Local Shape of Random Invariant Manifolds

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Consider here:

- equation of Burgers type
- multiplicative noise
- deterministic fixed point 0
- local random invariant manifolds using a cut-off
- shape of the manifold near 0

Contents

- SPDE of Burgers type
- RDS – Random Dynamical Systems
- RIM – Random Invariant Manifolds
- Main results on LRIM (local RIM)
- Some ideas of proofs

An Equation of Burgers type

Equation of Burgers type

$$du = [Lu + \nu u + B(u,u)]dt + \sigma u \circ d\omega \qquad \text{(B)}$$

- L - non positive differential operator on Hilbert-space H
  Exp. $\partial_x^2 + 1$ on $[0, \pi]$ Dirichlet b.c.
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Known: (B) defines a Random Dynamical System on H.

Wiener space and Shift

Consider the probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P})$, where

$$\Omega_0 = \{\omega \in C^0(\mathbb{R}, \mathbb{R}) \ : \ \omega(0) = 0\} \ .$$

Define the Shift $\theta_\tau : \Omega_0 \to \Omega_0$

$$\theta_\tau \omega(t) = \omega(t - \tau) - \omega(\tau) \ ,$$

which is measure preserving/ergodic with respect to the two-sided Wiener measure $\mathbb{P}$ on Ω_0.

Random Dynamical System (RDS)

[L. Arnold, Crauel, Schmalfuß, Flandoli, Scheutzow, Chueshov, Duan, Caraballo, Kloeden, Robinson,....]

A RDS on H over θ_t on $(\Omega_0, \mathcal{F}_0, \mathbb{P})$ is a measurable map

$$\begin{aligned} \varphi : \mathbb{R}^+ \times \Omega_0 \times H &\to H \\ (t, \omega, u) &\mapsto \varphi(t, \omega)u \end{aligned}$$

with the cocycle property

$$\varphi(0, \omega) = Id, \qquad \varphi(t, \theta_\tau \omega)\varphi(\tau, \omega) = \varphi(t + \tau, \omega)$$

for all $t, \tau \in \mathbb{R}^+$ and $\omega \in \Omega_0$.

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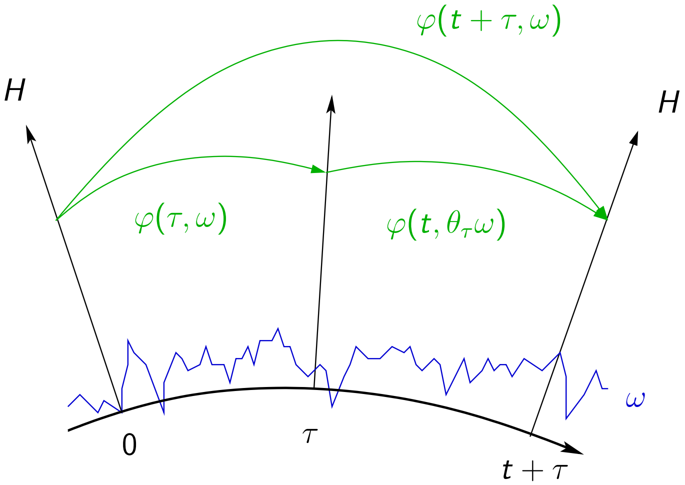
Remark: Usually, $\varphi(t, \omega)u$ is continuous in t and in u.

Cocycle Property $\varphi(t,\theta_\tau\omega)\varphi(\tau,\omega) = \varphi(t+\tau,\omega)$



Ornstein-Uhlenbeck process

Definition (OU-process)

Define

$$z(t) = z(\theta_t \omega)$$

on Ω_0 , where

$$z(\omega) = \sigma \int_{-\infty}^{0} e^s \omega(s) ds \, .$$

$t \mapsto z(\theta_t \omega)$ is continuous and solves

$$dz = -z dt + \sigma d\omega .$$

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Remark: $z(t)$ is a stationary OU-process on the Wiener space.

Transformation

Theorem

The solution to (B) generates a RDS.

Using the standard transformation

$$v(t) = e^{-z(t)}u(t)$$

Equation (B) becomes:

$$\partial_t v = -Lv + zv + \nu v + e^z B(v,v),$$

The solution defines a RDS, which by the transformation defines the RDS φ for (B).

Random Invariant Manifold (RIM)

Definition (Random Invariant Manifold)

A random set $M(\omega)$ is positive invariant for the RDS φ, if

$$\varphi(t,\omega)M(\omega) \subset M(\theta_t\omega), \quad \text{for } t \geq 0.$$

If

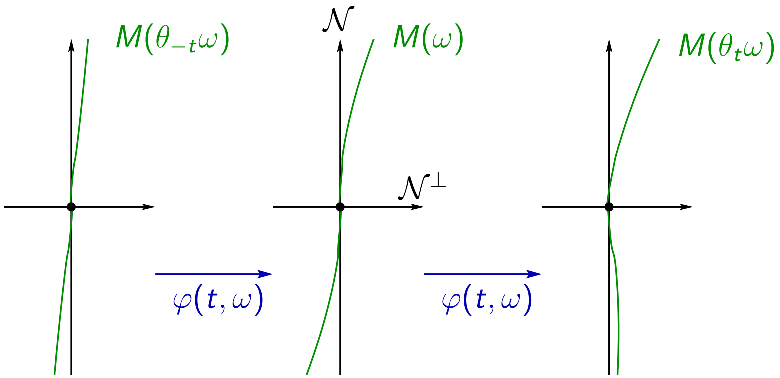
$$M(\omega) = \{u + \psi(\omega, u) | u \in \mathcal{N}\}$$

is the graph of a random Lipschitz mapping

$$\psi(\omega, \cdot) : \mathcal{N} \to \mathcal{N}^\perp \ ,$$

then $M(\omega)$ is called a Lipschitz invariant manifold (RIM).

RIM are moving in time!



Lipschitz Condition

Theorem

If the nonlinearity is globally Lipschitz with sufficiently small Lipschitz-constant, then there exists a RIM.
The RIM is pull-back attracting.

See for example:
[Duan,Lu, Schmalfuß '03, '04]
[Duan, Wang '07]
[Mohammed, Zhang, Zhao, 08]

Local Random Invariant Manifold (LRIM)

compare [Lu, Schmalfuß, 07]

A random set $M^R(\omega)$ is a LRIM with radius $R > 0$ of (B), if it is graph of a random function $\psi(\omega,\cdot) : \mathcal{N} \to \mathcal{N}^\perp$ such that for all bounded sets $B \subset B_R(0) \subset H$

$$\varphi(t,\omega)[M^R(\omega) \cap B] \subset M^R(\theta_t\omega)$$

for all $t \in [0, \tau_0(\omega))$ with

$$\tau_0(\omega) = \inf\{t \geq 0 : \varphi(t,\omega)[M^R(\omega) \cap B] \not\subset B_R(0)\}.$$

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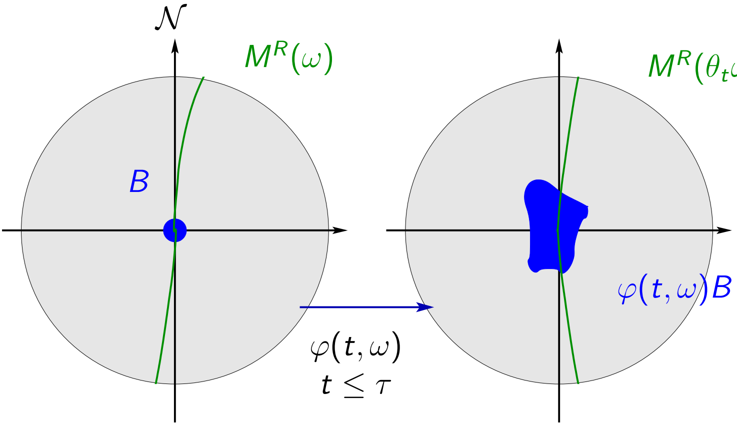
Key Idea

Take a cut-off at radius $R > 0$ for (B) such that the nonlinearity is Lipschitz with small constant.

A Sketch of a LRIM



Existence

Theorem (DB, Wang '09)

The RDS φ defined by (B) has a LRIM $M^R(\omega)$ for sufficiently small $R > 0$.
It is given as the graph of a random Lipschitz map defined by $h(\omega, \cdot) : \mathcal{N} \to \mathcal{N}^\perp$:

$$M^R(\omega) = \Big\{ \Big(\xi, e^{z(\omega)} h(\omega, e^{-z(\omega)} \xi) \Big) \in B_R(0) \ : \ \xi \in \mathcal{N} \Big\} \ .$$

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Remark: The LRIM is locally exponentially attracting in the pullback sense. (compare [Duan, Wang '07] for RIM)

Local Shape

Denote by P_s the projection onto $\mathcal{N}^\perp$
and let $B_s = P_s B$ and $L_s = P_s L$.

Theorem (DB, Wang '09)

Suppose $\sigma > 0$, $|\nu| < \sigma$ and $R \leq 1$, and let h be the LRIM given by the previous theorem.
Then

$$\|e^{z(\omega)} h(\omega, e^{-z(\omega)}\xi) - L_s^{-1} B_s(\xi, \xi)\| \leq C(\|\xi\| + R^2 + \sqrt{\sigma}) \cdot \|\xi\|^2$$

holds for all $\|\xi\| \leq \frac{1}{2} R$
with probability larger than $1 - C \exp\{-1/\sqrt{\sigma}\}$.

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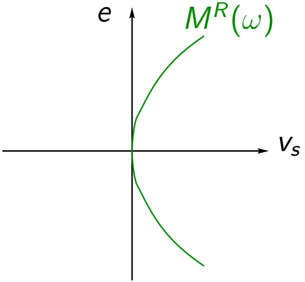
Remark: This result will not apply to deterministic equations, as we always assume $|\nu| < \sigma$.

Example

Suppose $\mathcal{N} = \text{span}\{e\}$ and fix $\xi = \alpha \cdot e \in \mathcal{N}$.
The LRIM $\mathcal{M}^R(\omega)$ is given (with high probability) as the graph of

$$L_s^{-1} B_s(\xi, \xi) = \alpha^2 L_s^{-1} B_s(e, e) =: \alpha^2 v_s \perp \mathcal{N}.$$



Cut Off

compare [Caraballo, Langa, Robinson '01], [Lu, Schmalfuß, 07]

Let $\chi : H \to \mathbb{R}$ be a bounded smooth function such that $\chi(u) = 1$ if $\|u\| \leq 1$ and $\chi(u) = 0$ if $\|u\| \geq 2$.
$\forall\ R > 0$ define

$$\chi_R(u) = \chi(u/R) \text{ for all } u \in H$$

$$B^{(R)}(u) = \chi_R(u) B(u,u).$$

Now $B^{(R)}$ is globally Lipschitz-continuous with constant

$$\text{Lip}(B^{(R)}) = C_B C_\chi R \to 0 \text{ for } R \to 0\,.$$

Consider the following cut-off equation

$$du = [Lu + \nu u + B^{(R)}(u)]dt + \sigma u \circ d\omega\,, \qquad u(0) = u_0\,.$$

Cut off

Transformation $v = ue^{-z}$ yields

$$\partial_t v = Lv + zv + \nu v + e^{-z} B^{(R)}(e^z v)\,, \qquad v(0) = u_0 e^{-z(0)}\,.$$

In order to obtain a RIM for the RDS $\varphi^R(t,\omega)$ of the cut-off equation, we consider the RIM of the transformed equation above.

We use the Ljapunov-Perron Method.

The fixed point space

Let $-\lambda_* < 0$ be the largest non-zero eigenvalue of L.
For $-\nu < \eta < \lambda_* - \nu$ define the Banach space

$$C_\eta^- = \left\{ v \in C((-\infty, 0], H) : \|v\|_{C_\eta^-} < \infty \right\}$$

with norm

$$\|v\|_{C_\eta^-} = \sup_{t \leq 0} \left\{ e^{\eta t - \int_0^t z(\tau) d\tau} \|v(t)\| \right\} < \infty.$$

Ljapunov-Perron Operator

Define the nonlinear operator $\mathcal{T}$ on C_η^- by

$$\mathcal{T}(v,\xi)(t) = e^{\nu t+\int_0^t z(s)ds}\xi + \int_0^t e^{\nu(t-\tau)+\int_\tau^t z(r)dr} e^{-z(\tau)} B_c^{(R)}\left(v(\tau)e^{z(\tau)}\right) d\tau + \int_{-\infty}^t e^{(-L_s+\nu)(t-\tau)+\int_\tau^t z(r)dr} e^{-z(\tau)} B_s^{(R)}\left(\left(v(\tau)e^{z(\tau)}\right) d\tau\right.$$

with $v \in C_\eta^-$, $\xi \in \mathcal{N}$, $\omega \in \Omega_0$, and $t \leq 0$.

Fixed Point

The operator $\mathcal{T}$ has a unique fixed point $v^* = v^*(\omega, \xi) \in C_\eta^-$.
One can check that this defines a RIM for the transformed cut-off equation.

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Define $h(\omega, \xi) = P_s v^*(0, \omega; \xi)$.

Thus, by Definition of $\mathcal{T}$

$$h(\omega, \xi) = \int_{-\infty}^0 e^{(L_s - \nu)\tau + \int_\tau^0 z(r)dr} e^{-z(\tau)} P_s B^{(R)}(v^*(\tau, \xi) e^{z(\tau)}) d\tau.$$

This allows for estimates on h, where estimates on v^* in C_η^- are necessary.

Manifold

$$\mathcal{M}^R_{cut}(\omega) = \{(\xi, e^{z(\omega)} h(\omega, e^{-z(\omega)}\xi)) : \xi \in \mathcal{N}\}$$

is a RIM for the RDS φ^R of the cut-off equation.
Define a Lipschitz mapping ψ by

$$\begin{array}{rcl} \psi(\omega,\cdot) : \mathcal{N} & \to & \mathcal{N}^\perp \,, \\ \xi & \mapsto & \psi(\omega,\xi) = e^{z(\omega)} h(\omega, e^{-z(\omega)}\xi) \,. \end{array}$$

Now

$$\mathcal{M}^R(\omega) = \mathcal{M}^R_{cut}(\omega) \cap B_R(0)$$

defines a LRIM of the RDS $\varphi(t,\omega)$.

Conclusions

Results:

- Existence of LRIM using cut-off
- LRIM locally a parabola
- flow on the manifold (amplitude equations)

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To do:

- Relation of RIM of (B) to LRIM?
- Is M^R a RIM in a small neighboorhood of 0?
- Formulation of LRIM without cut-off?