BSDEs and fractional smoothness

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Theorem 1.1 (Pardoux-Peng, 1990)

There exists a unique strong solution of the BSDE that consists of **adapted** *progressively measurable processes* (Y, Z) *with*

$$\sup_{t\in[0,T]} \|Y_t\|_{L_2}^2 + \int_0^1 \|Z_t\|_{L_2}^2 dt < \infty.$$

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  - Then there exist $u \in C^{0,1}([0,T) \times \mathbb{R})$ with

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  - Under some regularity: u is viscosity solution of semi-linear PDE

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + f\left(t, x, u, \frac{\partial u}{\partial x}\sigma\right) = 0 \quad \text{und} \quad u(1,x) = g(x).$$

  - Non-linear Feynman-Kac theory

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- Assumption: There exists $0 = r_0 < r_1 < \cdots < r_L = T$ (spectrum points) and a measurable and polynomially bounded $g : \mathbb{R}^L \to \mathbb{R}$ such that

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- Zhang (2005): On $[r_{l-1}, r_l)$ we can write

$$\begin{aligned} Y_t &= u_l(X_{r_1}, ..., X_{r_{l-1}}; t, X_t), \\ Z_t &= \partial_{x_l} u_l(X_{r_1}, ..., X_{r_{l-1}}; t, X_t), \end{aligned}$$

where $X_{r_1}, ..., X_{r_{l-1}}$ are considered as parameters and a jointly measurable

$$u_l : \mathbb{R}^{l-1} \times [r_{l-1}, r_l) \times \mathbb{R} \to \mathbb{R} \in C^{0,1}([r_{l-1}, r_l) \times \mathbb{R})$$

such that terminal condition in x_{l-1} on $[r_{l-2}, r_{l-1}]$ equals: $u_l(x_1, ..., x_{l-2}, x_{l-1}; r_{l-1}, x_{l-1})$.

Definition 3.1 (Fractional smoothness)

Let $\Theta = (\theta_1, ..., \theta_L) \in (0,1]^L$ and $p \in [2,\infty)$. Then we let $(\xi, f) \in B^{\Theta}_{p,\infty}(X)$ provided that there is some $c > 0$ such that

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Given a normed space E and $A \subseteq E$ we define $e_n(A|E) := \inf \varepsilon$, such that there are $x_1, ..., x_n \in E$ with

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Moreover, we set $e^E_{2,\infty}(A) := \sup_{n \ge 1} \sqrt{n} e_n(A|E)$.

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For $p \in [2, \infty)$, $\Theta \in (0,1)^L$, $\varphi_\Theta(t) = (r_l - t)^{\frac{\theta_l - 1}{2}}$ *on* $[r_{l-1}, r_l)$
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(C1) $\|Z_t - Z_s\|_p \le c_1 \left(\int_s^t (r_l - r)^{\theta_l - 2} dr \right)^{\frac{1}{2}}$ *for* $r_{l-1} \le s < t < r_l$.

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- $(C4) \Longrightarrow (C2)$ in Gobet and Makhlouf (2008).
- $(C4) \Longrightarrow$ upper bound of $\sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|Z_t - Z_{t_{k-1}}\|_2^2 dt$ in Gobet and Makhlouf (2008).

Definition 3.4 (Adapted splines)

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Recall that

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$$|\tau|_\Theta := \sup_{k=1,...,n} \int_{t_{k-1}}^{t_k} \varphi_\Theta(u)^2 du.$$

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- It would be desirable to have a condition only on ξ which works for all f, so that one would not need to solve the BSDE to describe the fractional smoothness.

4. A first sufficient condition for $B^{\Theta}_{p,\infty}(X)$

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$$W^{\theta}{}_t = \int_0^t \theta(s) dW_s + \int_0^t \sqrt{1 - \theta(s)^2} dB_s.$$

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- Measurable $\theta : [0, T] \mapsto [-1, 1]$ and

$$W^{\theta}{}_t = \int_0^t \theta(s) dW_s + \int_0^t \sqrt{1 - \theta(s)^2} dB_s.$$

- $(\mathcal{F}_t^{\theta})_{t \in [0,T]}$ augmentation of the natural filtration of W^{θ}.

4. A first sufficient condition for $B^{\Theta}_{p,\infty}(X)$

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$$W^{\theta}{}_t = \int_0^t \theta(s) dW_s + \int_0^t \sqrt{1 - \theta(s)^2} dB_s.$$

- $(\mathcal{F}^{\theta}_t)_{t \in [0,T]}$ augmentation of the natural filtration of W^{θ}.
- $X^{\theta}_t = x_0 + \int_0^t b(s, X^{\theta}_s) ds + \int_0^t \sigma(s, X^{\theta}_s) dW^{\theta}_s$.

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$$Y_t^\theta = \xi^\theta + \int_t^T f(s, X_s^\theta, Y_s^\theta, Z_s^\theta) ds - \int_t^T Z_s^\theta dW_s^\theta.$$

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Theorem 4.1

For $1 < p < \infty$ we have that

$$\left\| (Y^\theta - Y)_T^* \right\|_p + \left\| \left(\int_0^T |Z_t^\theta - Z_t|^2 dt \right)^{1/2} \right\|_p \leq c \left[\|\xi^\theta - \xi\|_p + \left(\int_0^T \theta(t)^2 dt \right)^{1/2} \right]$$

for a constant $c > 0$ depending on p, T, f, b, σ and ξ.

Now we fix $0 \leq A < B \leq T$, define the function

$$\theta_{A,B}(t) := \chi_{(A,B]}(t),$$

and write

$$(F(X_{s_1}, ..., X_{s_N}))^{A,B} := F(X_{s_1}^{\theta_{A,B}}, ..., X_{s_N}^{\theta_{A,B}}).$$

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Corollary 4.2

Let $\xi = g(X_{r_1}, ..., X_{r_L})$, $p \in [2, \infty)$, *and* $\Theta = (\theta_1, ..., \theta_L) \in (0, 1]^L$. *If there is a constant* $c > 0$ *such that one has that*

$$\|\xi - \xi^{t,r_l}\|_p \leq c(r_l - t)^{\frac{\theta_l}{2}}$$

for all $l = 1, ..., L$ *and* $r_{l-1} < t < r_l$, *then*

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Proof.

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$$\|Y_{r_l} - \mathbb{E}(Y_{r_l}|\mathcal{F}_t)\|_p \quad \leq \quad 2\|Y_{r_l} - \mathbb{E}_B Y_{r_l}^{t,r_l}\|_p$$

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Let $g_1, \ldots, g_L \in BV$ *and*

$$\xi = \Phi(g_1(X_{r_1}), \ldots, g_L(X_{r_L}))$$

where, for some $0 < \alpha \leq 1$, *the function* Φ *satisfies*

$$|\Phi(x_1, \ldots, x_L) - \Phi(y_1, \ldots, y_L)| \leq \kappa \left(|x_1 - y_1|^\alpha + \cdots + |x_L - y_L|^\alpha \right).$$

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Then one has that $(\xi, f) \in \bigcap_{0 < \theta < \frac{\alpha}{2p}} B_{p,\infty}^{(\theta,\ldots,\theta)}(X)$.

Proof. According to Avikainen (Finance and Stochastics 2009) one has

$$\mathbb{E}|g(X) - g(Y)|^p \leq c(p, q, g, X)\|X - Y\|_p^{\frac{q}{q+1}}$$

whenever $g \in BV$, $1 \leq p, q < \infty$, where X has a bounded density.

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Now we take a large q. □

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5. A second sufficient condition for $B^{\Theta}_{p,\infty}(X)$

Theorem 5.1

Assume polynomially bounded Borel functions $g_1, ..., g_L$ such that

$$\|g_l(X_{r_l}) - \mathbb{E}(g_l(X_{r_l})|\mathcal{F}_t)\|_p \leq c(r_l - t)^{\frac{\theta_l}{2}} \tag{1}$$

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Example 5.2 (Geiss-Geiss 2004)

Let $\mu_t = \mathrm{law}(X_t)$, $0 < \theta_l < 1$, and $p = 2$. Then

$$g_l \in (L_2(\mathbb{R}, \mu_{r_l}), D_{1,2}(\mathbb{R}, \mu_{r_l}))_{\theta_l,\infty} \Longrightarrow (1).$$

Given two Banach spaces $X_1 \subseteq X_0$ (continuous embedding), we let

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Let $0 < \theta < 1$, $g = \sum_{k=0}^{\infty} \alpha_k h_k$, where $(h_k)_{k=0}^{\infty} \subseteq L_2(\mathbb{R}, \gamma)$ is the ONB consisting of Hermite polynomials, then

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$$g \in B^{\theta}_{2,\infty}(\mathbb{R}, \gamma) \Longleftrightarrow \sum_{k=1}^{\infty} k t^{k-1} \alpha_k^2 \leq \frac{c^2}{(1-t)^{\theta}}.$$

6. What did we do?

Considered for the path-dependent BSDE

$$\begin{aligned} X_t &= x_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \\ Y_t &= g(X_{r_1}, ..., X_{r_L}) + \int_t^T f(r, X_r, Y_r, Z_r) dr - \int_t^T Z_s dW_s : \end{aligned}$$

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- Two sufficient conditions for $(\xi, f) \in B^{\Theta}_{p,\infty}(X)$.
- Link to real interpolation theory.

7. What is open?

For example:

- For what generators f is it true that

$$(\xi, 0) \in B^{\Theta}_{p,\infty}(X) \Longrightarrow (\xi, f) \in B^{\Theta}_{p,\infty}(X).$$

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- Limiting case $L \to \infty$, i.e. for example $\xi = (\sup_t X_t)$.
- K-closedness in the following situation: If $\mu = \mathrm{law}(X_T)$ and the diffusion X is regular enough, is it then true that

7. What is open?

For example:

- For what generators f is it true that

$$(\xi, 0) \in B^{\Theta}_{p,\infty}(X) \Longrightarrow (\xi, f) \in B^{\Theta}_{p,\infty}(X).$$

- Limiting case $L \to \infty$, i.e. for example $\xi = (\sup_t X_t)$.
- K-closedness in the following situation: If $\mu = \mathrm{law}(X_T)$ and the diffusion X is regular enough, is it then true that

$$(L_2(\mathbb{R}, \mu), D_{1,2}(\mathbb{R}, \mu))_{\theta,\infty} = (L_2, D_{1,2})_{\theta,\infty} \cap L_2(\mathbb{R}, \mu).$$

8. The Idea of the proof of Theorem 3.1

For $1 < p < \infty$ we have to show

$$\left\| (Y^\theta - Y)_T^* \right\|_p + \left\| \left(\int_0^T |Z_t^\theta - Z_t|^2 dt \right)^{1/2} \right\|_p \le c \left[\|\xi^\theta - \xi\|_p + \left(\int_0^T \theta(t)^2 dt \right)^{1/2} \right].$$

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Choose a C^1-function $\varphi : [-1,1] \mapsto [0,1]$ such that $\varphi(\theta) = 1$ for $|\theta| \le \frac{1}{3}$ and $\varphi(\theta) = 0$ for $|\theta| \ge \frac{2}{3}$. Get

$$\sup_{\theta \in [-1,1]} \left(\frac{\varphi(\theta)}{\sqrt{1-\theta^2}} + \frac{1-\varphi(\theta)}{|\theta|} \right) < \infty$$

using the convention $\frac{0}{0} = 0$.

Consider the generator

$$f^{\theta}(t,\omega,y,z) = f\left(t, X_t^{\theta}, z^W \frac{\varphi(\theta(t))}{\sqrt{1-\theta(t)^2}} + z^B \frac{1-\varphi(\theta(t))}{\theta(t)}\right)$$

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- Comparison of the two solutions.

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- Comparison of the two solutions.
- And some computations.

9. References

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