1}^{n,\tau} = u_i^{n,\tau} + \tau\, A_{i+1}^{n,\tau}(u_{i+1}^{n,\tau})$$

$$+B_{k,i}^{n,\tau}(u_i^{n,\tau})\left(W^k(t_{i+1}) - W^k(t_i)\right), \qquad (E^{n,\tau})$$

$i = 0, \ldots, m-1$, for V_n-valued $\mathcal{F}_{t_i}$-measurable r.v. $u_i^{n,\tau}$, $i = 1, \ldots, m$, where $u_0^{n,\tau}$ is a given V_n-valued $\mathcal{F}_0$-measurable r.v., and

$$A_j^{n,\tau} : \Omega \times V_n \to V_n^*, \quad \mathcal{F}_{t_j} \otimes \mathcal{B}(V_n)\text{-measurable},$$

$$(B_{k,i}^{n,\tau}) : \Omega \times V_n \to l_2(H_n), \quad \mathcal{F}_{t_i} \otimes \mathcal{B}(V_n)\text{-measurable}$$

mappings for $j = 1, \ldots, m$, $i = 0, \ldots m-1$.

Let $\lambda > 0$, $K \geq 0$, $L \geq 0$ and $L_1 \geq 0$ be constants and make the following assumptions.

(ST1) *Strong monotonicity:* Almost surely for all $u, v \in V_n$

$$2\left\langle u - v\,,\, A_j^{n,\tau}(u) - A_j^{n,\tau}(v)\right\rangle_n + \sum_k \left|B_{k,j}^{n,\tau}(u) - B_{k,j}^{n,\tau}(v)\right|_{H_n}^2 \leq -\lambda \|u - v\|_{V_n}^2 + L|u - v|_{H_n}^2$$

(ST2) Almost surely for all $u \in V_n$

$$\left|A_j^{n,\tau}(u)\right|_{V_n^*}^2 \leq K\|u\|_{V_n}^2 + f_j^{n,\tau},$$
$$\sum_k \left|B_{k,i}^{n,\tau}(u)\right|_{H_n}^2 \leq K\|u\|_{V_n}^2 + g_i^{n,\tau},$$

where $f_j^{n,\tau} \geq 0$ and $g_i^{n,\tau} \geq 0$ are r. variables,

$$\sup_{n,m} \sum_j \tau E f_j^{n,\tau} < \infty, \quad \sup_{n,m} \max_i E g_i^{n,\tau} < \infty.$$

(ST3) Almost surely for all $u, v \in V_n$

$$\left|A_j^{n,\tau}(u) - A_j^{n,\tau}(v)\right|_{V_n^*}^2 \leq L_1 \|u - v\|_{V_n}^2, \tag{L1}$$

for $j = 1, \ldots, m$.

Remarks.

- *Clearly,* **(ST1)**&**(ST3)**$\Rightarrow \exists L_2 = L_2(L, \lambda, L_1)$

$$\sum_k \left| B_{k,j}^{n,\tau}(u) - B_{k,j}^{n,\tau}(v) \right|_{H_n}^2 \le L_2 \, \|u - v\|_{V_n}^2 \qquad (L2)$$

for all $u, v \in V_n$, $n \ge 1$, $m \ge 0$ *and* $j = 1, \cdots, m$.

- **(ST1)–(ST2)**$\Rightarrow \exists C = C(\lambda, K)$

$$2\langle u, A_j^{n,\tau}(u)\rangle_n + \sum_k |B_{k,j}^{n,\tau}(u)|_{H_n}^2$$
$$\le -\frac{\lambda}{2} \|u\|_{V_n}^2 + L|u|_{H_n}^2 + C(f_j^{n,\tau} + g_j^{n,\tau})$$

for all $u \in V_n$, $n \ge 0$, $m \ge 1$ *and* $j = 1, \ldots, m$

- **(ST3)** *(Lipschitz)* $\Rightarrow A_i^{n,\tau}$ *is hemicont.*

Theorem. *Assume (ST1)-(ST3) hold and* $E\|u_0^{n,\tau}\|_{V_n}^2 < \infty$ *for all* $n \geq 0$ *and* $m \geq 1$*. Then for* $\tau < 1/L$ *equation* $(E^{n,\tau})$ *has a unique* V_n*-valued solution* $(u_i^{n,\tau})_{j=1}^m$*, such that* $u_j^{n,\tau}$ *is* $\mathcal{F}_{t_j}$*-measurable and* $E\|u_j^{n,\tau}\|_{V_n}^2 < \infty$ *for each* j*,* n*.*

Proof: $(E^{n,\tau})$ can be rewritten as

$$D_{i+1}(u_{i+1}^{n,\tau}) = u_i^{n,\tau} + B_{k,i}^{n,\tau}\left(u_i^{n,\tau}\right)\left(W^k(t_{i+1}) - W^k(t_i)\right),$$

with $D_i : V_n \to V_n^*$, $D_i(v) = v - \tau A_i^{n,\tau}(v)$ for each $i = 1, 2, \cdots m$. By the above conditions and remark D_i is monotone, coercive and hemi-continuous with linear growth. □

To get accuracy estimates, assume more regularity on (E) and relate $A(t_i,)$ to $A_i^{n,\tau}$ and $B_k(t_i)$ to $B_{k,i}^{n,\tau}$ by a consistency condition. Let $\nu \in]0, 1/2]$ and $c \geq 0$ be constants and make the following assumptions.

(R3) The solution u of (E) has an $\mathcal{H}$-valued stochastic modification, denoted also by u, and $\sup_{t\in[0,T]} E|u(t)|^2_{\mathcal{H}} < \infty$.

(R4) There is a r.v. $\eta \geq 0$, $E\eta < \infty$, such that

$$\|A(s,v) - A(t,v)\|_V^2 \leq (K\,|v|_{\mathcal{V}}^2 + \eta)\,|t-s|^{2\nu},$$

$$\sum_k |B_k(s,u) - B_k(t,u)|_V^2 \leq (K\,|u|_{\mathcal{V}}^2 + \eta)\,|t-s|^{2\nu}$$

almost surely for all $v \in \mathcal{V}$, $0 \leq s < t \leq T$.

Consistency (C2). For a sequence $\varepsilon_n \downarrow 0$

$$|\Pi_n A(t_j,u) - A_j^{n,\tau}(\Pi_n u)|_{V_n^*}^2 \leq c\Big(|u|_{\mathcal{V}}^2 + \xi_j^{n,\tau}\Big)\Big(\tau^{2\nu} + \varepsilon_n^2\Big),$$

$$\sum_k \Big|\Pi_n B_k(t_i,u) - B_{k,i}^{n,\tau}(\Pi_n u)\Big|_{H_n}^2 \leq c(|u|_{\mathcal{V}}^2 + \eta_i^{n,\tau})\Big(\tau^{2\nu} + \varepsilon_n^2\Big)$$

a.s. for $u \in \mathcal{V}$, $j = 1, \ldots m$, $i = 0, \cdots, m-1$, with random variables $\xi_j^{n,\tau} \geq 0$, $\eta_i^{n,\tau} \geq 0$,

$$\sup_{n,m} \sum_j \tau E \xi_j^{n,\tau} < \infty \qquad \sup_{n,m} \sum_i \tau E \eta_i^{n,\tau} < \infty.$$

Set $e_i^{n,\tau} = \Pi_n u(t_i) - u_i^{n,\tau}$.

Theorem 3. *Let* $(R1)-(R4)$, $(ST1)-(ST3)$ *and* $(C2)$ *hold. Then for* $\tau < 1/L$

$$E \max_{0\le i\le m} |e_i^{n,\tau}|^2_{H_n} + \sum_{1\le i\le m} \tau\, E\|e_i^{n,\tau}\|^2_{V_n} \le CE|e_0^{n,\tau}|^2_{H_n} + C(\tau^{2\nu} + \varepsilon_n^2)$$

with a constant C *independent of* n, m.

Proof: Fix n, τ, and to ease notation write $e_i := e_i^{n,\tau}$, $A_i := A_i^{n,\tau}$, $B_{k,i} := B_{k,i}^{n,\tau}$ $u_i := u_i^{n,\tau}$.

Then $|e_{i+1}|_n^2 - |e_i|_n^2$

$$= 2 \int_{t_i}^{t_{i+1}} \left\langle e_{i+1}\,, \Pi_n A(s,u(s)) - A_{i+1}(u_{i+1}) \right\rangle_n ds$$
$$+ \Big| \sum_k \int_{t_i}^{t_{i+1}} F_k(s)\, dW^k(s) \Big|_n^2 + 2 \int_{t_i}^{t_{i+1}} \Big(e_i\,,\, F_k(s)\Big)_n dW^k(s)$$
$$- \Big| \int_{t_i}^{t_{i+1}} \big[\Pi_n A(s,u(s)) - A_{i+1}(u_{i+1})\big]\, ds \Big|_n^2,$$

with $F_k(s) = \Pi_n B_k\big(s, u(s)\big) - B_{k,i}(u_i^{n,\tau})$ for s in $]t_i, t_{i+1}]$, $i = 0, 1, \cdots, m-1$.

Step 1. Show

$$\max_{0\le l\le m} E|e_l|_n^2 + E\sum_{1\le i\le m} \tau\|e_i\|_n^2 \le CE|e_0|_n^2 + C(\tau^{2\nu} + \varepsilon_n^2).$$

Step 2. Show

$$E\max_{1\le i\le m} |e_i|_n^2 \le CE|e_0|_n^2 + C(\tau^{2\nu} + \varepsilon_n^2)$$

by using Davis' inequality and Step 1. (See [GM3] for details). □

(b) Explicit scheme. Let $V_n \hookrightarrow H_n \hookrightarrow V_n^*$ with Π_n be approximations for $V \hookrightarrow H \hookrightarrow V^*$. Assume moreover that for each $n \geq 0$ as sets

$$V_n = H_n = V_n^*,$$

and there is a constant $\vartheta(n)$ such that

$$\|u\|_{V_n}^2 \leq \vartheta(n)\,|u|_{H_n}^2\,, \quad \forall u \in H_n.$$

Consider for each n and $i = 0, 1, \ldots, m-1$

$$\begin{aligned} u^n_{\tau,i+1} =& u^n_{\tau,i} + \tau\, A^{n,\tau}_i\big(u^n_{\tau,i}\big) \\ &+ B^{n,\tau}_{k,i}\big(u^n_{\tau,i}\big)\Big(W^k(t_{i+1}) - W^k(t_i)\Big), \qquad (E^n_\tau) \end{aligned}$$

for V_n-valued $\mathcal{F}_{t_i}$-measurable random variables $u^n_{\tau,i}$, $i = 1, \cdots, m$, where $u^n_{\tau,0}$ is a given V_n-valued $\mathcal{F}_0$-measurable r.v., and

$$A^{n,\tau}_i : \Omega \times V_n \to V^*_n \quad \text{and} \quad B^{n,\tau}_{k,i} : \Omega \times V_n \to H_n$$

are given $\mathcal{F}_{t_i} \otimes \mathcal{B}(V_n)$-measurable mappings satisfying (ST1)-(ST3).

Theorem. *Assume (ST1)-(ST3). Then for any V-valued $\mathcal{F}_0$-measurable $u^n_{\tau,0}$ such that $E\|u^n_{\tau,0}\|^2_{V_n} < \infty$, (E^n_τ) has a unique solution $(u^n_{\tau,i})^m_{i=1}$ such that $u^n_{\tau,i}$ is $\mathcal{F}_{t_i}$-measurable and $E\|u^n_{\tau,i}\|^2_{V_n} < \infty$ for all i, m and n.*

Proof: Use that $\|u\|^2_n \le \vartheta(n)\,|u|^2_n$, and hence by duality $|u|^2_n \le \vartheta(n)\,|u|^2_{n*}$, where $\|\cdot\|_n := \|\cdot\|_{V_n}$, $|\cdot|_n := |\cdot|_{H_n}$, $|\cdot|_{n*} := |\cdot|_{V^*_n}$.

Thus $\|u^n_{\tau,i+1}\|^2_n \le \vartheta(n)|u^n_{\tau,i+1}|^2_n$,

$$E|u^n_{\tau,i+1}|^2_n \le 3E|u^n_{\tau,i}|^2_n + 3\tau E|A^{n,\tau}_i(u^n_{\tau,i})|^2_n \\ + 3\tau\sum_k E|B^{n\tau}_i(u^n_{\tau,i})|^2_n$$

$$\le \Big(\vartheta(n)+\vartheta(n)\tau K+\tau K\Big)\,E\|u^n_{\tau,i}\|^2_n+3\tau\Big(\vartheta(n)+1\Big)M,$$

and induction on i finishes the proof. □

Set $e^n_{\tau,i} := \Pi_n u(t_i) - u^n_{\tau,i}$. Then we get the following generalization of a result from [Y].

Theorem 4. *Let (R1)-(R4), (ST1)-(ST3) and (C2) hold. Let* n *and* τ *satisfy*

$$L_1\tau\vartheta(n) + 2\sqrt{L_1 L_2 \tau\vartheta(n)} \le q$$

for some constant $q < \lambda$, *where* L_1 *and* L_2 *are the Lipschitz constants of* $A_i^{n,\tau}$ *and* $B_{k,i}^{n,\tau}$ *in (L1) and (L2) respectively. Then*

$$E \max_{0\le i\le m} |e^n_{\tau,i}|^2_{H_n} + \sum_{0\le i<m} \tau E\|e^n_{\tau,i}\|^2_{V_n}$$
$$\le CE|e^n_{\tau,0}|^2_{H_n} + C(\tau^{2\nu} + \varepsilon_n^2).$$

Proof: See [GM3].

Applications

Consider the linear SPDE from Example 2

$$du(t,x) = \Big(D_p(a^{pq}D_q u(t,x)) + f(t,x)\Big)\,dt$$

$$+\Big(b^p_k D_p u(t,x) + g_k(t,x)\Big)\,dW^k(t), \qquad u(0,x) = u_0(x),$$

for $t \in (0,T]$, $x \in \mathbb{R}^d$, where $p,q = 0,1,\ldots,d$,

$$a^{pq}, f : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R},$$

$$b^p = (b^p_k), g = (g_k) : \Omega \times [0,T] \times \mathbb{R}^d \to l_2(\mathbb{R}),$$

u_0 is L_2-valued $\mathcal{F}_0$-measurable.

Assume

(1) *Stochastic λ-parabolicity:* With $\lambda > 0$

$$\left(a^{ij} - \tfrac{1}{2} b^i_k b^j_k\right) z_i z_j \geq \lambda |z|^2 \quad \text{for all } z \in \mathbb{R}^d,\ t, \omega$$

(2) *Smoothness:* Let $l \geq 0$ be an integer.

Almost surely

- $|D^\alpha a^{pq}| \leq K$, $|D^\alpha b^p|_{l_2} \leq K$

  for $|\alpha| \leq l$, $p, q = 0, \ldots, d$.
- $|f|^2_l + \sum_k |g_k|^2_l \leq \eta$, with a r.v. $\eta \geq 0$, $E\eta < \infty$.
- $u_0 \in W^l_2$.

The implicit and the explicit approximations, $v^{h,\tau}$ and v^h_τ are given by the systems of equations defined for $i = 0, \cdots, m-1$ by

$$\begin{aligned} v^{h,\tau}_{i+1} =& v^{h,\tau}_i + \tau\Big(L_h(t_{i+1})v^{h,\tau}_{i+1} + f(t_{i+1})\Big) \\ & + \Big(M_{k,h}(t_i)v^{h,\tau}_i + g(t_i)\Big)\Big(W^k(t_{i+1}) - W^k(t_i)\Big), \\ v^{h,\tau}_0(z) =& u(0,z), \quad z \in \mathbb{G}, \end{aligned} \qquad (E^{h,\tau})$$

and

$$
\begin{aligned}
v^h_{\tau,i+1} =& v^h_{\tau,i} + \tau(L_h(t_i)v^h_{\tau,i} + f(t_i)) \\
&+ \sum_k \big(M_{k,h}(t_i)v^h_{\tau,i} + g(t_i)\big)\big(W^k(t_{i+1}) - W^k(t_i)\big), \\
v^h_{\tau,0}(z) =& u(0,z), \quad z \in \mathbb{G} \qquad\qquad (E^h_\tau)
\end{aligned}
$$

respectively, where $t_i = i\tau = iT/m$, and

$$
L_h(t)\varphi = \delta^-_p(a^{pq}(t)\delta^+_q\varphi), \quad M_{k,h}\varphi = b^p_k(t)\delta^+_p\varphi.
$$

Definition. A sequence $(v_i^{h,\tau})_{i=0}^m$ of $W_{h,2}^0(\mathbb{G})$-valued random variables $v_i^{h,\tau}$ is a solution to $(E^{h,\tau})$ if $v_i^{h,\tau}$ is $\mathcal{F}_{t_i}$-measurable for each i, and $(E^{h,\tau})$ holds (a.s.). A solution $(v_{\tau,i}^h)_{i=0}^m$ to (E_τ^h) is defined similarly.

From the existence and uniqueness result for the corresponding abstract schemes it is easy to deduce the existence of a unique solution to the above finite difference schemes.

To get an accuracy estimate for these schemes we ensure the regularity condition (R4) by the following assumption.

(3) Almost surely, for all $s, t \in [0, T]$

- $\sum_k |D^\gamma(b_k^p(t) - b_k^p(s))|^2 \leq K|t - s|$,
- $\sum_k |g_k(s) - g_k(t)|_l^2 \leq \eta\, |t - s|$
- $|D^\gamma(a^{pq}(t) - a^{pq}(s))| \leq K|t - s|^{\frac{1}{2}}$,
- $|f(t) - f(s)|_l^2 \leq \eta\, |t - s|$

for $|\gamma| \leq l$, $p, q = 0, ..., d$, $x \in \mathbb{R}^d$ with constant K and r. v. η, $E\eta < \infty$.

Let $r \geq 0$ be an integer such that

$$l > r + 2 + \frac{d}{2} \quad \text{for } l \text{ from Condition (2).} \qquad (\star)$$

Theorem 5. *Let* $(\star)$ *hold. Assume (1)-(3). Then for sufficiently small* τ

$$E \max_{1 \leq i \leq m} |v_i^{h,\tau} - u(t_i)|_{h,r}^2 + E \sum_{1 \leq i \leq m} \tau \, |v_i^{h,\tau} - u(t_i)|_{h,r+1}^2 \leq C(h^2 + \tau)$$

for all $h \in (0,1)$*, with constant* C *independent of* h *and* τ*.*

Proof: Note that $W^m_{h,2} = W^0_{h,2}$ as sets, and $|\varphi|_{h,m+1} \le \frac{\kappa}{h}|\varphi|_{h,m}$ for $\varphi \in W^0_{h,2}$, $m \ge 0$, $h > 0$, with a constant $\kappa = \kappa(m,d)$. Take $H_n = W^r_{h,2}$, $H = W^{l-2}_2$, $\mathcal{H} = W^l_2$ and the normal triples

$$V_n \hookrightarrow H_n \hookrightarrow V_n^*, \quad V \hookrightarrow H \hookrightarrow V^*,$$

$$\mathcal{V} \hookrightarrow \mathcal{H} \equiv \mathcal{H}^* \hookrightarrow \mathcal{V}^*,$$

with $V_n = W^{r+1}_{h,2}$, $V = W^{l-1}_2$, $\mathcal{V} = W^{l+1}_2(\mathbb{R}^d)$. Cast the equations into the abstract framework and verify the conditions of Thm 3. $\square$

Theorem 6. Let $(\star)$ and (1)-(3) hold. Then there is a constant $\rho > 0$ such that for h and τ satisfying

$$\frac{\tau}{h^2} \leq \rho \qquad (\star\star)$$

we have

$$E \max_{1\leq i\leq m} |v^h_{\tau,i} - u(t_i)|^2_{h,r} + E \sum_{i<m} \tau |v^h_{\tau,i} - u(t_i)|^2_{h,r+1} \leq C(h^2 + \tau)$$

for all $h \in (0, 1)$, where C is a constant independent of τ and h.

Proof: As in the proof of the previous theorem take $H_n = W^r_{h,2}$, $H = W^{l-2}_2$, $\mathcal{H} = W^l_2$ and the normal triples

$$V_n \hookrightarrow H_n \equiv H_n^* \hookrightarrow V_n^*, \quad V \hookrightarrow H \equiv H^* \hookrightarrow V^*,$$
$$\mathcal{V} \hookrightarrow \mathcal{H} \equiv \mathcal{H}^* \hookrightarrow \mathcal{V}^*$$

with $V_n = W^{r+1}_{h,2}$, $V = W^{l-1}_2$ and $\mathcal{V} = W^{l+1}_2$. Cast the finite difference scheme in the abstract scheme and use Thm 4. $\Box$

Corollary. *Let Assumptions (1)-(3) hold with an integer* $l > 2 + d$. *Then the following statements are valid:*

(i) For sufficiently small τ

$$E \max_{1 \le i \le m} \sup_{z \in \mathbb{G}} |v_i^{h,\tau}(z) - u(t_i, z)| \le C(h + \sqrt{\tau})$$

for all $h \in (0, 1)$, *with a constant* C *independent of* h *and* τ.

(ii) Assume also that τ *and* h *satisfy* $(\star\star)$.

Then

$$E \max_{1\le i\le m} \sup_{z\in\mathbb{G}} |(v^h_{\tau,i}(z) - u(t_i,z))| \le C(h+\sqrt{\tau})$$

for all $h \in (0,1)$, *with a constant* C *independent of* h *and* τ.

Proof: Note that for $r > d/2$ by the 'discrete' Sobolev's embedding for all $\varphi \in W^r_{h,2}$

$$\sup_{z\in\mathbb{G}} |\varphi(z)| \le C|\varphi|_{W^r_{h,2}}, \quad h \in (0,1)$$

with a constant $C = C(d)$, and use Thms 3-4.

□

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II. Splitting-up approximations

In the context of linear SPDEs the splitting-up method and our approach to it can be loosely described as follows.

Suppose we want to solve

$$du(t,x) = Lu(t,x)\,dt + M_k u(t,x) \circ dW_t^k, \qquad (E)$$

on $[0,T] \times \mathbb{R}^d$, with initial data $u_0 = u_0(x)$, where '$\circ$' is for the Stratonovich differential,

W^k are independent Wiener processes, L is a second order and M_k are first order differential operators. For simplicity of presentation let the coefficients of L and M_k be time independent.

Suppose we have effective methods for solving

$$dv(t,x) = Lv(t,x)\,dt, \qquad (E1)$$

$$d\tilde{v}(t,x) = M_k\tilde{v}(t,x) \circ dW_t^k \qquad (E2)$$

on small time intervals.

Then it is natural to set

$$T_n := \{t_i = i\delta : i = 0, 1, 2, ..., n\}, \quad \tau := T/n,$$

for an integer $n \geq 1$, and define the approximation $u_n(t)$ for $t \in T_n$ recursively by $u_n(0) = u_0$,

$$u_n(t_{i+1}, \cdot) = \mathbb{S}_\tau \mathbb{Q}_{t_i t_{i+1}} u_n(t_i, \cdot)$$

where $\{\mathbb{S}_t : t \geq 0\}$ and $\{\mathbb{Q}_{st} : 0 \leq s \leq t\}$ are the solution operators of equations (E1) and (E2), respectively.

In this way the approximation of (E) in each interval $[t_i, t_{i+1}]$ is split into two steps: solving the degenerate SPDE (E2) and taking its solution at time t_{i+1} as the initial value at time t_i while solving PDE (E1) again on $[t_i, t_{i+1}]$. We want to estimate the error $u - u_n$. The usual approach to this problem compares the solution operator of (E) at $t = t_j$ with

$$\mathbb{S}_\tau \mathbb{Q}_{t_{j-1}t_j} \cdots \mathbb{S}_\tau \mathbb{Q}_{t_1 t_2} \mathbb{S}_\tau \mathbb{Q}_{t_0 t_1},$$

and uses semigroup methods (See [BGR1], [BGR2], [FL], [MQ] and the references therein.)

The approach from [GK1]-[GK2] is based on looking at the splitting-up method differently: Instead of going back and forth in time we propose to 'stretch out the time', by the time scales $A_t(n)$, $B_t(n)$, defined by

$$A_t(n) := k\tau \quad \text{for} \quad t \in [2k\tau, (2k+1)\tau),$$
$$A_t(n) := t-(k+1)\tau \quad \text{for} \quad t \in [(2k+1)\tau, (2k+2)\tau),$$
$$B_t(n) := A_{t+\tau}(n), \qquad k = 0, 1, ...,$$

and to consider the equation

$$dv_n(t,x) = Lv_n(t,x)\,dA_t(n) + M_k v_n(t,x) \circ dW^k_{B_t(n)}. \qquad (1)$$

Obviously, $v_n(2t) = u_n(t)$ for $t \in T_n$. Instead of the solution u of (E) consider

$$\bar{u}_n := u(B_t(n), x),$$

that satisfies

$$d\bar{u}_n(t,x) = L\bar{u}_n(t,x)\,dB_t(n) + M_k \bar{u}_n(t,x) \circ dw^k_{B_t(n)}. \qquad (2)$$

Equations (1) and (2) suggest and make possible to use stochastic calculus to estimate

$$E \sup_{t \leq 2T} ||v_n(t) - \bar{u}_n(t)||^q,$$

in appropriate norms $\| \cdot \|$, that gives an estimate for

$$E \max_{t \in T_n} ||u_n(t) - u(t)||^q,$$

since, clearly,

$$E \max_{t \in T_n} ||u_n(t) - u(t)||^q \leq E \sup_{t \leq 2T} ||v_n(t) - \bar{u}_n(t)||^q.$$

One of the results of [GK1] says, see Thm 1 below, that under natural conditions for each $T > 0$ and $q > 0$ there is a constant N such

$$E \max_{t \in T_n} |u_n(t) - u(t)|_0^q \leq N/n^q \quad \text{for all } n \geq 1,$$

where $|\cdot|_0$ denotes the $L_2(\mathbb{R}^d)$ norm.

In the above explanation of our approach to the splitting-up method we use the Stratonovich differential in the equations above.

In fact in [GK1]-[GK2] more general equations then (E) are considered. In place of the Stratonovich diffferential

$$M_k u_k(t,x) \circ dW_t^k,$$

which is just a short notation for

$$\frac{1}{2} M_k M_k u(t,x)\,dt + M_k u(t,x)\,dW_t^k$$

with Itô's differential $M_k u_k(t,x)\,dW_t^k$, the more general term

$$L_0 u(t,x)\,dt + M_k u(t,x)\,dW_t^k$$

with a second order differential operator L_0 is considered. Correspondingly, in place of (E2)

$$d\tilde{v}(t,x) = L_0\tilde{v}(t,x)\,dt + M_k\tilde{v}(t,x)\,dw_t^k, \qquad (E2')$$

is considered and the stochastic parabolicity for this equation is assumed, which is satisfied in the special case $L_0 := \frac{1}{2}M_kM_k$ in equation (E2). Note that it is well-known that, in general, equation (E2') is not solvable if the stochastic parabolicity is not satisfied. In particular, it is not well-posed when $L_0 = 0$.

To formulate our result for *multistage* splittings and for equations with free tems, instead of (E) consider

$$du(t) = (Lu(t) + f)\,dt + (M_k u(t) + g_k)\,dW_t^k, \qquad (E')$$
$$u(0) = u_0,$$

where $L = \sum_{r=0}^{d_1} L_r, \quad f = \sum_{r=0}^{d_1} f_r,$ for some $d_1 \geq 1$, with Borel functions f_r, g_k on $\mathbb{R}^d$ and

$$L_r = a_r^{ij}(x) D_i D_j + a_r^i(x) D_i + a_r(x),$$
$$M_k = b_k^i(x) D_i + b_k(x), \quad x \in \mathbb{R}^d.$$

Assumption 1. Let $m \geq 0$ be an integer. The derivatives of a_r^{ij}, a_r^i, a_r, b_k^i, b_k, $r = 0, ..., d_1$, $i, j = 1, ..., d$, up to order $m+3$ are continuous,

$$|D^\alpha a_r^{ij}| + |D^\alpha a_r^i| + |D^\alpha a_r| \leq K,$$

$$\sum_k (|D^\alpha b_k^i|^2 + |D^\alpha b_k|^2) \leq K^2,$$

$$u_0 \in W_2^{m+3}, \quad f_r \in W_2^{m+3}, \quad g_k \in W_2^{m+4},$$

$$|u_0|_{m+3} \leq K, \quad |f_r|_{m+3} \leq K, \quad \sum_k |g_k|^2_{m+4} \leq K^2$$

for $|\alpha| \leq m+3$, with a constant K, where $|\cdot|_k$ denotes the norm in $W_2^k(\mathbb{R}^d)$.

To approximate equation (E') consider the splitting-up equations

$$dv(t) = (L_\gamma v(t) + f_\gamma)\, dt, \qquad (E\gamma)$$

$$d\tilde{v}(t) = (L_0 \tilde{v}(t) + f_0)\, dt + (M_k \tilde{v}(t) + g_k)\, dW_t^k, \qquad (E0)$$

for $\gamma = 1, \ldots, d$.

Assumption 2.(*Stochastic parabolicity*) The matrices

$$(a_\gamma^{ij}), \quad (a_0^{ij} - \tfrac{1}{2} b_k^i b_k^j) \qquad \gamma = 1, \ldots, d_1.$$

are positive semidefinite.

Notice that by Assumption 2 the matrix

$$\Big(\sum_{r=0}^{d_1} a_r^{ij} - \tfrac{1}{2} b_k^i b_k^j\Big)$$

is also positive semidefinite. Thus by the L_2-theory of SPDEs it is known that Assumptions 1 and 2 imply the existence of a unique solution $(u(t))_{t\geq 0}$ to (E'). Furthermore, for each $s \geq 0$ there is a unique solution to (Er), $r = 0, 1, ..., d_1$, on $[s, \infty)$, for any W_2^1-valued $\mathcal{F}_s$-measurable initial data given at s.

Define a multistage approximation $u^{(n)}$ for (E') on T_n recursively by $u^{(n)}(0) := u_0$,

$$u^{(n)}(t_{i+1}) := \mathbb{S}_\tau^{(d_1)} \ldots \mathbb{S}_\tau^{(2)} \mathbb{S}_\tau^{(1)} \mathbb{Q}_{t_i t_{i+1}} u^{(n)}(t_i),$$

$i = 0, \ldots, n-1$, where $\mathbb{S}_t^{(\gamma)}$ and $\mathbb{Q}_{st}$ denote the solution operators of $(E\gamma)$ and $(E0)$, respectively.

Theorem 1. *Assume Assumptions 1-2. Then for any real* $q > 0$

$$\varepsilon_n^q := E \max_{t \in T_n} |u^{(n)}(t) - u(t)|_m^q \leq C n^{-q} \quad \text{for all } n \geq 1$$

with a constant C *independent of* n.

Proof: Set $d' := d_1 + 1$, fix $n \geq 1$ and let $\tau := T/n$. According to our idea we change time by using the following function

$$\kappa(t) = \begin{cases} t - kd_1\tau & t \in [kd'\tau, (kd'+1)\tau], k = 0, 1, ..., \\ (k+1)\tau & t \in [(kd'+1)\tau, (k+1)d'\tau], k = 0, 1, ..., \\ 0 & t \leq 0. \end{cases}$$

Define

$$Y^k(t) = W^k_{\kappa(t)}, \quad \bar{\mathcal{F}}_t = \mathcal{F}_{\kappa(t)}, \quad V^0_{t,0} = V^0_{t,1} = \kappa(t),$$

$$V^r_{t,0} = \kappa(t), \quad V^r_{t,1} = \kappa(t - r\tau) \quad \text{for} \quad r = 1, 2, ..., d_1.$$

Consider for $\varepsilon = 0, 1$ the equations

$$du_\varepsilon(t) = (L_r u_\varepsilon(t) + f_r)\, dV^r_{t,\varepsilon} + (M_k u_\varepsilon(t) + g_k)\, dY^k_t$$

with $u_0(0, x) = u_1(0, x) = u_0(x)$, and note

$$u_0(d't) = u(t), \quad u_1(d't) = u^{(n)}(t) \quad \text{for } t \in T_n.$$

Note also that the martingale part of the driving processes are the same, Y, in both equations. Hence it is possible to show that for any $T' > 0$

$$\bar{\varepsilon}_n^q(T') := E \sup_{t \in [0,T']} |u_1(t) - u_0(t)|_m^q \leq N A^q(T'),$$

with constant N independent of n, where

$$A(T') := \sup_{\omega \in \Omega} \max_{t \in [0,T]} \max_r |V_{t,1}^r - V_{t,0}^r|.$$

(See Theorem 2.2 and its proof in [GK1]). Hence for $T' := d'T$

$$\varepsilon_n^q \le \bar{\varepsilon}_n^q(T') \le N \sup_{t\in[0,Td']} \sup_{r\le d_1} |\kappa(t+r\tau) - \kappa(t)|^q$$
$$= NT^q n^{-q}. \qquad \square$$

Remark. *For l from $\{1, ..., d_1\}$ define the approximation $u^{(n)}$ by splitting-up as follows:*
$u^{(n)}(t_{i+1}) = \mathbb{S}_\tau^{(d_1)}...\mathbb{S}_\tau^{(l+1)}\mathbb{Q}_{t_i t_{i+1}}\mathbb{S}_\tau^{(l)}...\mathbb{S}_\tau^{(1)}u^{(n)}(t_i)$.
Then one can easily see from its proof that Theorem 1 remains valid.

Let $C^l = C^l(\mathbb{R}^d)$ denote the Banach space of functions $f = f(x)$, $x \in \mathbb{R}^d$, having continuous derivatives up to the order l, such that $\|f\|_{C^l} := \sup_{x\in\mathbb{R}^d} \sum_{|\beta|\le l} |D^\beta f(x)| < \infty$. We get the following corollary from the previous theorem by Sobolev's theorem on embedding of $W_2^m(\mathbb{R}^d)$ into C^l.

Corollary 1. *Let $l \geq 0$ be an integer and let Assumptions 1-3 hold with $m > l + d/2$. Then*

$$E \max_{t \in T_n} ||u^{(n)}(t) - u(t)||_X^q \leq N n^{-q}$$

for all $n \geq 1$, where $X := C^l$, and $\|\cdot\|_X$ denotes the norm in X.

The next corollary can be obtained easily by a standard application of the Borel-Cantelli lemma.

Corollary 2. *Let Assumptions 1-2 hold. Then for each $\varepsilon > 0$ then there is a random variable ξ_ε, such that almost surely*

$$\max_{t \in T_n} ||u^{(n)}(t) - u(t)||_X \leq \xi_\varepsilon n^{-1+\varepsilon}$$

for all $n \geq 1$, where X is H^m, or we can also take $X := C^l$ if $m > l + d/2$.

Remark. *The above results can be extended to the case of equations with time dependent data (see [GK1]).*

Sharpness of the accuracy estimate

Here we show that, the rate $1/n$ is sharp for equation (E') with $d_1 = 1$.

First we study the case of deterministic PDEs.

Consider

$$\frac{\partial}{\partial t} u(t,x) = (L_0 + L_1) u(t,x) + f(x) \qquad (\bar{E})$$

with initial condition $u(0,x) = u_0(x)$, where $f : \mathbb{R}^d \to \mathbb{R}$ and $L_r = a_r^{ij} D_i D_j + a_r^i D_i + a_r$, with coefficients a_r^{ij}, a_r^i, a_r, depending *only* on $x \in \mathbb{R}^d$. We split this PDE into two:

$$\frac{\partial}{\partial t} u(t,x) = L_0 u(t,x) + f_0(x), \qquad (\bar{E}0)$$

$$\frac{\partial}{\partial t} u(t,x) = L_1 u(t,x) + f_1(x), \qquad (\bar{E}1)$$

such that $f = f_0 + f_1$.

We fix an integer $m \geq 0$ and make the following assumptions.

Assumption 1*. The matrices (a_r^{ij}) are non-negative definite for all $x \in \mathbb{R}^d$, $r = 0, 1$.

Assumption 2*. For an integer $m \geq 0$ the partial derivatives of the coefficients a_r^i, a_r up to order m and for a^{ij} up to order $2 \vee m$ exist, are continuous, and by magnitude are bounded by K.

Assumption 3*. u_0, f_0, f_1 are in W_2^m,

$$|u_0|_m + |f_0|_m + |f_1|_m \leq K.$$

Under Assumptions (1*)-(2*) with $m \geq 1$, equations ($\bar{E}$), ($\bar{E}$0) and ($\bar{E}$1), with initial data u_0 admit a unique solution.

Let u be the solution $(\bar{E})$, and denote the solution operator of $(\bar{E}r)$ by $\mathbb{S}_t^{(r)}$ for $r = 0, 1$. Let Assumptions 1*-3* hold with $m + 4$ in place of m. Then by Thm 1 the splitting-up approximations, $u^{(n)}(t_0) = u_0$,

$$u^{(n)}(t_{k+1}) = \mathbb{S}_\tau^{(1)}\mathbb{S}_\tau^{(0)}u^{(n)}(t_k), \quad k = 0, .., n-1$$

converge in W_2^m to the true solution u with rate at least $1/n$. The next theorem is a special case of a statement of Thm 6.1 in [GK2].

Theorem 2. *Let Assumptions 1*-3* hold with* $m+6$ *in place of* m. *Let* $L_1 u_0 + f_1 = 0 \quad (a.e.)$. *Then for* $0 \le r_1 \le r_2 \le T$

$$\lim_{n\to\infty} n \max_{t\in T_n \cap [r_1, r_2]} |u(t) - u^{(n)}(t)|_m = (T/2) \sup_{t\in[r_1,r_2]} |L_1 u(t) + f_1|_m,$$

In particular, if

$$L_1 u(t) + f_1 \neq 0 \quad \text{for some } t \in [0, T], \qquad (\bullet)$$

then $\lim_{n\to\infty} n \max_{t\in T_n} |u(t) - u^{(n)}(t)|_m > 0$.

Remark 1. *Assume* u_0 *is the* unique *solution of* $L_1 u_0 + f_1 = 0$ *in* W_2^{m+6}. *Then* $(\bullet)$ *holds if and only if* $u(t) \not\equiv u_0$ *on* $[0, T]$. *Note that the uniqueness holds if* $-a_1 \geq K$ *for a sufficiently big constant* K, *that can be ensured by changing* a_0 *such that* $a_0 + a_1$ *remains unchanged. In fact both* $-a_0$ *and* $-a_1$ *can be made as large as one likes by replacing* u *in equation* $(\bar{E})$ *with* $ue^{\lambda t}$, *where* λ *is an appropriate constant.*

Remark 2. *Assume* $L_1 u_0 + f_1 = 0$. *Then* $u(t) \not\equiv u_0$ *on* $[0, T]$ *if and only if* $L_0 u_0 + f_0 \neq 0$.

Proof: Indeed, if $L_0 u_0 + f_0 \neq 0$ and $u(t) \equiv u_0$ then $\partial u / \partial t = 0$ and $(L_0 + L_1) u_0 + f = 0$, which contradicts $L_0 u_0 + f_0 \neq 0$. On the other hand, if $L_0 u_0 + f_0 = 0$, then $v(t) := u_0$ obviously is a solution of $(\bar{E})$ with initial data u_0. By uniqueness $v = u$ and hence $u(t) \equiv u_0$. $\square$

Remark. By Remarks 1-2 if u_0 is a unique solution of $L_1 u_0 + f_1 = 0$ in W_2^{m+6}, then condition (•) is fulfilled if and only if $L_0 u_0 + f_0 \neq 0$.

Consider now SPDE (E') with $d_1 = 1$,

$$du(t,x) = (L_1 u(t,x) + f_1(x))\, dt$$

$$+(L_0 u(t,x) + f_0(x))\, dt + (M_k u(t,x) + g_k(t,x))\, dW_t^k,$$

where $L_r = a_r^{ij} D_{ij} + a_r^i D_i + a_r,\ M_k = b_k^i D_i + b_k.$ for $r = 0, 1,\ k = 1, \ldots.$

Assume that the coefficients a_r^{ij}, a_r^i, a_r, b_k^i, b_k and free terms f_r, g_k depend only on $x \in \mathbb{R}^d$.

Theorem 3. *Let Assumptions 1-2 hold with $m+3$ in place of m. Assume that u_0 is the unique (nonrandom) solution of $L_1 u_0 + f_1 = 0$ in W_2^{m+6} (see Remark 1). Assume also that $L_0 u_0 + f_0 \neq 0$. Define the splitting-up approximation $u^{(n)}$ for equation (E') as before. Then* $\liminf_{n \to \infty} n^q \max_{t \in T_n} E|u^{(n)}(t) - u(t)|_m^q > 0$.

Proof: Since by Jensen's inequality, Minkowski's inequality, and the triangle inequality

$$(E|u^{(n)}(t) - u(t)|_m^q)^{1/q} \geq E|u^{(n)}(t) - u(t)|_m$$

$$\geq |Eu^{(n)}(t) - Eu(t)|_m \geq \big|\,|Eu^{(n)}(t)|_m - |Eu(t)|_m\big|,$$

it suffices to prove that

$$\liminf_{n\to\infty} n \max_{t\in T_n} \big|\,|Eu^{(n)}(t)|_m - |Eu(t)|_m\,\big| > 0.$$

To this end notice that $\bar{u}(t) := Eu(t)$ is a generalized solution of the problem

$$d\bar{u}(t) = (L_1\bar{u}(t) + f_1)\,dt + (L_0\bar{u}(t) + f_0)\,dt,$$
$$\bar{u}(0) = u_0,$$

and $\bar{u}^{(n)} := Eu^{(n)}$ is the splitting-up approximation

$$\bar{u}^{(n)}(t_0) = u_0, \qquad \bar{u}^{(n)}(t_{k+1}) = \mathbb{S}_\tau^{(1)}\mathbb{S}_\tau^{(0)}\bar{u}^{(n)}(t_k)$$

for $\bar{u}$, where $\mathbb{S}_t^{(r)}$ is the solution operator of the equation

$$du(t) = (L_r u(t) + f_r)\, dt.$$

Hence the theorem follows from the previous theorem. $\square$

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III. Accelerated numerical schemes

In the previous part we have seen that the estimate, C/n for the accuracy of the splitting-up approximations for SPDEs is sharp.

Can one improve the method to get higher order approximations? To discuss this question consider the deterministic PDE

$$du(t) = (Lu(t) + f(t))\,, dt \qquad (E)$$

$$u(0) = u_0,$$

and the splitting-up equations

$$dv(t) = (L_r + f_r)\, dt \qquad (Er)$$

with solution operators $\mathbb{S}_t^{(r)}$, where

$$L = L_1 + \dots L_{d_1}, \qquad f = f_1 + \dots f_{d_1},$$

and, as before, u_0, f_r are from $W_2^m(\mathbb{R}^d)$, and

$$L_r = a_r^{ij} D_i D_j + a_r^i D_i + a_r, \qquad r = 1, ..., d_1,$$

are elliptic differential operators with sufficiently smooth and bounded coefficients.

It is known that there are splitting-up approximations which are more accurate. For example, the error of the symmetric splitting-up,

$$u_n(t_i) := (\mathbb{S}^{(1)}_{\tau/2}\mathbb{S}^{(2)}_{\tau/2}...\mathbb{S}^{(d_1)}_{\tau/2}\mathbb{S}^{(d_1)}_{\tau/2}...\mathbb{S}^{(2)}_{\tau/2}\mathbb{S}^{(1)}_{\tau/2})^i u_0, \qquad (Sy)$$

$$t_i \in T_n = \{t_i = i\tau : i = 0, ..., n\},\ \tau = T/n,$$

is proportional to τ^2.

This example suggests looking for compositions of splittings, often called *fractional step* approximations,

$$\prod_{i=1}^{m}\prod_{j=1}^{d_1}\mathbb{S}^{(j)}_{c^{ij}\tau},$$

with appropriate real numbers c^{ij} and integer $m \geq 1$, such that

$$u(\tau) - \prod_{i=1}^{m}\prod_{j=1}^{d_1}\mathbb{S}^{(j)}_{c^{ij}\tau}u_0, \qquad (FS)$$

the local error, is proportional to τ^{k+1} in appropriate norms. Such local error leads to a global error proportional to τ^k, i.e., such fractional step approximations represent methods of (at least) *order* k.

Remark 1. *Note that in fact the fractional step method (FS) is just a splitting-up method corresponding to the splitting*

$$L = \sum_{r=1}^{d'} L'_r, \quad f = \sum_{r=1}^{d'} f'_r,$$

$$L'_1 = c^{11} L_1,\ L'_2 = c^{12} L_2,\ ...,\ L'_{d_1} = c^{1d_1} L_r,$$
$$L'_{d_1+1} = c^{21} L_1, ...,\ L'_{d'} = c^{md_1} L_{d_1}$$
$$f'_1 = c^{11} L_1,\ f'_2 = c^{12} f_2,\ ...,\ f'_{d_1} = c^{1r} f_{d_1},$$
$$f'_{d_1+1} = c^{21} f_1, ...,\ f'_{d'} = c^{md_1} f_{d_1}, \quad d' = md_1,$$

where the summation convention with respect to the repeated indices is not in force.

The conditions on c^{ij} and m which lead to splitting methods of high order have been studied in the literature. Such methods are obtained for Hamiltonian systems of ODEs, and for some classes of linear and for nonlinear equations.

It is shown, however, that the numbers c^{ij} in each scheme of order $k \geq 3$ cannot be all nonnegative. Therefore these methods cannot be used to approximate the solution of partial differential equations of parabolic type.

Thus the natural question arises, if there exists for parabolic equations a different way from the multiplicative one to accelerate the convergence of splitting-up approximations to an order higher than 2.

Inspired by *Richardson's method* ([Ri], [RiGa]), it is proved in [GK3] that for any given integer $k \geq 0$ there exist absolute constants $b_0, b_1, ..., b_k$, expressed by simple formulas, such that the accuracy of the approximation

$$v_n := b_0 u_n + b_1 u_{2n} + b_2 u_{4n} + ... + b_k u_{2^k n}$$

is of order τ^{k+1}. Here $u_{2^j n}$ is the splitting-up approximation along the grid $T_{2^j n}$. This is Theorem 2 below. We obtain it by expanding the error $u - u_n$ of the splitting-up approximations in powers of $\tau = T/n$, and by choosing the coefficients $b_0, ..., b_k$ to cancel the terms with τ^j, $j \leq k$. To formulate this expansion let u_n be the splitting-up approximation

$$u_n(t_i) = (\mathbb{S}_\tau^{(d_1)} ... \mathbb{S}_\tau^{(2)} \mathbb{S}_\tau^{(1)})^i u_0, \quad t_i \in T_n.$$

Denote by W_p^m the Sobolev space defined as the closure of C_0^∞ functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ in the norm

$$\|\varphi\|_{m,p} := \Big(\sum_{|\gamma| \le m} \int_{\mathbb{R}^d} |D^\gamma \varphi(x)|^p \, dx \Big)^{1/p}.$$

Let $l \ge 0$ be an integer, $p \in [2, \infty)$, and make the following assumptions.

Assumption 1. The derivatives of a_r^{ij}, a_r^i and a_r up to order l are continuous on $\mathbb{R}^d$, for $|\gamma| \leq l$

$$|D^\gamma a_r^{ij}| + |D^\gamma a_r^i| + |D^\gamma a_r| \leq K, \quad i, j = 1, \ldots d,$$

and $u_0 \in W_p^l$, $f_r \in W_p^l$ for $r = 1, ..., d_1$, with a constant K.

Assumption 2. The matrices (a_r^{ij}) are positive semidefinite for $r = 1, \ldots, d_1$.

Definition. By a solution of (Er) with initial condition $v(0) = u_0$ we mean an W_p^1-valued weakly continuous function $v(t) = v(t, \cdot)$ defined on $[0, T]$ such that for all $\phi \in C_0(\mathbb{R}^d)$ and $t \in [0, T]$

$$(u(t,\cdot),\phi) = (u(0,\cdot),\phi) + \int_0^t [-(a^{ij} D_i u(s), D_j \phi)$$

$$+((a^i - D_j a^{ij}) D_i u(s) + a u(s) + f_r(s), \phi)]\, ds,$$

where (,) denotes the usual inner product in $L^2(\mathbb{R}^d)$.

It is known from PDE theory that if Assumptions 1-2 hold with $m \geq 1$ then equations (E) and (Er) have a unique solution u and v_r with initial value u_0.
Now we present a result on the expansion of the error $u - u_n$ from [GK3].

Theorem 1. *Let $m \geq 0$ and $k \geq 0$ be integers. Let Assumptions 1-2 above hold with*

$$l \geq 4 + m + 4k.$$

Then for $t \in T_n$ and $x \in \mathbb{R}^d$

$$u_n(t,x) = u(t,x) + \tau u^{(1)}(t,x)$$
$$+ \tau^2 u^{(2)}(t,x) + ... + \tau^k u^{(k)}(t,x) + R_n^{(k)}(t,x), \qquad (Ex)$$

with W_p^m-valued weakly continuous functions $u^{(1)}$,...,$u^{(k)}$ and $R_n^{(k)}$ on $[0,T]$, such that

$u^{(j)}$, $j = 1, 2, ..., k$, *are independent of* n, *and*

$$\sup_{t \in T_n} \|R_n^{(k)}(t)\|_{m,p} \leq N\tau^{k+1}$$

for all n, *where* $N = (m, k, p, d, d_1, K, T)$.

Proof: Using the idea explained in Part II we rearrange the splitting-up method in forward time by considering the equation

$$dw(t,x) = \sum_{r=1}^{d_1} (L_r w(t,x) + f_r)\, dA_r(t),$$
$$w(0,x) = u_0(x),$$

where A_r is absolutely continuous, $A_r(0) = 0$, $\dot{A}_r := dA_r/dt$ is periodic with period $d_1\tau$, and

$$\dot{A}_r(t) = 1_{[r-1,r]}(t/\tau), \quad t \in [0, d_1\tau] \quad \text{(a.e.)}$$
$$\text{for } r = 1, ..., d_1.$$

Instead of (E) consider

$$dv(t,x) = (Lv(t,x)+f)\, dA_0(t), \quad v(0,x) = u_0(x),$$

with $A_0(t) := t/d_1$. Clearly, $v(t) = u(A_0(t))$, and

$$v(d_1t) = u(t), \quad w(d_1t) = u_n(t) \quad \text{for all } t \in T_n.$$

Hence we get Thm 1 by proving that it holds with v and w in place of u, and u_n, respectively, for all $t = id_1\tau$, $i = 0, 1, ...n$. To finish the proof see Theorem 2.2 and its proof in [GK3].

□

Now from the above expansion we deduce a result on acceleration of the splitting-up method. Let V denote the square matrix defined by $V^{ij} := 2^{-(i-1)(j-1)}$, $i, j = 1, ..., k+1$.

Notice that the determinant of V is Vandermonde, generated by $1, 2^{-1}, ..., 2^{-k}$, and hence it is different from 0. Thus V is invertible. Set $(b_0, b_1, ..., b_k) := (1, 0, ..., 0)V^{-1}$, and define

$$v_n(t) := \sum_{j=0}^{k} b_j u_{2^j n}(t), \quad t \in T_n,$$

where $u_{2^j n}$ is the splitting-up approximation based on the grid

$$T_{2^j n} := \{iT/(2^j n) : i = 0, 1, .., 2^j n\}.$$

Theorem 2. *Under the conditions of the previous theorem*

$$\max_{t \in T_n} \|v_n(t) - u(t)\|_{m,p} \leq N\tau^{k+1},$$

with constant $N = N(m, k, p, d, d_1, K, T)$.

Proof: By the previous theorem

$$u_{2^j n} = u + \sum_{i=1}^{k} \frac{\tau^i}{2^{ji}} u^{(i)} + R_{2^j n}^{(k)}, \quad j = 0, 1, ..., k.$$

Hence for all $n \geq 1$

$$
\begin{aligned}
v_n =& (\sum_{j=0}^{k} b_j)u + \sum_{j=0}^{k} \sum_{i=1}^{k} b_j \frac{\tau^i}{2^{ij}} u^{(i)} + \sum_{j=0}^{k} b_j R_{2^j n}^{(k)} \\
=& u + \sum_{i=1}^{k} \tau^i u^{(i)} \sum_{j=0}^{k} \frac{b_j}{2^{ij}} + \sum_{j=0}^{k} b_j R_{2^j n}^{(k)} \\
=& u + \sum_{j=0}^{k} b_j R_{2^j n}^{(k)},
\end{aligned}
$$

since $\sum_{j=0}^{k} b_j = 1$ and $\sum_{j=0}^{k} b_j 2^{-ij} = 0$ for $i = 1, 2, ...k$ by the definition of $(b_0, ..., b_k)$.

Thus

$$\max_{t\in T_n}\|v_n(t)-u(t)\|_{m,p} = \max_{t\in T_n}\|\sum_{j=0}^{k} b_j R^{(k)}_{2^j n}(t)\|_{m,p}$$

$$\leq \sum_{j=0}^{k} |b_j| \max_{t\in T_n}\|R^{(k)}_{2^j n}(t)\|_{m,p} \leq N\tau^{k+1},$$

with a constant $N = (m, k, p, d, d_1, K, T)$.

Remark 2. *Assume that $u^{(1)} = 0$ in expansion (Ex) in Thm 1. This happens, e.g., for the symmetric splitting, that is a special case* *of our splitting-up scheme, as it is explained in Remark 1. In this case we need only take* k *terms in the linear combination to achieve accuracy of order* $k+1$. *Namely, we now define* $v_n(t)$ *by*

$$v_n(t) := \sum_{j=0}^{k-1} \lambda_j u_{2^j n}(t), \qquad t \in T_n,$$

where

$$(\lambda_0, \lambda_1, \ldots, \lambda_{k-1}) := (1, 0, \ldots, 0)V^{-1},$$

and V *is now a* $k \times k$ *matrix with entries* $V_{i1} := 1$, $V_{i,j} := 2^{-(i-1)j}$ *for* $i = 1, 2, \ldots, k$ *and* $j = 2, \ldots, k$. *Then Thm 2 remains valid, and can be proved in the same way as above. For example,*

$$v_n(t) := -\frac{1}{3}u_n(t) + \frac{4}{3}u_{2n}(t), \quad t \in T_n, \quad n = 1, \ldots$$

is of accuracy τ^3 *when* u_n, u_{2n} *are obtained by the symmetric splitting (Sy).*

We are interested now in accelerated schemes for other numerical methods. In [GK3] a class of numerical methods based on ‘time discretization’ is described for a variety of linear evolution equations, and accelerated numerical schemes are obtained by Richardson’s method.

To present some results from [GK3] let

$$V_0, V_1, V_2, ..., V_l$$

be a sequence of separable Banach spaces for a fixed integer $l \geq 1$ such that V_i is continuously embedded into V_{i-1}, for every $i = 1, 2, ..., l$, and V_1 is dense in V_0. Below we use the notation $\| \cdot \|_i := \| \cdot \|_{V_i}$ and K denotes a constant.

Consider the abstract Cauchy problem

$$dv(t) = (Lv(t) + f)\, dt, \quad v(0) = v_0 \in V_l, \quad \text{(Eq)}$$

on $[0,T]$, where $f \in V_l$, and L is a bounded linear operator from V_1 into V_0, such that newline $\|Lv\|_i \le K\|v\|_{i+1}$ for all $v \in V_{i+1}$, $i = 0,...,l$.

Definition. A V_1-valued weakly continuous function $v = \{v(t) : t \in [0,T]\}$ is called a solution of (Eq) if for all $t \in [0,T]$

$$v(t) = v_0 + \int_0^t (L(s)v(s) + f(s))\,ds,$$

where the integral is understood in Bochner's sense.

Here are two examples of operators and Banach spaces which fall into our scheme.

Example 1. (*Second order parabolic PDEs*)

$$L := a^{ij}(x)D_iD_j + a^i(x)D_i + a(x), \qquad (L)$$

with bounded, sufficiently smooth a^{ij}, a^i, a and positive semidefinite matrix (a^{ij}). Take $V_i = W_p^{\nu+2i}$ for some $\nu \geq 0$ and $i = 0, ..., l$.

Example 2. (*Parabolic system of PDEs*) The unknown function, $v = (v^1, ..., v^q)$, is a function of $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ with values in $\mathbb{R}^q$. The operator L has the form of (L) above but now the 'coefficients' a^{ij}, a^i, a are $q \times q$ matrices for every i, j and x. For every i and x the matrix a^i is symmetric, for every i, j and x

$$a^{ij} = diag(a^{ij1}, a^{ij2}, ..., a^{ijq})$$

is a diagonal matrix, and for each $l = 1, 2, ..., q$

$$a^{ijl} z^i z^j \geq 0 \quad \forall (z^1, z^2, ..., z^q) \in \mathbb{R}^q.$$

The coefficients are assumed to be sufficiently smooth functions of x with bounded derivatives. Take for $i = 0, ..., l$

$$V_i := (W_2^{\nu+2i})^q := W_2^{\nu+2i} \times ... \times W_2^{\nu+2i}$$

for a fixed $\nu \geq 0$. Notice that L can be completely degenerate, i.e., $a^{ij} \equiv 0 \in \mathbb{R}^{q\times q}$. In this case (Eq) is a *symmetric hyperbolic system* of first order PDEs.

To describe a class of approximation schemes for (Eq) let $H_r = H_r(t)$, $r = 1, ..., d_1$, be right-continuous functions on $\mathbb{R}$, with finite variation on every finite interval, $H_r(0) = 0$, and

$$H_r(t+1) - H_r(t) = H_r(1) = 1, \quad \forall t \in \mathbb{R}.$$

Let Θ_r, $r = 1, ..., d_1$ be bounded linear operators on $\mathbb{V}_1$, such that for $i = 1, ..., l$

$$\Theta : \mathbb{V}_i \to \mathbb{V}_i, \quad \sup_{t \leq T} \|\Theta_r \varphi(t)\|_i \leq K \sup_{t \leq T} \|\varphi(t)\|_i$$

for all $\varphi \in \mathbb{V}_i$, the space, with the sup norm, of V_i-valued weakly right-continuous functions on $[0, T]$ having weak left limit at each $t \in (0, T]$.

As an approximation for (Eq) consider the problem

$$dw(t) = \sum_{r=1}^{d_1} (L_r \Theta_r w(t) + f_r) \, dA_r(t), \quad t \in [0, T] \quad (N)$$
$$w(0) = v_0$$

for $\tau \in (0, 1]$, where for each $r = 1, ...d_1$

$$A_r(t) := \tau H_r(t/\tau),$$

$f_r \in V_l$, and L_r is a bounded linear operator from L_1 into L_0, such that

$$L = \sum_{r=1}^{d_1} L_r, \quad f = \sum_{r=1}^{d_1} f_r,$$

and

$$|L_r v|_i \leq K|v|_{i+1} \quad \text{for all } v \in V_{i+1},\ i = 0, ..., l.$$

Definition. A function $w \in \mathbb{V}_1$ is called a solution to (N) if for all $t \in [0, T]$

$$w(t) = v_0 + \int_0^t \sum_{r=1}^{d_1} (L_r \Theta_r w(s) + f_r)\, dA_r(s).$$

Let us now consider some examples of numerical methods represented by (N).

Let $d_1 = 1$, $L_1 = L$, $H_1(t) := \lfloor t \rfloor$. Then (N) is the *implicit Euler method* with step size τ. If $\Theta_1 = \Theta$ is the operator defined by

$$(\Theta\varphi)(t) := \vartheta\varphi(t-) + (1 - \vartheta)\varphi(t)$$

with some $\vartheta \in \mathbb{R}$, then (N) is the Θ-*method*, which is often called the *Crank-Nicolson* method when $\vartheta = 1/2$, and the *explicit Euler method* when $\vartheta = 1$. When $d_1 > 1$, then (N) represents various kind of *splitting-up methods* applied to (Eq) as explained in part II.

Assumption I. (Eq) has a unique solution v and for each $\tau \in (0,1]$ problem (N) has a unique solution w_τ. Moreover, v, $w_\tau \in \mathbb{V}_l$, and

$$\sup_{\tau \in (0,1]} \sup_{t \in [0,T]} \|w_\tau(t)\|_l < \infty.$$

Assumption II. For each $k = 0, 1, ..., d_1$ there is a bounded linear operator $\mathcal{R}_k : \mathbb{V}_0 \to \mathbb{V}_0$, such that for $k = 0, ..., d_1$, and $A_0(t) := t$

(i) $\sup_{t\in[0,T]} \|(\mathcal{R}_k g)(t)\|_j \le K \sup_{t\in[0,T]} \|g(t)\|_j$, $g \in \mathbb{V}_j$, $j = 0, 1, ..., l$;

(ii) (existence) for any $g \in \mathbb{V}_1$ the function $u = \mathcal{R}_k g$ satisfies

$$u(t) = \int_0^t Lu(s)\, ds + \int_0^t g(s)\, dA_k(s), \quad t \in [0, T];$$

(iii) (uniqueness) If $g_0, ..., g_{d_1} \in \mathbb{V}_0$ and $u \in \mathbb{V}_1$ satisfies

$$u(t) = \int_0^t Lu(s)\,ds + \sum_{k=0}^{d_1} \int_0^t g_k(s)\,dA_k(s), \qquad t \in [0,T],$$

then $u = \sum_{k=0}^{d_1} \mathcal{R}_k g_k$.

Now we formulate a result on expansion of $w_\tau(t)$ in τ valid for

$$t \in \mathbb{T}_\tau := \{i\tau : i = 0, 1, ...,\} \cap [0,T].$$

Theorem 3. *Let* $k \geq 0$ *be an integer. Let Assumptions I and II hold with* $l \geq 2(k+1)$*. Then there exist functions* $v^{(1)}, ..., v^{(k)} \in \mathbb{V}_0$*, independent from* τ*, and* $R_k(\tau) \in \mathbb{V}_0$*, such that for all* $\tau \in (0, 1]$

$$w_\tau(t) = v(t) + \tau v^{(1)}(t) + ... + \tau^k v^{(k)}(t) + R_k(\tau, t)$$

for all $t \in T_\tau$*, and* $\sup_{t \in \mathbb{T}_\tau} \|R_k(\tau, t)\|_0 \leq N\tau^{k+1}$*, where* N *is a constant, independent of* τ*.*

Hence, as before, we get a result on acceleration of the numerical schemes represented by (N).

Set $\bar{w}_\tau(t) := \sum_{j=0}^{k} \lambda_j w_{2^{-j}\tau}(t)$, with

$$(\lambda_0, ..., \lambda_k) := (1, 0, ..., 0)V^{-1},$$

where V is the $(k+1) \times (k+1)$ matrix defined by $V_{ij} := 2^{-(i-1)(j-1)}$.

Theorem 4. Under the conditions of the previous theorem for all $\tau \in (0, 1]$

$$\sup_{t \in \mathbb{T}_\tau} \|v(t) - \bar{w}_\tau(t)\|_0 \leq N\tau^{k+1}$$

with a constant N independent of τ.

A general framework for accelerated temporal discretization schemes is given in [GK5], that can also be adapted to a class of SPDEs.

Taylor expansions for SPDEs and higher order numerical schemes for SPDEs with additive noise are presented in [J1], [J2] and [JK]. For accelerated finite difference schemes in spatial variables for PDEs and SPDEs we refer to [GK6], and [GK7].

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