Solving the Zakai equation using cubature methods

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- The class of SPDEs
- How do we define an approximation ?
- A classical particle approximation
- The application of Lyons-Victoir cubature method
- Final remarks

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- D.C., Saadia Ghazali, *On the convergence rates of a general class of weak approximations of SDEs*, Stochastic differential equations: theory and applications, 221–248, Interdiscip. Math. Sci., 2, World Sci. Publ., Hackensack, NJ, 2007.
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$(\Omega, \mathcal{F}, P)$ probability space
$W = \left\{ \left(W_t^k \right)_{k=1}^m, t \geq 0 \right\}$ standard Brownian motion.
$\rho = \{\rho_t, t \geq 0\}$, $\mathcal{M}_F\left(\mathbb{R}^d\right)$-valued stochastic process.

The Zakai equation

$$d\rho_t(\varphi) = \rho_t(A\varphi)dt + \sum_{k=1}^m (\rho_t(\gamma_k \varphi) + \rho_t(B^k \varphi))dW_t^k. \tag{1}$$

where

$$\begin{aligned} A\varphi(x) &= \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\,\partial_i\partial_j\varphi(x) + \sum_{i=1}^d b_i(x)\,\partial_i\varphi(x) \\ B^k\varphi(x) &= \sum_{i=1}^d \alpha_{ik}(x)\,\partial_i\varphi(x) \quad k = 1, ..., m \\ \gamma_k &: \mathbb{R}^d \to \mathbb{R} \end{aligned}$$

Also define $\vartheta = \{\vartheta_t, t \geq 0\}$ $\mathcal{P}\left(\mathbb{R}^d\right)$-valued stochastic process.

$$\vartheta_t(\varphi) = \frac{\rho_t(\varphi)}{\rho_t(\mathbf{1})},$$

where $\rho_t(\mathbf{1}) = \int_{\mathbb{R}^d} 1\rho_t(dx) = \rho_t(\mathbb{R}^d)$.

Fujisaki-Kallianpur-Kunita, Kushner-Stratonovitch equation

$$d\vartheta_t(\varphi) = \vartheta_t(A\varphi)dt + \sum_{k=1}^{m} (\vartheta_t(\gamma_k\varphi) - \vartheta_t(\gamma_k)\vartheta_t(\varphi) + \vartheta_t(B^k\varphi))(dW_t^k - \vartheta_t(\gamma_k)dt) \quad (2)$$

$(\Omega, \mathcal{F}, \tilde{P}, (\mathcal{F}_t)_{t\geq 0})$ filtered probability space
$V = (V_t^i)_{i=1}^p, t \geq 0\}$, $U = \{(U_t^i)_{i=1}^m, t \geq 0\}$ $\mathcal{F}_t$-adapted independent Bm

$$X_t = X_0 + \int_0^t b(X_s)\,ds + \int_0^t \alpha(X_s)\,dU_s + \int_0^t \sigma(X_s)\,dV_s \quad (3)$$

$$W_t = \int_0^t \gamma(X_s)\,ds + U_t, \quad (4)$$

The filtering problem: Find the conditional law of the *signal* X_t given $\mathcal{W}_t = \sigma(W_s, s \in [0,t]) \subset \mathcal{F}_t$, i.e.,

$$\vartheta_t(\varphi) = \tilde{\mathbb{E}}[\varphi(X_t)|\mathcal{W}_t], \qquad t \geq 0, \qquad \varphi \in \mathcal{B}(\mathbb{R}^d).$$

Theorem

$\vartheta = \{\vartheta_t, t \geq 0\}$ is a solution of equation (2) (with $a = \sigma\sigma^\top + \alpha\alpha^\top$).

W becomes a Bm via a change of measure (Girsanov's theorem)

$$\left.\frac{dP}{d\tilde{P}}\right|_{\mathcal{F}_t} = Z_t, \quad t \geq 0,$$

where $Z = \{Z_t, t \geq 0\}$ is the exponential martingale

$$Z_t = \exp\left(-\int_0^t \sum_{k=1}^m \gamma_k(X_s)\, dU_s^k - \frac{1}{2}\int_0^t \sum_{k=1}^m \gamma_k(X_s)^2\, ds\right), t \geq 0.$$

Under P, W is now a Bm and X satisfies ($\tilde{b} = b - \alpha\gamma$):

$$X_t = X_0 + \int_0^t \tilde{b}(X_s)\, ds + \int_0^t \alpha(X_s)\, dW_s + \int_0^t \sigma(X_s)\, dV_s.$$

The Feynman-Kac formula

$$\rho_t(\varphi) = E\left[\varphi(X_t)\exp\left(\int_0^t \sum_{k=1}^m \gamma_k(X_s)\, dW_s^k - \frac{1}{2}\int_0^t \sum_{k=1}^m \gamma_k(X_s)^2\, ds\right)\middle| \mathcal{W}_t\right]$$

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The description of a numerical approximation for a stochastic PDE should contain three parts:

1. The subset of $\mathcal{P}(\mathbb{S})$ from where the approximation is chosen.
   E.g. for particle approximations

$$\rho_t^n = \sum_{j=1}^{n} \tilde{a}_j^n(t)\,\delta_{v_j^n(t)}. \qquad \vartheta_t^n = \sum_{j=1}^{n} \bar{a}_j^n(t)\,\delta_{v_j^n(t)}.$$

2. The law of evolution of the approximation.
   E.g. The algorithm

3. The choice in which the approximating error is measured.
   E.g. for particle approximations

$$\mathbb{E}\left[\left|\rho_t^n(\varphi) - \rho_t(\varphi)\right|\right]. \quad \varphi \in C_b(\mathbb{R}^d).$$

This description enables us to compare and analyse various methods.

The approximation will take the form

$$\rho_t^n = \sum_{j=1}^n \tilde{a}_j^n(t)\,\delta_{v_j^n(t)}. \qquad \vartheta_t^n = \sum_{j=1}^n \bar{a}_j^n(t)\,\delta_{v_j^n(t)}.$$

Partition the time interval $[0,\infty)$: $0<\delta<...<i\delta<(i+1)\delta<....$

Time 0.

- choose $v_j^n(0)$ i.i.d., $\mathcal{L}\left(v_j^n(0)\right)=\vartheta_0$,

$$\vartheta_t^n = \sum_{j=1}^n \frac{1}{n}\delta_{v_j^n(t)} \qquad \rho_t^n = \sum_{j=1}^n \frac{\rho_t(\mathbf{1})}{n}\delta_{v_j^n(t)}.$$

Time interval $[i\delta,(i+1)\delta]$.

- The particles move with the same law as X

$$v_j^n(t) = v_j^n(i\delta) + \int_{i\delta}^t \tilde{b}\left(v_j^n(s)\right)ds + \int_{i\delta}^t \alpha\left(v_j^n(t)\right)dW_t + \int_{i\delta}^t \sigma\left(v_j^n(s)\right)dV_s^j$$

V^j mutually independent, $\mathcal{F}_t$-adapted Brownian motions, independent of W and independent of all other random variables in the system.

The approximation will take the form

$$\rho_t^n = \sum_{j=1}^n \tilde{a}_j^n(t)\,\delta_{v_j^n(t)}. \qquad \vartheta_t^n = \sum_{j=1}^n \bar{a}_j^n(t)\,\delta_{v_j^n(t)}.$$

Partition the time interval $[0,\infty)$: $0 < \delta < \ldots < i\delta < (i+1)\,\delta < \ldots.$

Time 0.

- choose $v_j^n(0)$ i.i.d., $\mathcal{L}\left(v_j^n(0)\right) = \vartheta_0$,

$$\vartheta_t^n = \sum_{j=1}^n \frac{1}{n}\delta_{v_j^n(t)} \qquad \rho_t^n = \sum_{j=1}^n \frac{\rho_t(\mathbf{1})}{n}\delta_{v_j^n(t)}.$$

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$$v_j^n(t) = v_j^n(i\delta) + \int_{i\delta}^t \tilde{b}\left(v_j^n(s)\right)ds + \int_{i\delta}^t \alpha\left(v_j^n(t)\right)dW_t + \int_{i\delta}^t \sigma\left(v_j^n(s)\right)dV_s^j$$

V^j mutually independent, $\mathcal{F}_t$-adapted Brownian motions, independent of W and independent of all other random variables in the system.

- The weights $a_j^n(t)$, respectively $\bar{a}_j^n(t)$ are of the form

$$
\begin{aligned}
a_j^n(t) &= \exp\left(\sum_{k=1}^m \int_{i\delta}^t \gamma^k\left(v_j^n(s)\right) dW_s^k - \frac{1}{2}\sum_{k=1}^m \int_{i\delta}^t \left(\gamma^k\left(v_j^n(s)\right)\right)^2 ds\right) \\
\bar{a}_j^n(t) &= \frac{a_j^n(t)}{\sum_{j=1}^n a_j^n(t)}
\end{aligned}
$$

- At time $(i+1)\delta$, each particle branches into a random number of particles. The offsprings inherit the space position of its parent. All the (unnormalized) weights $a_j^n((i+1)\delta)$ are reinitialized back to 1.
- $o_{\bar{j}}^{n,(i+1)\delta}$, the number of offspring produced by the $\bar{j}$ particle at time $(i+1)\delta$ is $\mathcal{F}_{(i+1)\delta}$-adapted, $\mathbb{E}[o_{\bar{j}}^{n,(i+1)\delta}] = n\bar{a}_{\bar{j}}^{n,(i+1)\delta}$ and

$$
o_{\bar{j}}^{n,(i+1)\delta} = \begin{cases} \left[n\bar{a}_{\bar{j}}^{n,(i+1)\delta}\right] & \text{with prob. } 1 - \left\{n\bar{a}_{\bar{j}}^{n,(i+1)\delta}\right\} \\ \left[n\bar{a}_{\bar{j}}^{n,(i+1)\delta}\right] + 1 & \text{with prob. } \left\{n\bar{a}_{\bar{j}}^{n,(i+1)\delta}\right\} \end{cases},
$$

where $\bar{a}_{\bar{j}}^{n,(i+1)\delta} = \bar{a}_{\bar{j}}^n((i+1)\delta-) = \lim_{t\to(i+1)\delta, t<(i+1)\delta} \bar{a}_{\bar{j}}^n(t)$.

Remarks

- Since we keep the number of particle constant, the random variables $o_{\bar{j}}^{n,(i+1)\delta}$, $\bar{j} = 1, ..., n$ are correlated.
- The branching correction procedure is introduced to cull the unlikely particles and multiply those situated in the right areas. However the branching procedure introduces randomness into the system, by replacing the corresponding weights with random number of offsprings.
- The offsprings' distribution has to be chosen with great care to minimize this effect. The random number of offsprings should have minimal variance. That is, as the mean number of offsprings is pre-determined, we should choose the $o_{\bar{j}}^{n}$s to have the smallest possible variance amongst all *integer* valued random variables with the given mean $n\bar{a}_{\bar{j}}^{n}$. Solution: use the Tree Based Branching Algorithm of *Crisan and Lyons* [2002] or the genetic algorithm of *Whitley* [1993] and *Carpenter, Clifford and Fearnhead* [1999].

The algorithm that produces $o_{\bar{j}}^{n,(i+1)\delta}$, $\bar{j} = 1, ..., n$.

$u_{\bar{j}}^{n,(i+1)\delta}$ $\bar{j} = 1, ..., n-1$ i.i.d., uniformly distributed on $[0,1]$.

$g = n; \; h = n; \; \bar{j} = 1;$

do while $\bar{j} < n$

if $\left\{ n\bar{a}_{\bar{j}}^{n} \right\} + \left\{ g - n\bar{a}_{\bar{j}}^{n} \right\} < 1$ *then*

if $u_{\bar{j}}^{n} < 1 - \frac{\left\{ n\bar{a}_{\bar{j}}^{n} \right\}}{\{g\}}$ *then*

$o_{\bar{j}}^{n} = \left[n\bar{a}_{\bar{j}}^{n} \right]$

else

$o_{\bar{j}}^{n} = \left[n\bar{a}_{\bar{j}}^{n} \right] + (h - [g])$

end if

else

if $u_{\bar{j}}^{n} < 1 - \frac{1 - \left\{ n\bar{a}_{\bar{j}}^{n} \right\}}{1 - \{g\}}$ *then*

$o_{\bar{j}}^{n} = \left[n\bar{a}_{\bar{j}}^{n} \right] + 1$

else

$o_{\bar{j}}^{n} = \left[n\bar{a}_{\bar{j}}^{n} \right] + (h - [g])$

end if

$g = g - n\bar{a}_{\bar{j}}^{n}$

$h = h - o_{\bar{j}}^{n}$

$\bar{j} = \bar{j} + 1$

end do

$o_{\bar{j}}^{n} = h$

Define

$$\vartheta_t^n = \sum_{j=1}^n \bar{a}_j^n(t)\,\delta_{v_j^n(t)} \qquad \rho_t^n = \sum_{j=1}^n \xi^n(t)\bar{a}_j^n(t)\,\delta_{v_j^n(t)},$$

where $\xi^n = \{\xi^n(t), t \geq 0\}$ is defined as $\xi_0^n = \rho_0(\mathbf{1})$ and

$$\xi_t^n = \xi_{i\delta}^n \left(\frac{1}{n}\sum_{j=1}^n a_j^n(t)\right), \quad \text{for } t \in [i\delta, (i+1)\,\delta).$$

Theorem

Assume that $\sigma_{ij}, b_i, \alpha_{ik}, \gamma_k$ are bounded and Lipschitz. Then there exists a constant c independent of n such that for any $\varphi \in C_b\left(\mathbb{R}^d\right)$

$$\mathbb{E}\left[\left(\rho_t^n(\varphi) - \rho_t(\varphi)\right)^2\right] \leq \frac{c}{n}\,||\varphi||_\infty^2 .$$

$$\mathbb{E}\left[\left|\vartheta_t^n(\varphi) - \vartheta_t(\varphi)\right|\right] \leq \frac{c}{\sqrt{n}}\,||\varphi||_\infty .$$

Assume $\alpha = 0$. Recall **The Feynman-Kac formula**

$$\begin{aligned} \rho_t(\varphi) &= \mathbb{E}\left[\varphi(X_t) Z_t(X, W) \middle| \mathcal{W}_t\right], \\ Z_t(X, W) &= \exp\left(\int_0^t \sum_{k=1}^m \gamma_k(X_s)\, dW_s^k - \frac{1}{2}\int_0^t \sum_{k=1}^m \gamma_k(X_s)^2\, ds\right) \end{aligned}$$

where $X = \{X_t, t \geq 0\}$ is independent of W

$$X_t = X_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dV_s.$$

Remarks:

- $\rho_t(\varphi)$ is the expected value of a functional of X which depends on the driving noise W
- to approximate numerically $\rho_t(\varphi)$ we need to
  - approximate the functional with a version, say $\tilde{\Lambda}_t$, that depends only on a finite set of values for W.
  - approximate the law of the process X
  - integrate $\tilde{\Lambda}_t$ with respect to the approximation of the law of X.
- the approximation of the solution of the SPDE can be viewed as a problem of weak approximation of the forward diffusion.
- we use the Lyons-Victoir cubature method to approximate the law of X.

Consider equidistant partition $\{\frac{iT}{n}\}_{i=0}^{n}$ and w_i be the noise increments

$$\{w_i := W_{\frac{it}{n}} - W_{\frac{(i-1)t}{n}}\}_{i=1}^{n},$$

Let g_i be the noise dependent functions $g_i = \sum_{j=1}^{m}(\gamma^j w_i^j - \frac{t}{2n}(\gamma^j)^2)$.

Define the operators

$$\begin{aligned} R_s^n\varphi(x) &= P_s\varphi(x) = \mathbb{E}[\varphi(X_s(x))] \\ R_s^i\varphi(x) &= P_s(\varphi\exp(g_i))(x) = \mathbb{E}[\varphi(X_s(x))\exp(g_i(X_s(x)))] \end{aligned}$$

for i=0,..,n-1. Let ρ_t^n be the approximate measure

$$\begin{aligned} \rho_t^n(\varphi) &= \mathbb{E}[\varphi(X_t)Z_t^n(X,W)|\mathcal{W}_t] = \mathbb{E}\left[R_{\frac{t}{n}}^0 R_{\frac{t}{n}}^1 \ldots R_{\frac{t}{n}}^n \varphi(X_0)\middle|\mathcal{W}_t\right] \\ Z_t^n(X,W) &= \exp(\sum_{i=1}^{n} g_i(X_{\frac{(i-1)t}{n}})) \end{aligned}$$

Finally define $\vartheta_t^n(\varphi)$ by the formula $\vartheta_t^n(\varphi) = \frac{\rho_t^n(\varphi)}{\rho_t^n(\mathbf{1})}$.

M. All moments of X_0 are finite. The functions b, σ are Lipschitz.
FLp. $\mathbb{E}\left[Z_t(X,W)^p\right] < \infty$ and $\sup_n \mathbb{E}\left[Z_t^n(X,W)^p\right] < \infty$ for some $p > 2$.

Condition **FLp** holds true if γ is bounded. If γ is unbounded, but it has linear growth, then the condition is satisfied if X has exponential moments uniformly bounded on $[0,t]$.

Theorem

*Assume that conditions **M** and **FLp** hold true and γ_k, $k = 1,...,m$ are Lipschitz. Then, if φ has polynomial growth, there exists a constant $c = c(\varphi, t)$ independent of n such that*

$$\mathbb{E}\left[\left|\rho_t^n\varphi - \rho_t\varphi\right|^2\right] \leq \frac{c}{n}.$$

Moreover, if $\sup_n \mathbb{E}[(\vartheta_t^n(\varphi))^2] < \infty$, *then*

$$\mathbb{E}\left[\left|\vartheta_t^n\varphi - \vartheta_t\varphi\right|\right] \leq \frac{c}{\sqrt{n}},$$

where, again, $c = c(\varphi, t)$ is a constant independent of n.

AP. Let $(P_s)_{s\geq 0}$ be the semigroup associated to the Markov process X. We will assume that, for any Lipschitz continuous function $\psi : \mathbb{R}^d \to \mathbb{R}$, $P_s\psi$ is twice differentiable for any $s \in [0,t]$. Moreover, if

$$P_{a,b}\psi \triangleq P_a\psi - P_b\psi, a, b \in [0,t],$$

we will assume that there exists a constant $c_7 = c_7(t)$ independent of a and b such that

$$\sup_{x\in\mathbb{R}^d} |P_{a,b}\psi(x)| \leq ck_\psi\left(\sqrt{a} - \sqrt{b}\right) \quad (5)$$

$$\sup_{x\in\mathbb{R}^d} |\partial_i P_{a,b}\psi| \leq \frac{c}{b}k_\psi(a-b), \quad i = 1, ..., d, \quad (6)$$

where k_ψ is the Lipschitz constant of ψ.

Inequalities (5) and (6) are satisfied if, for example, $f, \sigma = \left(\sigma^i\right)_{i=1}^d \in C_b^\infty(\mathbb{R}^d)$ and the vector fields $\left(\sigma^i\right)_{i=1}^d$ satisfy the Hörmander condition.

Theorem

*Assume that conditions **M, FLp** and **AP** hold true. Assume also that the functions φ and γ_i $i = 1,..,m$ are Lipschitz. Then there exists $N > 0$ such that for all $n > N$ and $\varepsilon \in (0,1)$, there exists a constant $c = c(\varphi, t, N, \varepsilon)$ independent of the partition such that*

$$\mathbb{E}\left[\left|\rho_t^n \varphi - \rho_t \varphi\right|^2\right] \leq \frac{c}{n^{2-\varepsilon}}.$$

Moreover, if $\sup_n \mathbb{E}[(\vartheta_t^n(\varphi))^2] < \infty$, then for all $n > N$ and $\varepsilon \in (0,1)$, there exists a constant $c = c(\varphi, t, N, \varepsilon)$ independent of the partition such that

$$\mathbb{E}\left[\left|\vartheta_t^n \varphi - \vartheta_t \varphi\right|\right] \leq \frac{c}{n^{1-\varepsilon}}.$$

Theorem

Assume that conditions ***M, FLp*** *are satisfied and that the functions* $\gamma_i \in C_b^2(\mathbb{R}^d)$ *for* $i = 1, .., m$. *Then, if* φ *has polynomial growth, there exists a constant* $c = c(\varphi, t)$ *independent of* n *such that*

$$\mathbb{E}\left[\left|\rho_t^n \varphi - \rho_t \varphi\right|^2\right] \leq \frac{c}{n^2}. \tag{7}$$

Moreover, if $\sup_n \mathbb{E}[(\vartheta_t^n(\varphi))^2] < \infty$,

$$\mathbb{E}\left[\left|\vartheta_t^n \varphi - \vartheta_t \varphi\right|\right] \leq \frac{c}{n},$$

where, again, $c = c(\varphi, t)$ *is a constant independent of* n.

The theorem recovers a result of Picard [1984]. However, Picard's result is proved under the condition

$$\mathbb{E}\left[\exp\left(1+\varepsilon\right)tH^2\left(X\right)\right] < \infty, \tag{8}$$

where

$$H^2\left(M\right) = \sup\left\{\sum_{i=1}^m \left|\gamma_i^2\left(x\right)\right|,\ \left|x\right| \leq M\right\}$$

and $X^* = (X_i^*)_{i=1,\ldots,m}$ is the random vector with entries $X_i^* = \sup_{s\in[0,t]}\left|X_s^i\right|$. We have, for $p > 2$

$$\mathbb{E}\left[Z_t(X,W)^p\right] = \mathbb{E}\left[\exp\left(\frac{p^2-p}{2}\sum_{i=1}^m\int_0^t \gamma_i^2\left(X_s\right)ds\right)\right] \leq \mathbb{E}\left[\exp\left(\frac{p^2-p}{2}tH^2\left(X\right)\right)\right]$$

and by choosing $p = \dfrac{1+\sqrt{9+8\varepsilon}}{2} > 2$ so that $p^2 - p = 2\left(1+\varepsilon\right)$, we get that $\mathbb{E}\left[Z_t^p\right] < \infty$. Similarly

$$\mathbb{E}[(Z_t(X,W)^n)^p] < \mathbb{E}\left[\exp\left(1+\varepsilon\right)tH^2\left(X\right)\right]$$

for any n, so (8) implies condition **FLp**.

This is not enough ! A second step is required as the operators R_i, are not explicitly computable.

Recall that that X solves SDE (written here in Stratonovitch form)

$$X_t^x = x + \int_0^t \tilde{b}\,(X_s^x)\,ds + \int_0^t \sigma\,(X_s^x) \circ dV_s.$$

Replace the Wiener measure on the path space $C_{\mathbb{R}^d}[0,t]$ with a new measure, called cubature measure, denoted by $\mathbb{Q}_t^m$. The measure $\mathbb{Q}_t^m$ is given by i.e $\mathbb{Q}_t^m := \sum_{i=1}^N \lambda_i \delta_{\omega_i}$ where $\omega_i \in C_{\mathbb{R}^d}[0,t]$ are bounded variation paths.

Cubature of order 3: We use $N = 2^d$ paths with equal weights $\lambda_j = \frac{1}{2^d}$ defined as

$$\omega_t^j = t(z_j^i, \ldots, z_j^d),$$

where $z_j^i, \ldots, z_j^d \in \{-1, 1\}$.

Introduce the operators

$$\begin{aligned} \bar{R}_s^n \varphi\,(x) &= \mathbb{Q}^m[\varphi(X_s^x)] \\ \bar{R}_s^i \varphi\,(x) &= \mathbb{Q}^m[\varphi(X_s^x)\exp\,(g_i(\,X_s^x\,))] \end{aligned}$$

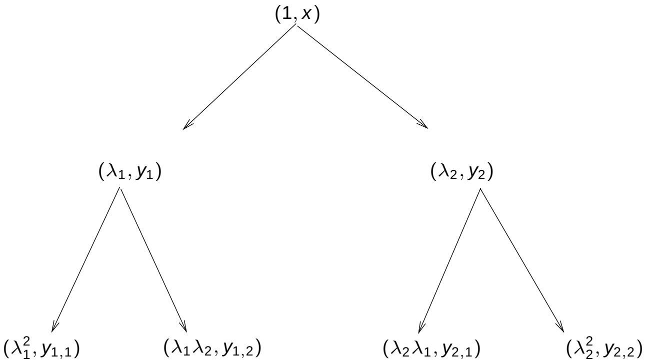
In other words, $\mathbb{Q}^m[\varphi(X_s^x)] = \sum_{i=1}^{N} \lambda_i \varphi(X_t^{j,x})$ where $X^{j,x}$ is the solution of the following ODE

$$X_t^{j,x} = x + \int_0^t b\left(X_s^{j,x}\right) ds + \int_0^t \sigma\left(X_s^{j,x}\right) d\omega_s^i \quad i = 1, .., N$$

Recall that $\rho_t^n(\varphi) = \mathbb{E}\left[R^0_{\frac{t}{n}} R^1_{\frac{t}{n}} ... R^n_{\frac{t}{n}} \varphi(X_0) \middle| \mathcal{W}_t\right]$ with corresponding approximation

$$\begin{aligned} \bar{\rho}_t^n(\varphi) &= \mathbb{E}\left[\bar{R}^0_{\frac{t}{n}} \bar{R}^1_{\frac{t}{n}} ... \bar{R}^n_{\frac{t}{n}} \varphi(X_0) \middle| \mathcal{W}_t\right] \\ \bar{\vartheta}_t^n &= \frac{\bar{\rho}_t^n}{\bar{\rho}_t^n(1)} \end{aligned}$$

Cubature with two paths and two time steps



where

$$
\begin{aligned}
y_i &= X^{i,x}_{\frac{1}{n}} = x + \int_0^{\frac{1}{n}} b\left(X^{i,x}_s\right) ds + \int_0^{\frac{1}{n}} \sigma\left(X^{i,x}_s\right) d\omega^i_s, \quad i = 1, 2 \\
y_{i,j} &= X^{i,j,x}_{\frac{1}{n}} = y_i + \int_0^{\frac{1}{n}} b\left(X^{i,j,x}_s\right) ds + \int_0^{\frac{1}{n}} \sigma\left(X^{i,j,x}_s\right) d\omega^i_s \quad i, j = 1, 2.
\end{aligned}
$$

Theorem

If the operators $\bar{R}^i$ are defined using a cubature measure $\mathbb{Q}^m$ of order m, then for all $\varphi \in C_b^{m+2}(\mathbb{R}^d)$ and $p \geq 1$,

$$||\rho_t^n(\varphi) - \bar{\rho}_t^n(\varphi)||_p \leq \frac{c}{n^{-(m-1)/2}} \sum_{i=1}^{m+2} ||\nabla^i \varphi||_\infty .$$

where $c = c(t, m, p)$ is independent of n. Moreover

$$\left|\left|\vartheta_t(\varphi) - \bar{\vartheta}_t^n(\varphi)\right|\right|_p \leq \frac{c}{n},$$

where $c = c(t, m, p, \varphi)$ is independent of n.

- The cubature method is essentially *deterministic*. It uses a set of ODEs to approximate the distribution of the solution of the SDE (2^d for the cubature of degree 3).
- Their application to the solution of the filtering problem leads to a deterministic algorithm as long as the law of ξ_0 is replaced by a deterministic approximation.
- The (exponentially) increase in the computational effort can be controlled by one of the following methods:
  - naive Monte-Carlo method
  - the Tree Based Branching Algorithm of Crisan and Lyons (see Ninomiya [2004])
  - the recombination scheme of Schmeiser, Sorreff and Teichmann
  - the recombining algorithm of Lyons and Litterer