

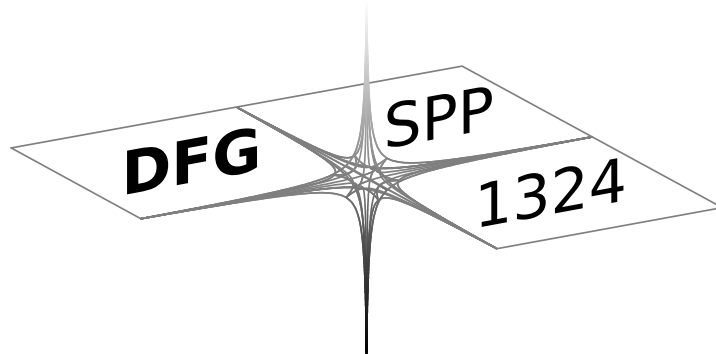
DFG-Schwerpunktprogramm 1324

„Extraktion quantifizierbarer Information aus komplexen Systemen“

Weak convergence of finite element approximations of linear stochastic evolution equations with additive Lévy noise

M. Kovács, F. Lindner, R. L. Schilling

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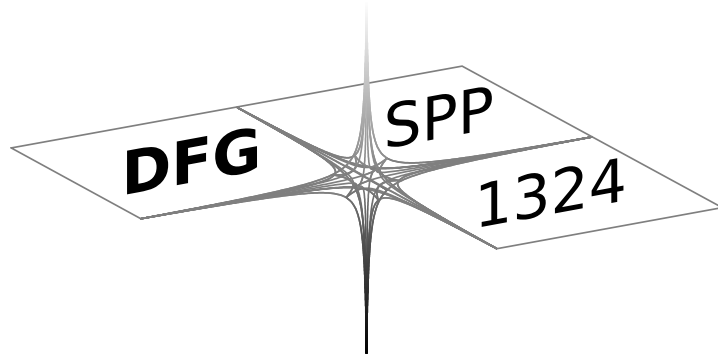
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Weak convergence of finite element approximations of linear stochastic evolution equations with additive Lévy noise *

Mihály Kovács, Felix Lindner and René L. Schilling

Abstract

We present an abstract framework to study weak convergence of numerical approximations of linear stochastic partial differential equations driven by additive Lévy noise. We first derive a representation formula for the error which we then apply to study space-time discretizations of the stochastic heat and wave equations. We use the standard continuous finite element method as spatial discretization and the backward Euler method and I -stable rational approximations to the exponential function, respectively, as time-stepping for the heat and wave equations. For twice continuously differentiable bounded test functions with bounded first and second derivatives, with some additional condition on the second derivative for the wave equation, the weak rate of convergence is found to be twice the strong rate. The results extend earlier work by two of the authors as we consider general square-integrable infinite-dimensional Lévy processes with no additional assumptions on the jump intensity measure. Furthermore, the present framework is applicable to hyperbolic equations as well.

Keywords: Stochastic partial differential equation, infinite-dimensional Lévy process, cylindrical Lévy process, Poisson random measure, finite elements, error estimate, weak convergence, backward Kolmogorov equation

MSC 2010: 60H15, 65M60; secondary: 60H35, 60G51, 65C30, 35R60

1 Introduction

Let H be a real separable Hilbert space and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a stochastic basis satisfying the usual conditions, $L = (L(t))_{t \geq 0}$ be a square-integrable cylindrical Lévy process in a real separable Hilbert space U with respect to the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$,

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taking values in a possibly larger Hilbert space $U_1 \supset U$, and $B : U \rightarrow H$ is a bounded linear operator. Consider an H -valued stochastic convolution process

$$X(t) = E(t)X_0 + \int_0^t E(t-s)B \, dL(s) \quad (1.1)$$

where $(E(t))_{t \in [0, T]}$ is a family of bounded linear operators on H and X_0 is an \mathcal{F}_0 -measurable H -valued random variable. Without loss of generality, all Hilbert spaces are assumed to be infinite-dimensional. Important examples of such processes are weak solutions $(X(t))_{t \geq 0}$ of certain stochastic partial differential equations (SPDEs, for short) driven by additive Lévy noise; these can be written as abstract Itô stochastic differential equations

$$dX(t) + AX(t) \, dt = B \, dL(t), \quad t \geq 0; \quad X(0) = X_0, \quad (1.2)$$

where $-A$ is the generator of a strongly continuous semigroup $(E(t))_{t \geq 0}$ on H . In particular, we consider the stochastic heat equation

$$dX(t) + \Lambda X(t) \, dt = dL(t), \quad t \geq 0; \quad X(0) = X_0, \quad (1.3)$$

and the stochastic wave equation, written as a first order system,

$$\begin{aligned} dX_1(t) - X_2(t) \, dt &= 0, & t \geq 0; & \quad X_1(0) = X_{0,1}, \\ dX_2(t) + \Lambda X_1(t) \, dt &= dL(t), & t \geq 0; & \quad X_2(0) = X_{0,2}. \end{aligned} \quad (1.4)$$

In both cases $\Lambda := -\Delta = -\sum_{j=1}^d \partial^2 / \partial \xi_j^2$ is the Laplace operator on $L^2(\mathcal{O})$ with domain $D(\Lambda) := H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$ where $\mathcal{O} \subset \mathbb{R}^d$ is a sufficiently nice bounded domain. For the precise abstract setup of these equations we refer to Sections 4 and 5. In general, however, we do not require that $(E(t))_{t \geq 0}$ enjoys the semigroup property so that the abstract framework can accommodate Volterra type evolution equations as well.

Consider an approximation $\tilde{X} = (\tilde{X}(t))_{t \in [0, T]}$ of the process $(X(t))_{t \in [0, T]}$ given by

$$\tilde{X}(t) = \tilde{E}(t)X_0 + \int_0^t \tilde{E}(t-s)B \, dL(s), \quad (1.5)$$

where $(\tilde{E}(t))_{t \in [0, T]}$ is a family of bounded linear operators on H , which is again not necessarily (extendable to) an operator semigroup. For example, the family $(\tilde{E}(t))_{t \in [0, T]}$ may be a time-interpolated solution operator family of a space-time discretized stochastic evolution problem, when H is an L^2 -space of some spatial domain \mathcal{O} . We study the so-called weak error

$$e(T) := \mathbb{E}(G(\tilde{X}(T)) - G(X(T))) \quad (1.6)$$

for suitable test functions $G : H \rightarrow \mathbb{R}$. At the heart of the paper are the error representation formulae for $e(T)$, Theorem 3.3 and Corollary 3.5. The proof of Theorem 3.3 is based on Kolmogorov's backward equation for the martingale $Y(t) = E(T)X_0 + \int_0^t E(T-s)B \, dL(s)$, $t \in [0, T]$, which has the important property that $Y(T) = X(T)$. The introduction of such an auxiliary process Y is well-known for equations with Gaussian noise and has

been used by many authors in a weak error analysis, see, for example [12, 18, 19, 21] to mention just a few (compare also [9, 10]). However, the extension of those arguments is not straightforward and the resulting error representation formula differs from the one in the Gaussian case in [19]. One of the difficulties in the general Lévy case (in contrast to the Gaussian case) is that there are no readily available, sufficiently general results on Kolmogorov's backward equation to suit our analysis. We remedy this, at least for Y as above, in Proposition 3.6. Another complication arises from the fact that we use tools from the theory of stochastic integration based on two different settings. One, where we integrate operator valued processes w.r.t. a Hilbert space valued Lévy process, promoted in the monographs [32, Chapter 8], [28, 29], and another one where we integrate Hilbert space-valued integrands w.r.t. a Poisson random measure [26, 34]. The problem occurs because our setting for stochastic differential equations is based on the first approach while the proof of the error representation formula is based on an Itô formula which appears in [26, Theorem 3.6]; the latter form is well suited for our purposes, but it is formulated using the second approach for stochastic integration. Therefore, in the appendix we link the two stochastic integrals so that we can use the results from both theories.

Using the abstract error representation we study the weak error of a space-time discretization for the stochastic heat and wave equations. As space discretization we employ a standard continuous finite element method. For the stochastic heat equation we use a backward Euler method and for the stochastic wave equation an I -stable rational approximation of the exponential function, such as the Crank-Nicolson scheme, as time integrators. For both equations, the Hilbert-Schmidt norm condition $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathcal{L}_2(L^2(\mathcal{O}))} < \infty$, $\beta > 0$, determines the rate of convergence. Here $U = L^2(\mathcal{O})$ and $Q \in \mathcal{L}(L^2(\mathcal{O}))$ is the covariance operator of L as introduced in Section 2.1.

For the stochastic heat equation, we show in Theorem 4.5 that for twice continuously differentiable bounded test functions with bounded first and second derivatives the rate of weak convergence is twice that of strong convergence and it is at least $O(h^{2\beta} + (\Delta t)^\beta)$, $\beta \in (0, 1]$, modulo a logarithmic term, where h and Δt are the space- and time-discretization parameters, respectively. This extends the corresponding result from [25], where, in contrast to the present paper, the analysis is restricted to so-called impulsive cylindrical processes on $L^2(\mathcal{O})$ as driving noise. Moreover, there is a serious restriction on the jump size intensity measure in [25, Section 6] admitting only processes of bounded variation (on finite time intervals). Here, the only restriction we have on L is that it is square-integrable, non-Gaussian and has mean zero.

For the stochastic wave equation the additional technical condition (5.8) has to be imposed in order to prove that the weak order is twice that of the strong order and at least $O(h^{\min(2\beta \frac{r}{r+1}, r)} + (\Delta t)^{\min(2\beta \frac{p}{p+1}, 1)})$, see Theorem 5.3. Here p and r are the classical orders of the time-discretization and of the finite element method. We would like to point out that, while the extra condition (5.8) on the second derivative on the test function is restrictive, it trivially holds for the important function $g(x) = \|x\|_{L^2(\mathcal{O})}^2$. Although the results in the present paper, notably the error representation formulae, do not allow for such a test function, as it is unbounded with unbounded first derivative, we expect that they could be extended to cover this case as well with some more technical effort. This is non-trivial

in the Lévy setting and will be done in a follow-up paper since it does not lie within the scope of the present article. Furthermore, as far as the authors know, there are no results available in the literature concerning weak approximation of hyperbolic stochastic partial differential equations driven by Lévy noise.

Let us remark that weak error estimates for approximations of Lévy-driven stochastic ordinary differential equations have been considered by various authors, see, e.g. [17, 27, 33, 36] and the references therein. There also exists a series of papers on strong error estimates for approximations of SPDEs driven by Lévy processes or Poisson random measures, see, for example [4, 5, 6, 13, 15, 16, 23] and compare also Remarks 4.2 and 5.1 below. However, to the best of our knowledge, the first steps in a *weak* error analysis for Lévy-driven SPDEs have been done only recently in the already mentioned article [25].

The present paper is organized as follows. In Section 2 we describe the abstract framework of the paper, introduce infinite-dimensional Lévy processes with several examples and a framework for linear stochastic partial differential equations driven by additive Lévy noise. Assumption 2.6 summarizes the main assumptions for the general setting of the paper. In Section 3 we state and prove two representation formulae, Theorem 3.3 and Corollary 3.5, for $e(T)$ given by (1.6). The main ingredient in their proofs is Proposition 3.6 on Kolmogorov's backward equation. In Section 4 and Section 5 we use the representation formula from Corollary 3.5 to establish weak convergence rates for a space-time discretization scheme for the stochastic heat and wave equations. Section 6 contains some concluding remarks outlining how to remove some of the technical conditions imposed in the paper for keeping the presentation simple. In the appendix we link stochastic integration with respect to Poisson random measures to integration with respect to infinite-dimensional Lévy processes.

2 Setting and preliminaries

Here we describe in detail our abstract setting and collect some background material from infinite-dimensional stochastic analysis.

General notation. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ and $(\mathcal{G}, \langle \cdot, \cdot \rangle_{\mathcal{G}})$ be real, separable Hilbert spaces and denote by $\mathcal{L}(\mathcal{H}, \mathcal{G})$, $\mathcal{L}_1(\mathcal{H}, \mathcal{G})$ and $\mathcal{L}_2(\mathcal{H}, \mathcal{G})$ the spaces of linear and bounded operators, nuclear operators and Hilbert-Schmidt operators from \mathcal{H} to \mathcal{G} , respectively. The corresponding norms are denoted by $\| \cdot \|_{\mathcal{L}(\mathcal{H}, \mathcal{G})}$, $\| \cdot \|_{\mathcal{L}_1(\mathcal{H}, \mathcal{G})}$ and $\| \cdot \|_{\mathcal{L}_2(\mathcal{H}, \mathcal{G})}$. If $\mathcal{H} = \mathcal{G}$, we write $\mathcal{L}(\mathcal{H})$, $\mathcal{L}_1(\mathcal{H})$ and $\mathcal{L}_2(\mathcal{H})$ instead of $\mathcal{L}(\mathcal{H}, \mathcal{H})$, $\mathcal{L}_1(\mathcal{H}, \mathcal{H})$ and $\mathcal{L}_2(\mathcal{H}, \mathcal{H})$. Given a measure space (M, \mathcal{M}, μ) and $1 \leq p < \infty$, we denote by $L^p(M; \mathcal{H}) = L^p(M, \mathcal{M}, \mu; \mathcal{H})$ the space of all $\mathcal{M}/\mathcal{B}(\mathcal{H})$ -measurable mappings $f : M \rightarrow \mathcal{H}$ with finite norm $\|f\|_{L^p(M; \mathcal{H})} = (\mathbb{E}\|f\|_{\mathcal{H}}^p)^{1/p}$, where $\mathcal{B}(\mathcal{H})$ denotes the Borel σ -algebra on the Hilbert space \mathcal{H} . By $C_b^n(\mathcal{H}, \mathbb{R})$ we denote the space of all n -times continuously Fréchet differentiable functions $f : \mathcal{H} \rightarrow \mathbb{R}$, $x \mapsto f(x)$ which are bounded together with their derivatives. Identifying \mathcal{H} and $\mathcal{L}(\mathcal{H}, \mathbb{R})$ via the Riesz isomorphism, we consider for fixed $x \in \mathcal{H}$ the first derivative $f'(x)$ as an element of \mathcal{H} . Similarly, the second derivative $f''(x)$ is considered as an element of $\mathcal{L}(\mathcal{H})$. In

particular, the norm in $C_b^2(\mathcal{H}; \mathbb{R})$ reads $\|f\|_{C_b^2(\mathcal{H}, \mathbb{R})} = \sup_{x \in \mathcal{H}} |f(x)| + \sup_{x \in \mathcal{H}} \|f'(x)\|_{\mathcal{H}} + \sup_{x \in \mathcal{H}} \|f''(x)\|_{\mathcal{L}(\mathcal{H})}$. We also write f_x and f_{xx} instead of f' and f'' .

2.1 The driving Lévy process L

The process $L = (L(t))_{t \geq 0}$ in Eq. (1.2) is a Lévy process with values in a real and separable Hilbert space U_1 , defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions (cf. [32]). L is (\mathcal{F}_t) -adapted and for $t, h \geq 0$ the increment $L(t+h) - L(t)$ is independent of \mathcal{F}_t . We always consider a càdlàg (right continuous with left limits) modification of L , i.e., a modification such that $L(t) = \lim_{s \searrow t} L(s)$ for all $t \geq 0$ and $L(t-) := \lim_{s \nearrow t} L(s)$ exists for all $t > 0$, where the limits are pathwise limits in U_1 . Our standard reference for Hilbert space-valued Lévy processes is [32].

In order to keep the exposition simple, we assume that L is square-integrable, i.e., $\mathbb{E}\|L(t)\|_{U_1}^2 < \infty$, and that the Gaussian part of L vanishes. Moreover, we assume that L has mean zero, i.e., $\mathbb{E}L(t) = 0$ in U_1 . Let ν be the jump intensity measure (Lévy measure) of L . Note that the jump intensity measure ν of a general Lévy process in U_1 satisfies $\nu(\{0\}) = 0$ and $\int_{U_1} \min(1, \|y\|_{U_1}^2) \nu(dy) < \infty$, cf. [32, Section 4]. Due to our assumptions we have

$$\int_{U_1} \|y\|_{U_1}^2 \nu(dy) < \infty, \quad (2.1)$$

and the characteristic function of L is given by

$$\mathbb{E}e^{i\langle x, L(t) \rangle_{U_1}} = \exp \left\{ -t \int_{U_1} \left(1 - e^{i\langle x, y \rangle_{U_1}} + i\langle x, y \rangle_{U_1} \right) \nu(dy) \right\}, \quad t \geq 0, x \in U_1. \quad (2.2)$$

Conversely, any U_1 -valued Lévy process L satisfying (2.1) and (2.2) is square-integrable, with mean zero and vanishing Gaussian part.

Let $Q_1 \in \mathcal{L}(U_1)$ be the covariance operator of L . It is determined by the jump intensity measure ν via

$$\langle Q_1 x, y \rangle_{U_1} = \int_{U_1} \langle x, z \rangle_{U_1} \langle y, z \rangle_{U_1} \nu(dz), \quad x, y \in U_1, \quad (2.3)$$

see [32, Theorem 4.47]. Further, let

$$(U_0, \langle \cdot, \cdot \rangle_{U_0}) := \left(Q_1^{1/2}(U_1), \langle Q_1^{-1/2} \cdot, Q_1^{-1/2} \cdot \rangle_{U_1} \right)$$

be the reproducing kernel Hilbert space of L , where $Q_1^{-1/2}$ denotes the pseudo-inverse of $Q_1^{1/2}$, see [32, Section 7]. Recall that the operator B in Eq. (1.2) is defined on the Hilbert space U . We assume that

$$U_0 \subset U \subset U_1, \quad (2.4)$$

and that the inclusions (2.4) define continuous embeddings. We denote the embedding of U_0 into U by $J_0 \in \mathcal{L}(U_0, U)$ and set

$$Q := J_0 J_0^* \in \mathcal{L}(U). \quad (2.5)$$

The nonnegative and symmetric operator Q is the covariance operator of L considered as a cylindrical process in U , cf. Remark 2.1 below. As a consequence of Douglas' theorem as stated in [32, Appendix A.4], compare also [35, Corollary C.0.6], the reproducing kernel Hilbert space of L has the alternative representation

$$(U_0, \langle \cdot, \cdot \rangle_{U_0}) = (Q^{1/2}(U), \langle Q^{-1/2} \cdot, Q^{-1/2} \cdot \rangle_U).$$

Remark 2.1. Suppose w.l.o.g. that U is dense in U_1 , identify U and U^* via the Riesz isomorphism, and consider the Gelfand triple $U_1^* \subset U^* \equiv U \subset U_1$. Then it is not difficult to see that

$$\mathbb{E} \langle L(t), x \rangle \langle L(t), y \rangle = t \langle Qx, y \rangle_U, \quad t \geq 0, x, y \in U_1^*,$$

where $\langle \cdot, \cdot \rangle : U_1 \times U_1^* \rightarrow \mathbb{R}$ is the canonical dual pairing; compare [32, Proposition 7.7]. The unique continuous extensions of the linear mappings $U_1^* \ni x \mapsto \langle L(t), x \rangle \in L^2(\mathbb{P})$, $t \geq 0$, to the larger space U^* determine a 2-cylindrical U -process in the sense of [29], compare also [1], [37], [38].

Remark 2.2. Unlike in the case of a mean-zero (cylindrical) Q -Wiener process in U , the covariance operators $Q \in \mathcal{L}(U)$ and $Q_1 \in \mathcal{L}_1(U_1)$ do *not* determine the distribution of the Lévy process L , but the jump intensity measure ν does so according to (2.2). Note that the law of a general Lévy process is determined by its characteristics (Lévy triplet), cf. [32, Definition 4.28], and that the characteristics of L are $(-\int_{\{\|y\|_{U_1} \geq 1\}} y \nu(dy), 0, \nu)$. Nevertheless, the operator Q in (2.5) will play an important role in our error analysis. Let us shortly make the connection of our setting to the construction of a cylindrical Q -Wiener process in U as described in [11], [35]. To this end, let $(f_k)_{k \in \mathbb{N}}$ be an orthonormal basis of U_1 consisting of eigenvectors of Q_1 with eigenvalues $(\lambda_k)_{k \in \mathbb{N}}$ and consider the orthonormal basis $(e_k)_{k \in \mathbb{N}}$ of U_0 given by $e_k := \lambda_k^{1/2} f_k$. To simplify notation we suppose for the moment that all eigenvalues λ_k of Q_1 are strictly positive. Then, compare [32, Section 4.8], the real-valued Lévy processes $L_k = (L_k(t))_{t \geq 0}$, $k \in \mathbb{N}$, given by

$$L_k(t) := \lambda_k^{-1/2} \langle L(t), f_k \rangle_{U_1}$$

are uncorrelated, i.e., $\mathbb{E} L_k(t) L_j(s) = 0$ if $k \neq j$, they satisfy $\mathbb{E}(L_k^2(t)) = t$, and we have

$$L(t) = \sum_{k \in \mathbb{N}} L_k(t) e_k. \tag{2.6}$$

The infinite sum in (2.6) converges for all finite $T > 0$ in the space $\mathcal{M}_T^2(U_1)$ of càdlàg square-integrable U_1 -valued (\mathcal{F}_t) -martingales $M = (M(t))_{t \in [0, T]}$ with norm $\|M\|_{\mathcal{M}_T^2(U_1)} = (\mathbb{E} \|M(T)\|_{U_1}^2)^{1/2}$. In contrast to the Gaussian case, where uncorrelated coordinates are always independent, the coordinate processes L_k , $k \in \mathbb{N}$, are in general only uncorrelated but *not* independent.

Conversely, suppose that we are given an arbitrary symmetric and nonnegative operator $Q \in \mathcal{L}(U)$, an orthonormal basis $(e_k)_{k \in \mathbb{N}}$ of $U_0 = Q^{1/2}(U)$, and a family L_k , $k \in \mathbb{N}$, of real-valued Lévy processes on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ that satisfy the following conditions:

- Each L_k is (\mathcal{F}_t) -adapted and for $t, h \geq 0$ the increment $L_k(t+h) - L_k(t)$ is independent of \mathcal{F}_t ;
- each L_k is square-integrable with $\mathbb{E}L_k(t) = 0$ and $\mathbb{E}(L_k^2(t)) = t$;
- the processes $L_k, k \in \mathbb{N}$, are uncorrelated;
- for all $n \in \mathbb{N}$ the \mathbb{R}^n -valued process $((L_1(t), \dots, L_n(t))^\top)_{t \geq 0}$ is a Lévy process;
- the Gaussian part of each L_k is zero.

Then, if U_1 is a Hilbert space containing U such that the natural embedding of $U_0 = Q^{1/2}(U)$ into U_1 is Hilbert-Schmidt, the infinite sum in (2.6) converges in $\mathcal{M}_T^2(U_1)$ and defines a Lévy process L with reproducing kernel Hilbert space U_0 that fits into our setting.

We end this subsection with some examples of Lévy processes L . We suppose that all processes are defined relative to the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and that their increments on time intervals $[t, t+h]$ are independent of \mathcal{F}_t .

Example 2.3. (*Subordinate cylindrical \tilde{Q} -Wiener process*) Let $W = (W(t))_{t \geq 0}$ be a cylindrical \tilde{Q} -Wiener process in U in the sense of [35, Section 2.5.1], where $Q \in \mathcal{L}(U)$ is a given nonnegative and symmetric operator. Assume that W takes values in a possibly larger Hilbert space $U_1 \supset U$ such that the natural embedding of U into U_1 is dense and continuous. Let $\tilde{Q}_1 \in \mathcal{L}_1(U_1)$ be the covariance operator of W considered as a Wiener process in U_1 , i.e., $\mathbb{E}\langle W(t), x \rangle_{U_1} \langle W(s), y \rangle_{U_1} = \min(s, t) \langle \tilde{Q}_1 x, y \rangle_{U_1}$ for $x, y \in U_1, s, t \geq 0$. Let $Z = (Z(t))_{t \geq 0}$ be a subordinator, i.e., a real-valued increasing Lévy process in the sense of [40, Definition 21.4], [41]. Assume that W and Z are independent, that the drift of Z is zero, and that the jump intensity measure ρ of Z satisfies

$$\int_0^\infty s \rho(ds) < \infty. \quad (2.7)$$

The latter is equivalent to assuming that Z has first moments, $\mathbb{E}|Z(t)| < \infty$. According to [40, Remark 21.6], the Laplace transform of $Z(t)$ is given by

$$\mathbb{E}(e^{-rZ(t)}) = \exp\left(-t \int_0^\infty (1 - e^{-rs}) \rho(ds)\right), \quad r \geq 0. \quad (2.8)$$

In this situation, subordinate cylindrical Brownian motion

$$L(t) := W(Z(t)), \quad t \geq 0,$$

defines a U_1 -valued Lévy process $L = (L(t))_{t \geq 0}$ that fits into the general framework described above. Indeed, L has stationary and independent increments. Moreover, the independence of W and Z , the identity $\mathbb{E}e^{i\langle x, W(s) \rangle_{U_1}} = e^{-s \frac{1}{2} \langle \tilde{Q}_1 x, x \rangle_{U_1}}$, Eq. (2.8) and the symmetry of the distribution $\mathbb{P}_{W(1)} = N(0, Q_1)$ imply that characteristic function of $L(t)$ is given by

$$\mathbb{E}e^{i\langle x, L(t) \rangle_{U_1}} = \int_0^\infty e^{-s \frac{1}{2} \langle \tilde{Q}_1 x, x \rangle_{U_1}} \mathbb{P}_{Z(t)}(ds)$$

$$\begin{aligned}
&= \exp \left[-t \int_0^\infty (1 - e^{-\frac{1}{2} \langle \tilde{Q}_1 x, x \rangle_{U_1} s}) \rho(ds) \right] \\
&= \exp \left[-t \int_0^\infty \int_{U_1} (1 - e^{i \langle x, \sqrt{s} y \rangle_{U_1}} + i \langle x, \sqrt{s} y \rangle_{U_1}) \mathbb{P}_{W(1)}(dy) \rho(ds) \right].
\end{aligned}$$

As a consequence, (2.2) holds with

$$\nu = (\mathbb{P}_{W(1)} \otimes \rho) \circ \kappa^{-1}, \quad (2.9)$$

where $\kappa : U_1 \times (0, \infty) \rightarrow U_1$ is defined by $\kappa(y, s) = \sqrt{s}y$; compare [38, Lemma 4.8]. (Note that, by the scaling property of W , (2.9) is equivalent to the standard formula $\nu = \int_0^\infty \mathbb{P}_{W(s)} \rho(ds)$, where the measure-valued integral is defined in a weak sense, cf. [40, Section 30]). Moreover, (2.1) holds due to (2.7) as we have the equality $\int_{U_1} \|y\|_{U_1}^2 \nu(dy) = \int_0^\infty s \rho(ds) \mathbb{E}(\|W(1)\|_{U_1}^2)$ according to (2.9). It follows that L is a U_1 -valued, square-integrable, mean-zero Lévy process with vanishing Gaussian part. It is also not difficult to show that the covariance operators $Q_1 \in \mathcal{L}_1(U_1)$ and $Q \in \mathcal{L}(U)$ of L in (2.3) and (2.5) are given by $Q_1 = \int_0^\infty s \rho(ds) \tilde{Q}_1$ and $Q = \int_0^\infty s \rho(ds) \tilde{Q}$. Subordinate cylindrical Wiener processes have been considered, e.g., in [8].

Example 2.4. (*Independent one-dimensional Lévy processes*) Let $Q \in \mathcal{L}(U)$ be symmetric, nonnegative and let $(e_k)_{k \in \mathbb{N}}$ be an orthonormal basis of $U_0 := Q^{1/2}(U) \subset U$. Let $L_k = (L_k(t))_{k \in \mathbb{N}}$, $k \in \mathbb{N}$, be independent real-valued square-integrable Lévy processes with vanishing Gaussian part and $\mathbb{E}L_k(t) = 0$, $\mathbb{E}(L_k^2(t)) = t$. Let $U_1 \supset U$ be another Hilbert space such that the natural embedding of U_0 into U_1 is a Hilbert-Schmidt operator. Then, the series (2.6) converges for all $T \in (0, \infty)$ in the space $\mathcal{M}_T^2(U_1)$ and defines a Lévy process $L = (L(t))_{t \geq 0}$ satisfying (2.1) and (2.2) with jump intensity measure

$$\nu = \sum_{k \in \mathbb{N}} \nu_k \circ \pi_k^{-1},$$

where ν_k is the Lévy measure of L_k and $\pi_k : \mathbb{R} \rightarrow U_1$ is defined by $\pi_k(\xi) := \xi e_k$; compare [32, Section 4.8.1].

Example 2.5. (*Impulsive cylindrical process*) Let μ be a Lévy measure on \mathbb{R} such that $\int_{\mathbb{R}} \sigma^2 \mu(d\sigma) < \infty$. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain and $Z = (Z(t))_{t \geq 0}$ an impulsive cylindrical process on $U := L^2(\mathcal{O}) = L^2(\mathcal{O}, \mathcal{B}(\mathcal{O}), \lambda^d)$ with jump size intensity μ in the sense of [32, Definition 7.23]. Here, λ^d denotes d -dimensional Lebesgue measure. The process Z is a measure-valued process defined, informally, by $Z(t, d\xi) = \int_0^t \int_{\mathbb{R}} \sigma \hat{\pi}(ds, d\xi, d\sigma)$, where $\hat{\pi}$ is a compensated Poisson random measure on $[0, \infty) \times \mathcal{O} \times \mathbb{R}$ with reference measure $\lambda^1 \otimes \lambda^d \otimes \mu$; see [32, Section 7.2] for details. Let $\tilde{Q} \in \mathcal{L}(U)$ be symmetric and nonnegative, $(b_k)_{k \in \mathbb{N}}$ an orthonormal basis of U , and $U_1 \supset U$ a Hilbert space such that the natural embedding of $U_0 = \tilde{Q}^{1/2}(U) \subset U$ into U_1 is Hilbert-Schmidt. Then the series

$$L(t) := \tilde{Q}^{\frac{1}{2}} Z(t) := \sum_{k \in \mathbb{N}} \int_0^t \int_{\mathcal{O}} \int_{\mathbb{R}} \sigma b_k(\xi) \hat{\pi}(ds, d\xi, d\sigma) \tilde{Q}^{\frac{1}{2}} b_k, \quad t \geq 0, \quad (2.10)$$

converges for all $T \in (0, \infty)$ in $\mathcal{M}_T^2(U_1)$ and defines a Lévy process that fits into our general framework with $Q = \int_{\mathbb{R}} \sigma^2 \mu(d\sigma)Q$ and

$$\nu = (\lambda^d \otimes \mu) \circ \phi^{-1},$$

where $\phi \in L^2(\mathcal{O} \times \mathbb{R}, \lambda^d \otimes \mu; U_1)$ is defined by $\phi(\xi, \sigma) = \sum_{n \in \mathbb{N}} \sigma b_k(\xi) \tilde{Q}^{\frac{1}{2}} b_k$ (convergence in $L^2(\mathcal{O} \times \mathbb{R}, \lambda^d \otimes \mu; U_1)$). In [25] we considered the weak approximation of the stochastic heat equation driven by an impulsive process of the form (2.10). The results in Section 4 of the present article improve the results of [25] in several aspects.

2.2 Linear stochastic evolution equations with additive noise

We are mainly interested in equations of the type (1.2), where $A : D(A) \subset H \rightarrow H$ is an unbounded linear operator such that $-A$ is the generator of a strongly continuous semigroup $(E(t))_{t \geq 0} \subset \mathcal{L}(H)$, $B \in \mathcal{L}(U, H)$, $L = (L(t))_{t \geq 0}$ is a square-integrable Lévy process with reproducing kernel Hilbert space $U_0 \subset U$ as described in Subsection 2.1, and $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$. It is well known that if

$$\int_0^T \|E(t)B\|_{\mathcal{L}_2(U_0, H)}^2 dt < \infty \quad (2.11)$$

for some (and hence for all) $T > 0$, then there exists a unique weak solution $X = (X(t))_{t \geq 0}$ to (1.2) which is given by the variation-of-constants formula (1.1), see, e.g., [32, Chapter 9]. Similarly, if $(\tilde{E}(t))_{t \in [0, T]} \subset \mathcal{L}(H)$ is given by some approximation scheme such that $t \mapsto \tilde{E}(t)B$ is a measurable mapping from $[0, T]$ to $\mathcal{L}_2(U_0, H)$, then the condition

$$\int_0^T \|\tilde{E}(t)B\|_{\mathcal{L}_2(U_0, H)}^2 dt < \infty \quad (2.12)$$

ensures that the approximation $\tilde{X} = (\tilde{X}(t))_{t \in [0, T]}$ of $(X(t))_{t \in [0, T]}$ in (1.5) exists as a square-integrable H -valued process. We refer to [32, Chapter 8] for details on the construction and properties of the stochastic integral w.r.t. Hilbert space-valued Lévy processes.

It turns out that our general error representation formula for the weak error $e(T)$ in (1.6) does not require the semigroup property of the strongly continuous family of operators $(E(t))_{t \geq 0}$. This paves the way for analysing a more general class of Lévy-driven linear stochastic evolution equations, including for example stochastic Volterra type equations as considered in [20], [21] for the Gaussian case. For such equations, the weak solution still has the form (1.1) but the solution operator family $(E(t))_{t \geq 0} \subset \mathcal{L}(H)$ is not a semigroup anymore. Therefore, we weaken our abstract assumptions and summarize them as follows.

Assumption 2.6. We will use the following assumptions:

- (i) H , U and U_1 are real and separable Hilbert spaces;
- (ii) $L = (L(t))_{t \geq 0}$ is a U_1 -valued Lévy process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ admitting second moments and with reproducing kernel Hilbert space U_0 such that $U_0 \subset U \subset U_1$ as described in Subsection 2.1;

- (iii) $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$;
- (iv) $B \in \mathcal{L}(U, H)$ and $(E(t))_{t \in [0, T]} \subset \mathcal{L}(H)$ is a strongly continuous family of linear operators such that (2.11) holds;
- (v) for all $\varepsilon > 0$ there exists $\Phi_\varepsilon \in \mathcal{L}_2(U_0, H)$ such that

$$\|E(t)Bx\|_H \leq \|\Phi_\varepsilon x\|_H, \quad (t, x) \in [\varepsilon, T] \times U_0;$$

- (vi) $(\tilde{E}(t))_{t \in [0, T]} \subset \mathcal{L}(H)$ is a family of linear operators such that $t \mapsto \tilde{E}(t)B$ is a measurable mapping from $[0, T]$ to $\mathcal{L}_2(U_0, H)$ and (2.12) holds;
- (vii) $X = (X(t))_{t \in [0, T]}$ and $\tilde{X} = (\tilde{X}(t))_{t \in [0, T]}$ are H -valued stochastic processes given by (1.1) and (1.5).

Remark 2.7. If $(E(t))_{t \geq 0}$ is an operator semigroup, then 2.6(v) is a consequence of 2.6(iv).

To fix notation, let us briefly recall the Itô isometry for stochastic integrals w.r.t. L . It has the same form as the Itô isometry for stochastic integrals w.r.t. Hilbert space-valued Wiener processes. We set $\Omega_T := \Omega \times [0, T]$ and $\mathbb{P}_T := \mathbb{P} \otimes \lambda$, where λ is Lebesgue measure on $[0, T]$. The predictable σ -algebra on Ω_T w.r.t. $(\mathcal{F}_t)_{t \in [0, T]}$ is denoted by \mathcal{P}_T . For operator-valued processes $\Phi = (\Phi(t))_{t \in [0, T]}$ in

$$L^2(\Omega_T, \mathbb{P}_T; \mathcal{L}_2(U_0, H)) := L^2(\Omega_T, \mathcal{P}_T, \mathbb{P}_T; \mathcal{L}_2(U_0, H)),$$

we have

$$\mathbb{E} \left(\left\| \int_0^t \Phi(s) dL(s) \right\|_H^2 \right) = \mathbb{E} \int_0^t \|\Phi(s)\|_{\mathcal{L}_2(U_0, H)}^2 ds, \quad t \in [0, T], \quad (2.13)$$

and the integral process $(\int_0^t \Phi(s) dL(s))_{t \in [0, T]}$ belongs to the space $\mathcal{M}_T^2(H)$ of càdlàg square-integrable H -valued (\mathcal{F}_t) -martingales. The norm in $\mathcal{M}_T^2(H)$ is defined by $\|M\|_{\mathcal{M}_T^2(H)} = (\mathbb{E} \|M(T)\|_H^2)^{1/2}$, $M = (M(t))_{t \in [0, T]} \in \mathcal{M}_T^2(H)$. Note, however, that the integral processes given by the stochastic integrals in (1.1) and (1.5) are in general *not* martingales since the (deterministic) operator-valued integrands also depend on t .

We also recall the definition and some properties of Hilbert-Schmidt operators, cf. [44, Chapter 6]. Let \mathcal{H} and \mathcal{G} be real and separable Hilbert spaces. A linear and bounded operator $C \in \mathcal{L}(\mathcal{H}, \mathcal{G})$ belongs to the space $\mathcal{L}_2(\mathcal{H}, \mathcal{G})$ of Hilbert-Schmidt operators if

$$\|C\|_{\mathcal{L}_2(\mathcal{H}, \mathcal{G})} := \left(\sum_{k \in \mathbb{N}} \|Ch_k\|_{\mathcal{G}}^2 \right)^{1/2} < \infty$$

for some (and hence for every) orthonormal basis $(h_k)_{k \in \mathbb{N}}$ of \mathcal{H} . If $C \in \mathcal{L}(\mathcal{H}, \mathcal{G})$ and $C^* \in \mathcal{L}(\mathcal{G}, \mathcal{H})$ is the adjoint operator, then $C \in \mathcal{L}_2(\mathcal{H}, \mathcal{G})$ if and only if $C^* \in \mathcal{L}_2(\mathcal{G}, \mathcal{H})$ and one has

$$\|C\|_{\mathcal{L}_2(\mathcal{H}, \mathcal{G})} = \|C^*\|_{\mathcal{L}_2(\mathcal{G}, \mathcal{H})}. \quad (2.14)$$

Also, if $C \in \mathcal{L}_2(\mathcal{H}, \mathcal{G})$, $D \in \mathcal{L}(\mathcal{H})$ and $F \in \mathcal{L}(\mathcal{G})$, then obviously $CD \in \mathcal{L}_2(\mathcal{H}, \mathcal{G})$, $FC \in \mathcal{L}_2(\mathcal{H}, \mathcal{G})$ and

$$\|CD\|_{\mathcal{L}_2(\mathcal{H}, \mathcal{G})} \leq \|C\|_{\mathcal{L}_2(\mathcal{H}, \mathcal{G})} \|D\|_{\mathcal{L}(\mathcal{H})}, \quad \|FC\|_{\mathcal{L}_2(\mathcal{H}, \mathcal{G})} \leq \|F\|_{\mathcal{L}(\mathcal{G})} \|C\|_{\mathcal{L}_2(\mathcal{H}, \mathcal{G})}. \quad (2.15)$$

In particular, in our setting we have $\mathcal{L}(U_1, H) \subset \mathcal{L}_2(U_0, H)$ since

$$\|C\|_{\mathcal{L}_2(U_0, H)} = \|CQ_1^{1/2}\|_{\mathcal{L}_2(U_1, H)} \leq \|C\|_{\mathcal{L}(U_1, H)} \|Q_1^{1/2}\|_{\mathcal{L}_2(U_1)}$$

for all $C \in \mathcal{L}(U_1, H)$ and $\|Q_1^{1/2}\|_{\mathcal{L}_2(U_1)} = \text{Tr } Q_1 = \|Q_1\|_{\mathcal{L}_1(U_1)} < \infty$.

3 An error representation formula

In this section, we state and prove a general representation formula for the weak approximation error $e(T)$ in (1.6) within the abstract setting described above.

3.1 Formulation of the result

For the formulation and the proof of the error representation formula, we introduce auxiliary drift-free Itô processes $Y = (Y(t))_{t \in [0, T]}$ and $\tilde{Y} = (\tilde{Y}(t))_{t \in [0, T]}$ such that

$$X(T) = Y(T), \quad \tilde{X}(T) = \tilde{Y}(T).$$

The processes Y and \tilde{Y} are constructed by applying to X and \tilde{X} the deterministic operator-valued processes $(E(T-t))_{t \in [0, T]}$ and $(\tilde{E}(T-t))_{t \in [0, T]}$. That is, we set

$$Y(t) := E(T)X_0 + \int_0^t E(T-s)B \, dL(s), \quad t \in [0, T], \quad (3.1)$$

and

$$\tilde{Y}(t) := \tilde{E}(T)X_0 + \int_0^t \tilde{E}(T-s)B \, dL(s), \quad t \in [0, T]. \quad (3.2)$$

Moreover, we consider the auxiliary problem

$$dZ(t) = E(T-t)B \, dL(t), \quad t \in [\tau, T]; \quad Z(\tau) = \xi,$$

where $\tau \in [0, T)$ and ξ is an H -valued \mathcal{F}_τ -measurable random variable. Its solution is given by

$$Z(t, \tau, \xi) := \xi + \int_\tau^t E(T-s)B \, dL(s), \quad t \in [\tau, T], \quad (3.3)$$

and we use it to define for $G \in C_b^2(H, \mathbb{R})$ a function $u : [0, T] \times H \rightarrow \mathbb{R}$ by

$$u(t, x) := \mathbb{E} G(Z(T, t, x)), \quad (t, x) \in [0, T] \times H. \quad (3.4)$$

Then, u and its Fréchet partial derivatives u_x, u_{xx} are continuous and bounded on $[0, T] \times H$, cf. Proposition 3.6 below. We have

$$u_x(t, x) = \mathbb{E} G'(Z(T, t, x)), \quad u_{xx}(t, x) = \mathbb{E} G''(Z(T, t, x)). \quad (3.5)$$

Before stating the representation formula, we show in the following lemma how operators in $\mathcal{L}_2(U_0, H)$ can be identified with functions in

$$L^2(U_1, \nu; H) := L^2(U_1, \mathcal{B}(U_1), \nu; H)$$

and how processes in $L^2(\Omega_T, \mathbb{P}_T; \mathcal{L}_2(U_0, H))$ can be identified with elements in

$$L^2(\Omega_T \times U_1, \mathbb{P}_T \otimes \nu; H) := L^2(\Omega_T \times U_1, \mathcal{P}_T \otimes \mathcal{B}(U_1), \mathbb{P}_T \otimes \nu; H).$$

These identifications will be used implicitly throughout this article, see Remark 3.2 below. They also lead to a generic identification of integrals w.r.t. (cylindrical) Hilbert space-valued Lévy processes of jump type and integrals w.r.t. the associated Poisson random measures, cf. Appendix A.

Lemma 3.1. *Let $(f_k)_{k \in \mathbb{N}} \subset U_0$ be an orthonormal basis of U_0 consisting of eigenvectors of the covariance operator $Q_1 \in \mathcal{L}_1(U_1)$ of L and let $(\lambda_k)_{k \in \mathbb{N}} \subset [0, \infty)$ be the corresponding sequence of eigenvalues*

(i) *Given $\Phi \in \mathcal{L}_2(U_0, H)$, the series*

$$\iota(\Phi) := \sum_{k \in \mathbb{N}, \lambda_k \neq 0} \langle \cdot, f_k \rangle_{U_1} \Phi f_k$$

converges in $L^2(U_1, \nu; H)$. The linear mapping

$$\iota : \mathcal{L}_2(U_0, H) \rightarrow L^2(U_1, \nu; H), \quad \Phi \mapsto \iota(\Phi)$$

is an isometric embedding.

(ii) *Given $\Phi \in L^2(\Omega_T, \mathbb{P}_T; \mathcal{L}_2(U_0, H))$, the series*

$$\kappa(\Phi) := \sum_{k \in \mathbb{N}, \lambda_k \neq 0} \langle \cdot, f_k \rangle_{U_1} \Phi(\cdot) f_k$$

converges in $L^2(\Omega_T \times U_1, \mathbb{P}_T \otimes \nu; H)$. The linear mapping

$$\kappa : L^2(\Omega_T, \mathbb{P}_T; \mathcal{L}_2(U_0, H)) \rightarrow L^2(\Omega_T \times U_1, \mathbb{P}_T \otimes \nu; H), \quad \Phi \mapsto \kappa(\Phi)$$

is an isometric embedding. For \mathbb{P}_T -almost all $(\omega, t) \in \Omega_T$ we have $\kappa(\Phi)(\omega, t, \cdot) = \iota(\Phi(\omega, t))$ in $L^2(U_1, \nu; H)$, where ι is the embedding from (i).

Proof. (i) W.l.o.g. all eigenvalues λ_k are strictly positive. Let $(e_k)_{k \in \mathbb{N}}$ be the orthonormal basis of U_0 given by $e_k := \lambda_k^{1/2} f_k$. For $m, n \in \mathbb{N}$ with $m \leq n$ we have

$$\begin{aligned} \left\| \sum_{k=m}^n \langle \cdot, f_k \rangle_{U_1} \Phi f_k \right\|_{L^2(U_1, \nu; H)}^2 &= \int_{U_1} \left\| \sum_{k=m}^n \langle x, f_k \rangle_{U_1} \Phi f_k \right\|_H^2 \nu(dx) \\ &= \sum_{j,k=m}^n \lambda_j^{-1/2} \lambda_k^{-1/2} \int_{U_1} \langle x, f_j \rangle_{U_1} \langle x, f_k \rangle_{U_1} \nu(dx) \langle \Phi e_j, \Phi e_k \rangle_H \end{aligned}$$

$$= \sum_{k=m}^n \|\Phi e_k\|_H^2;$$

in the last step we used (2.3). Since $\sum_{k \in \mathbb{N}} \|\Phi e_k\|_H^2 = \|\Phi\|_{\mathcal{L}_2(U_0, H)}^2 < \infty$, this shows that the partial sums $\sum_{k=1}^n \langle \cdot, f_k \rangle_{U_1} \Phi f_k$, $n \in \mathbb{N}$, are a Cauchy sequence in $L^2(U_1, \nu; H)$ and

$$\left\| \sum_{k=1}^{\infty} \langle \cdot, f_k \rangle_{U_1} \Phi f_k \right\|_{L^2(U_1, \nu; H)} = \|\Phi\|_{\mathcal{L}_2(U_0, H)}.$$

(ii) The first two assertions can be shown as in the proof of (i). The last assertion is due the fact that the iterated integral

$$\int_{\Omega} \int_0^T \int_{U_1} \|\iota(\Phi(\omega, t))(x) - \kappa(\Phi)(\omega, t, x)\|_H^2 \nu(dx) dt \mathbb{P}(d\omega)$$

equals zero, which follows from an approximation argument. \square

Remark 3.2. From now on we will identify operators $\Phi \in \mathcal{L}_2(U_0, H)$ with the corresponding mappings $\iota(\Phi) \in L^2(U_1, \nu; H)$ and write

$$\Phi x = \iota(\Phi)(x), \quad x \in U_1.$$

Analogously, we identify processes $\Phi \in L^2(\Omega_T, \mathbb{P}_T; \mathcal{L}_2(U_0, H))$ with the corresponding mappings $\kappa(\Phi) \in L^2(\Omega_T \times U_1, \mathbb{P}_T \otimes \nu; H)$ and write

$$\Phi(\omega, t)x = \kappa(\Phi)(\omega, t, x), \quad (\omega, t, x) \in \Omega_T \times U_1.$$

For processes $\Phi \in L^2(\Omega_T, \mathbb{P}_T; \mathcal{L}_2(U_0, H))$ both identifications are compatible $\mathbb{P} \otimes \lambda$ -almost everywhere on Ω_T in the sense that we have $\kappa(\Phi)(\omega, t, \cdot) = \iota(\Phi(\omega, t))$ in $L^2(U_1, \nu; H)$ for $\mathbb{P} \otimes \lambda$ -almost all $(\omega, t) \in \Omega_T$.

Here is the main result of this section.

Theorem 3.3. *Under the Assumptions 2.6 and for $G \in C_b^2(H, \mathbb{R})$, the process $(\tilde{Y}(t))_{t \in [0, T]}$ from (3.2) and the function $u : [0, T] \times H \rightarrow \mathbb{R}$ from (3.4) it holds that*

$$\begin{aligned} \mathbb{E} \int_0^T \int_{U_1} & \left| u(t, \tilde{Y}(t) + \tilde{E}(T-t)By) - u(t, \tilde{Y}(t) + E(T-t)By) \right. \\ & \left. - \langle u_x(t, \tilde{Y}(t)), (\tilde{E}(T-t)B - E(T-t)B)y \rangle_H \right| \nu(dy) dt < \infty. \end{aligned} \quad (3.6)$$

The weak error $e(T)$ in (1.6) has the representation

$$\begin{aligned} e(T) &= \mathbb{E} \left\{ u(0, \tilde{E}(T)X_0) - u(0, E(T)X_0) \right\} \\ &+ \mathbb{E} \int_0^T \int_{U_1} \left\{ u(t, \tilde{Y}(t) + \tilde{E}(T-t)By) - u(t, \tilde{Y}(t) + E(T-t)By) \right. \\ &\quad \left. - \langle u_x(t, \tilde{Y}(t)), (\tilde{E}(T-t)B - E(T-t)B)y \rangle_H \right\} \nu(dy) dt. \end{aligned} \quad (3.7)$$

Remark 3.4. The terms $E(T-t)By$ and $\tilde{E}(T-t)By$ appearing in (3.6) and (3.7) are defined for $\lambda \otimes \nu$ -almost all $(t, y) \in [0, T] \times U_1$. This follows from (2.11), (2.12), Lemma 3.1 and Remark 3.2,

We will prove Theorem 3.3 in the next subsection. Let us briefly record an alternative representation of $e(T)$ which follows from Taylor's formula. It will be the starting point for our error estimates in Sections 4 and 5. For $t \in [0, T]$, $\theta \in [0, 1]$ and $y \in U_1$ set

$$\begin{aligned} F(t) &:= \tilde{E}(t)B - E(t)B, \\ \Psi_1(t, \theta, y) &:= (1 - \theta) \left\langle u_{xx} \left(t, \tilde{Y}(t) + E(T-t)By + \theta F(T-t)y \right) F(T-t)y, F(T-t)y \right\rangle_H, \\ \Psi_2(t, \theta, y) &:= \left\langle u_{xx} \left(t, \tilde{Y}(t) + \theta E(T-t)By \right) E(T-t)By, F(T-t)y \right\rangle_H. \end{aligned}$$

Corollary 3.5. *In the setting of Theorem 3.3 we have*

$$\mathbb{E} \int_0^T \int_{U_1} \int_0^1 \left\{ |\Psi_1(t, \theta, y)| + |\Psi_2(t, \theta, y)| \right\} d\theta \nu(dy) dt < \infty, \quad (3.8)$$

and the following alternative error representation holds:

$$\begin{aligned} e(T) &= \mathbb{E} \left\{ u(0, \tilde{E}(T)X_0) - u(0, E(T)X_0) \right\} \\ &\quad + \mathbb{E} \int_0^T \int_{U_1} \int_0^1 \left\{ \Psi_1(t, \theta, y) + \Psi_2(t, \theta, y) \right\} d\theta \nu(dy) dt. \end{aligned} \quad (3.9)$$

Proof. The integrand of the iterated integral in (3.7) can be rewritten as

$$\begin{aligned} &u \left(t, \tilde{Y}(t) + \tilde{E}(T-t)By \right) - u \left(t, \tilde{Y}(t) + E(T-t)By \right) - \left\langle u_x \left(t, \tilde{Y}(t) \right), F(T-t)y \right\rangle_H \\ &= \left\{ u \left(t, \tilde{Y}(t) + \tilde{E}(T-t)By \right) - u \left(t, \tilde{Y}(t) + E(T-t)By \right) \right. \\ &\quad \left. - \left\langle u_x \left(t, \tilde{Y}(t) + E(T-t)By \right), F(T-t)y \right\rangle_H \right\} \\ &\quad + \left\langle u_x \left(t, \tilde{Y}(t) + E(T-t)By \right) - u_x \left(t, \tilde{Y}(t) \right), F(T-t)y \right\rangle_H \\ &= \int_0^1 \left\{ \Psi_1(t, \theta, y) + \Psi_2(t, \theta, y) \right\} d\theta, \end{aligned}$$

where the last step is due to Taylor's formula. By (3.6) we have

$$\mathbb{E} \int_0^T \int_{U_1} \left| \int_0^1 \left\{ \Psi_1(t, \theta, y) + \Psi_2(t, \theta, y) \right\} d\theta \right| \nu(dy) dt < \infty.$$

The stronger assertion (3.8) follows from the boundedness of $G'' : H \rightarrow \mathcal{L}(H)$, Lemma 3.1 and (2.11), compare the estimates in the proof of Proposition 3.6 below. \square

3.2 Proof of the error representation formula

In this subsection, we give the proof of Theorem 3.3.

For $\xi \in L^0(\Omega, \mathcal{F}_t, \mathbb{P}; H)$ we have

$$\mathbb{E}\left(G(Z(T, t, \xi))\right) = \int_H \int_H G(x + y) \mathbb{P}_{\int_t^T E(T-s)B \, dL(s)}(\mathrm{d}y) \mathbb{P}_\xi(\mathrm{d}x) = \mathbb{E}\left(u(t, \xi)\right)$$

by (3.3), (3.4), the independence of $\int_t^T E(T-s)B \, dL(s)$ and \mathcal{F}_t , and Fubini's theorem. Since $X(T) = Y(T)$ and $\tilde{X}(T) = \tilde{Y}(T)$ it follows that

$$\begin{aligned} e(T) &= \mathbb{E}\left(G(\tilde{Y}(T)) - G(Y(T))\right) \\ &= \mathbb{E}\left(G(Z(T, T, \tilde{Y}(T))) - G(Z(T, 0, Y(0)))\right) \\ &= \mathbb{E}\left(u(T, \tilde{Y}(T)) - u(0, Y(0))\right) \\ &= \mathbb{E}\left(u(0, \tilde{Y}(0)) - u(0, Y(0))\right) + \mathbb{E}\left(u(T, \tilde{Y}(T)) - u(0, \tilde{Y}(0))\right). \end{aligned} \tag{3.10}$$

By (3.1) and (3.2), the first term in the last line equals $\mathbb{E}(u(0, \tilde{E}(T)X_0) - u(0, E(T)X_0))$.

To handle the second term in the last line of (3.10), we apply Itô's formula to the function $(t, x) \mapsto u(t, x)$ and the martingale $\tilde{Y} = (\tilde{Y}(t))_{t \in [0, T]}$. For this we need the following properties of u .

Proposition 3.6. *Let Assumption 2.6 hold and $G \in C_b^2(H, \mathbb{R})$. The function $u : [0, T] \times H \rightarrow \mathbb{R}$, $(t, x) \mapsto u(t, x)$ defined in (3.4) and its Fréchet partial derivatives u_x , u_{xx} are continuous and bounded on $[0, T] \times H$. The time derivative u_t of u exists on $[0, T] \times H$ and is continuous. Moreover, for every $\varepsilon > 0$ there exists some $C_\varepsilon < \infty$ such that*

$$\int_{U_1} \left| u(t, x + E(T-t)By) - u(t, x) - \langle u_x(t, x), E(T-t)By \rangle_H \right| \nu(\mathrm{d}y) < C_\varepsilon \tag{3.11}$$

for all $t \in [0, T - \varepsilon]$, and u satisfies the backward Kolmogorov equation

$$\left. \begin{aligned} u_t(t, x) &= - \int_{U_1} \left\{ u(t, x + E(T-t)By) - u(t, x) - \langle u_x(t, x), E(T-t)By \rangle_H \right\} \nu(\mathrm{d}y), \\ &\quad (t, x) \in [0, T] \times H, \\ u(T, x) &= G(x), \quad x \in H. \end{aligned} \right\} \tag{3.12}$$

Proof. We begin with the continuity and boundedness of u , u_x and u_{xx} . The boundedness is obvious by the definition (3.4) of u and by (3.5). Pick $0 \leq s \leq t \leq T$, $x, y \in H$. Using (3.4), Jensen's inequality, the mean value theorem, (3.3) and Itô's isometry, we have

$$\begin{aligned} |u(t, x) - u(s, y)|^2 &\leq \mathbb{E}\left(|G(Z(T, t, x)) - G(Z(T, s, y))|^2\right) \\ &\leq \sup_{x \in H} \|G'(x)\|_H^2 \mathbb{E}\left(\left\|x - y - \int_s^t E(T-r)B \, dL(r)\right\|_H^2\right) \\ &\leq 2 \sup_{x \in H} \|G'(x)\|_H^2 \left(\|x - y\|_H^2 + \int_s^t \|E(T-r)B\|_{\mathcal{L}_2(U_0, H)}^2 \, dr\right). \end{aligned}$$

Thus, the continuity of u follows from (2.11) and the boundedness of G' . Since $u_x(t, x) = \mathbb{E} G'(Z(T, t, x))$, the continuity of $u_x : [0, T] \times H \rightarrow H$ follows analogously from the boundedness of G'' . To show the continuity of $u_{xx} : [0, T] \times H \rightarrow \mathcal{L}(H)$, define $F \in C_b(H \times H; \mathbb{R})$ by

$$F(x, y) := \|G''(x) - G''(y)\|_{\mathcal{L}(H)}, \quad x, y \in H,$$

and fix $(t, x) \in [0, T] \times H$, $((t_k, x_k))_{k \in \mathbb{N}} \subset [0, T] \times H$ with $(t_k, x_k) \rightarrow (t, x)$ as $k \rightarrow \infty$. Note that $Z(T, t_k, x_k) \rightarrow Z(T, t, x)$ in $L^2(\Omega; H)$ by Itô's isometry. As a consequence, $(Z(T, t, x), Z(T, t_k, x_k)) \rightarrow (Z(T, t, x), Z(T, t, x))$ in distribution (on $H \times H$) and we obtain

$$\begin{aligned} \|u_{xx}(t, x) - u_{xx}(t_k, x_k)\|_{\mathcal{L}(H)} &\leq \mathbb{E} F(Z(T, t, x), Z(T, t_k, x_k)) \\ &\xrightarrow{k \rightarrow \infty} \mathbb{E} F(Z(T, t, x), Z(T, t, x)) = 0, \end{aligned}$$

yielding the continuity of u_{xx} .

By Taylor's formula and Lemma 3.1,

$$\begin{aligned} \int_{U_1} \left| u(t, x + E(T-t)By) - u(t, x) - \langle u_x(t, x), E(T-t)By \rangle_H \right| \nu(dy) \\ \leq \frac{1}{2} \sup_{x \in H} \|G''(x)\|_{\mathcal{L}(H)} \int_{U_1} \|E(T-t)By\|_H^2 \nu(dy) \\ = \frac{1}{2} \sup_{x \in H} \|G''(x)\|_{\mathcal{L}(H)} \|E(T-t)B\|_{\mathcal{L}_2(U_0, H)}^2. \end{aligned}$$

Using Assumption 2.6(v), this yields (3.11) with $C_\varepsilon = 1/2 \sup_{x \in H} \|G''(x)\|_{\mathcal{L}(H)} \|\Phi_\varepsilon\|_{\mathcal{L}_2(U_0, H)}^2$.

In order to verify the Kolmogorov equation (3.12), we first note that for fixed $t \in [0, T]$ the H -valued random variables $\int_0^t E(s)B dL(s)$ and $\int_{T-t}^T E(T-s)B dL(s)$ have the same distribution, so that

$$v(t, x) := \mathbb{E} G\left(x + \int_0^t E(s)B dL(s)\right) = u(T-t, x), \quad (t, x) \in [0, T] \times H. \quad (3.13)$$

Next, we fix $x \in H$ and apply Itô's formula [26, Theorem 3.6] to the function $y \mapsto G(x+y)$ and the martingale $M = (M(t))_{t \in [0, T]} := (\int_0^t E(s)B dL(s))_{t \in [0, T]} \in \mathcal{M}_T^2(H)$. Note that M fits into the setting of [26] since it has the representation

$$M(t) = \int_0^t \int_{U_1} E(s)By q(ds, dy), \quad t \in [0, T],$$

where q is the compensated Poisson random measure on $[0, \infty) \times U_1$ associated to L ; see the appendix for details. We obtain

$$\begin{aligned} G(x + M(t)) &= G(x) + \int_0^t \int_{U_1} \left\{ G(x + M(s-) + E(s)By) - G(x + M(s-)) \right\} q(ds, dy) \\ &\quad + \int_0^t \int_{U_1} \left\{ G(x + M(s) + E(s)By) - G(x + M(s)) \right. \\ &\quad \left. - \langle G'(x + M(s)), E(s)By \rangle_H \right\} \nu(dy) ds, \end{aligned} \quad (3.14)$$

where the integrand appearing in the integral w.r.t. q belongs to $L^2(\Omega_T \times U_1, \mathbb{P}_T \otimes \nu; \mathbb{R})$ as a consequence of Taylor's formula, the boundedness of G' , Lemma 3.1 and (2.11). Similarly, the second integral in (3.14) exists for all $\omega \in \Omega$ and belongs to $L^1(\Omega; \mathbb{R})$ since

$$\begin{aligned} & \int_0^t \int_{U_1} \left| G(x + M(s) + E(s)By) - G(x + M(s)) - \langle G'(x + M(s)), E(s)By \rangle_H \right| \nu(dy) ds \\ & \leq \frac{1}{2} \sup_{x \in H} \|G''(x)\|_{\mathcal{L}(H)} \int_0^t \|E(s)B\|_{\mathcal{L}_2(U_0, H)}^2 ds. \end{aligned}$$

Taking expectations in (3.14) and using the martingale property of the integral w.r.t. q yields

$$v(t, x) = G(x) + \int_0^t \int_{U_1} \left\{ v(s, x + E(s)By) - v(s, x) - \langle v_x(s, x), E(s)By \rangle_H \right\} \nu(dy) ds. \quad (3.15)$$

By the fundamental theorem of calculus, (3.12) follows from (3.13), (3.15) if the mapping

$$(0, T] \ni s \mapsto \int_{U_1} \left\{ v(s, x + E(s)By) - v(s, x) - \langle v_x(s, x), E(s)By \rangle_H \right\} \nu(dy) \in \mathbb{R}. \quad (3.16)$$

is continuous.

Note that we cannot apply directly the continuity theorem for parameter-dependent integrals to show the continuity of the mapping (3.16). The reason is that the term $E(s)By$ in the integral in (3.16) is defined only in an $L^2([0, T] \times U_1, \lambda \otimes \nu; H)$ -sense, cf. Lemma 3.1 and Remark 3.2, so that we have no information about the continuity of $(0, T] \ni s \mapsto E(s)By \in H$ for fixed $y \in U_1$. Therefore, we use an approximation argument: For $s \in (0, T]$, $x \in H$, $y \in U_1$ and $k \in \mathbb{N}$ set

$$\begin{aligned} f(s, x, y) &:= v(s, x + E(s)By) - v(s, x) - \langle v_x(s, x), E(s)By \rangle_H, \\ f_k(s, x, y) &:= f(s, x, \Pi_k y), \end{aligned}$$

where Π_k is the orthogonal projection of U_1 onto $\text{span}\{f_1, \dots, f_k\}$, $(f_k)_{k \in \mathbb{N}} \subset U_0$ being an orthonormal basis of U_1 as in Lemma 3.1. For fixed $x \in H$, $f(s, x, y)$ is defined in an $L^2([0, T] \times U_1, \lambda \otimes \nu; \mathbb{R})$ -sense whereas $f_k(s, x, y)$ is defined pointwise. The continuity theorem for parameter-dependent integrals and the strong continuity of $(E(t))_{t \geq 0}$ yield the continuity of $\int_{U_1} f_k(s, x, y) \nu(dy)$ in $(s, x) \in [0, T] \times H$. Moreover, we have $f_k(s, x, \cdot) \xrightarrow{k \rightarrow \infty} f(s, x, \cdot)$ in $L^1(U_1, \nu; \mathbb{R})$, uniformly in $(s, x) \in [\varepsilon, T] \times H$ for all $\varepsilon > 0$. Indeed, setting $\Pi^k y := y - \Pi_k y$ and using Taylor's theorem, Lemma 3.1 and Assumption 2.6(v), we obtain

$$\begin{aligned} & \int_{U_1} |f(s, x, y) - f_k(s, x, y)| \nu(dy) \\ & \leq \int_{U_1} \int_0^1 \left| \langle v_{xx}(s, x + E(s)B(\Pi_k y + \theta \Pi^k y)) E(s)B \Pi^k y, E(s)B \Pi^k y \rangle_H \right| (1 - \theta) d\theta \nu(dy) \\ & \quad + \int_{U_1} \int_0^1 \left| \langle v_{xx}(s, x + \theta E(s)B \Pi_k y) E(s)B \Pi_k y, E(s)B \Pi^k y \rangle_H \right| d\theta \nu(dy) \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{x \in H} \|G''(x)\|_{\mathcal{L}(H)} \left(\|E(s)B\Pi^k\|_{\mathcal{L}_2(U_0, H)}^2 + \|E(s)B\Pi_k\|_{\mathcal{L}_2(U_0, H)} \|E(s)B\Pi^k\|_{\mathcal{L}_2(U_0, H)} \right) \\
&\leq \sup_{x \in H} \|G''(x)\|_{\mathcal{L}(H)} \left(\|\Phi_\varepsilon \Pi^k\|_{\mathcal{L}_2(U_0, H)}^2 + \|\Phi_\varepsilon \Pi_k\|_{\mathcal{L}_2(U_0, H)} \|\Phi_\varepsilon \Pi^k\|_{\mathcal{L}_2(U_0, H)} \right)
\end{aligned}$$

for all $s \in [\varepsilon, T]$ and some $\Phi_\varepsilon \in \mathcal{L}_2(U_0, H)$. The expression in the last line tends to zero as $k \rightarrow \infty$. As a consequence, $\int_{U_1} f_k(s, x, y) \nu(dy) \xrightarrow{k \rightarrow \infty} \int_{U_1} f(s, x, y) \nu(dy)$ uniformly in $(s, x) \in [\varepsilon, T] \times H$. Thus, the continuity of $\int_{U_1} f_k(s, x, y) \nu(dy)$ in $(s, x) \in [0, T] \times H$ implies the continuity of $\int_{U_1} f(s, x, y) \nu(dy)$ in $(s, x) \in (0, T] \times H$. In particular, we obtain the continuity of the mapping (3.16) as well as the continuity of u_t on $[0, T] \times H$. \square

The regularity assertions in Proposition 3.6 allow us to apply Itô's formula [26, Theorem 3.6] to the function $(t, x) \mapsto u(t, x)$ and the martingale $\tilde{Y} = (\tilde{Y}(t))_{t \in [0, T]}$ defined in (3.2). Note that \tilde{Y} fits into the setting of [26] since it has the representation

$$\tilde{Y}(t) = \tilde{E}(T)X_0 + \int_0^t \int_{U_1} \tilde{E}(T-s)By q(ds, dy), \quad t \in [0, T], \quad (3.17)$$

where again q is the compensated Poisson random measure on $[0, \infty) \times U_1$ associated with L as described in the appendix. Equality (3.17) is a consequence of (2.12), Lemma 3.1, Remark 3.2 and Lemma A.2. For $T' \in (0, T)$ we obtain

$$\begin{aligned}
u(T', \tilde{Y}(T')) &= u(0, \tilde{Y}(0)) + \int_0^{T'} u_t(t, \tilde{Y}(t)) dt \\
&+ \int_0^{T'} \int_{U_1} \left\{ u(t, \tilde{Y}(t-) + \tilde{E}(T-t)By) - u(t, \tilde{Y}(t-)) \right\} q(ds, dy) \\
&+ \int_0^{T'} \int_{U_1} \left\{ u(t, \tilde{Y}(t) + \tilde{E}(T-t)By) - u(t, \tilde{Y}(t)) - \left\langle u_x(t, \tilde{Y}(t)), \tilde{E}(T-t)By \right\rangle_H \right\} \nu(dy) ds.
\end{aligned} \quad (3.18)$$

Using the boundedness of u , u_x and u_{xx} , (3.12), (2.11) and applying similar arguments as in the proof of Proposition 3.6, one sees that all terms in (3.18) are well-defined and integrable w.r.t. \mathbb{P} . Thus, we can take expectations and use the martingale property of the integral w.r.t. q and the backward Kolmogorov equation (3.12) to obtain

$$\begin{aligned}
\mathbb{E} \left(u(T', \tilde{Y}(T')) - u(0, \tilde{Y}(0)) \right) &= \\
\mathbb{E} \int_0^{T'} \int_{U_1} \left\{ u(t, \tilde{Y}(t) + \tilde{E}(T-t)By) - u(t, \tilde{Y}(t) + E(T-t)By) \right. \\
&\quad \left. - \left\langle u_x(t, \tilde{Y}(t)), (\tilde{E}(T-t)B - E(T-t)B)y \right\rangle_H \right\} \nu(dy) dt
\end{aligned} \quad (3.19)$$

for all $T' \in (0, T)$. Taking the limit $T' \rightarrow T$ on both sides of (3.19), we can replace T' by T . Here we used the stochastic continuity of \tilde{Y} and the continuity of u for the limit on the left hand side. The combination of (3.10) and (3.19) yields the error representation formula (3.7).

4 Application to the heat equation

In this section, we give a detailed error analysis of a space-time discretization of the linear stochastic heat equation with additive Lévy noise.

Let $\mathcal{O} \subset \mathbb{R}^d$ be a convex bounded domain with a C^∞ -boundary; if $d = 2$ we also allow for convex bounded domains with a polygonal boundary. Let $\Lambda := -\Delta = -\sum_{j=1}^d \partial^2 / \partial \xi_j^2$ be the Laplace operator on $L^2(\mathcal{O})$ with zero-Dirichlet boundary condition, i.e., with domain $D(\Lambda) := H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$. As usual, $H^n(\mathcal{O})$ denotes the L^2 -Sobolev space of order $n \in \mathbb{N}_0$ on \mathcal{O} and $H_0^1(\mathcal{O})$ is the $H^1(\mathcal{O})$ -closure of the space $C_c^\infty(\mathcal{O})$ of compactly supported test functions. Then, setting

$$H := U := L^2(\mathcal{O}), \quad (A, D(A)) := (\Lambda, D(\Lambda)), \quad B := \text{id}_{L^2(\mathcal{O})},$$

the abstract equation (1.2) becomes the stochastic heat equation (1.3). It is not difficult to see that the condition $\|\Lambda^{-1/2}Q^{1/2}\|_{\mathcal{L}_2(H)} < \infty$ implies (2.11), where

$$(E(t))_{t \geq 0} := (e^{-t\Lambda})_{t \geq 0} \subset \mathcal{L}(H) \tag{4.1}$$

is the semigroup generated by $-A = -\Lambda$. Hence, there exists a unique weak solution $X = (X(t))_{t \geq 0}$ to Eq. (1.3), given by the variation-of-constants formula (1.1). In the sequel, we use the smoothness spaces \dot{H}^α , $\alpha \in \mathbb{R}$, defined by

$$\begin{aligned} \dot{H}^\alpha &:= D(\Lambda^{\alpha/2}) \\ &:= \left\{ v = \sum_{k=1}^{\infty} v_k \varphi_k : (v_k)_{k \in \mathbb{N}} \subset \mathbb{R}, |v|_\alpha := \|\Lambda^{\alpha/2} v\|_{L^2(\mathcal{O})} = \left(\sum_{k=1}^{\infty} \lambda_k^\alpha v_k^2 \right)^{1/2} < \infty \right\}, \end{aligned}$$

where $(\varphi_k)_{k \in \mathbb{N}} \subset D(\Lambda)$ is an orthonormal basis of $L^2(\mathcal{O})$ consisting of eigenfunctions of Λ and $(\lambda_k)_{k \in \mathbb{N}} \subset (0, \infty)$ is the corresponding sequence of eigenvalues; compare [42, Chapters 3 and 19]. They are Hilbert spaces and one has the identities $\dot{H}^0 = H = L^2(\mathcal{O})$, $\dot{H}^1 = H_0^1(\mathcal{O})$ and $\dot{H}^2 = D(\Lambda) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$, where the natural norms of the respective spaces are equivalent. For negative α , the elements of \dot{H}^α are formal sums and we identify them with elements of $L^2(\mathcal{O})$ if $\sum_{k=1}^{\infty} v_k^2 < \infty$, so that \dot{H}^α is the closure of $L^2(\mathcal{O})$ w.r.t. the $|\cdot|_\alpha$ -norm.

Remark 4.1. The spaces \dot{H}^α , $\alpha \in \mathbb{R}$, can be obtained by both real and complex interpolation: For $\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1$, $\theta \in (0, 1)$, one has $\dot{H}^\alpha = (\dot{H}^{\alpha_0}, \dot{H}^{\alpha_1})_{\theta, 2} = [\dot{H}^{\alpha_0}, \dot{H}^{\alpha_1}]_\theta$ with equivalent norms, where $(\cdot, \cdot)_{\theta, 2}$ and $[\cdot, \cdot]_\theta$ denotes real interpolation with summability parameter $q = 2$ and complex interpolation, respectively. This follows, e.g., from [43, Theorem 1.18.5] and the fact that the spaces \dot{H}^α , $\alpha \in \mathbb{R}$, are isometrically isomorphic to weighted ℓ^2 -spaces. We will frequently use the corresponding interpolation inequalities in this and the next section.

For the spatial discretization of Eq. (1.3), we take a family of finite element spaces $(S_h)_{h>0} \subset H_0^1(\mathcal{O})$, consisting of piecewise linear functions with respect to a family of triangulations of \mathcal{O} . The parameter h corresponds to the maximal mesh size of the triangulation.

Unless otherwise stated, we endow the finite-dimensional spaces S_h with the inner product $\langle \cdot, \cdot \rangle_H$ and the norm $\|\cdot\|_H$. By $P_h : H \rightarrow S_h$ and $\Pi_h : \dot{H}^1 \rightarrow S_h$ we denote the orthogonal projections with respect to the inner products in H and \dot{H}^1 , respectively. The discrete Laplacian $\Lambda_h : S_h \rightarrow S_h$ is defined by

$$\langle \Lambda_h v, w \rangle_{L^2(\mathcal{O})} = \langle \nabla v, \nabla w \rangle_{L^2(\mathcal{O}; \mathbb{R}^2)}, \quad v, w \in S_h. \quad (4.2)$$

Under our assumptions on \mathcal{O} the Ritz projection Π_h satisfies the standard elliptic error estimate

$$\|\Pi_h v - v\|_{L^2(\mathcal{O})} \leq Ch^\beta |v|_\beta, \quad v \in \dot{H}^\beta, \quad 1 \leq \beta \leq 2, \quad (4.3)$$

see, e.g., [42, Lemma 1.1] or [24, Section 5.4].

The time discretization of Eq. (1.3) on a finite interval $[0, T]$ is done via the implicit Euler scheme with time step $\Delta t = T/N$, $N \in \mathbb{N}$, and grid points $t_n = n\Delta t$, $n = 0, \dots, N$. For $h > 0$ and $N \in \mathbb{N}$, the discretization $(X_{h,\Delta t}^n)_{n=1, \dots, N}$ of $(X(t))_{t \in [0, T]}$ in space and time is given as the solution to

$$X_{h,\Delta t}^n - X_{h,\Delta t}^{n-1} + \Delta t \Lambda_h X_{h,\Delta t}^n = P_h(L(t_n) - L(t_{n-1})), \quad n = 1, \dots, N; \quad X_{h,\Delta t}^0 = P_h X_0. \quad (4.4)$$

Remark 4.2 (strong error). If the covariance operator $Q \in \mathcal{L}(H)$ of L is such that

$$\|\Lambda^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathcal{L}_2(H)} < \infty \quad (4.5)$$

for some $\beta \geq 0$, then the solution $X(t)$ takes values in \dot{H}^β for all $t > 0$. For the Gaussian case, i.e., the case where L in (1.3) is a Q -Wiener process, it has been shown in [45, Theorem 1.2] that, if (4.5) holds and $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \dot{H}^\beta)$ for some $\beta \in (0, 1]$, then the scheme (4.4) has strong convergence of order β in space and $\beta/2$ in time:

$$\|X_{h,\Delta t}^n - X(t_n)\|_{L^2(\Omega; H)} \leq C(h^\beta + (\Delta t)^{\frac{\beta}{2}}), \quad n = 0, \dots, N.$$

Unlike weak error estimates, strong L^2 -error estimates are the same in the Gaussian case and in our setting, since the only stochastic tool that is needed is Itô's isometry (2.13) which looks the same if the driving noise is a Lévy process which is an L^2 -martingale. Thus the strong error result in [45, Theorem 1.2] carries over one-to-one to our setting.

Remark 4.3. The S_h -valued random variables $P_h(L(t_n) - L(t_{n-1}))$ in (4.4) can be defined in two ways. On the one hand, we may set

$$P_h(L(t_n) - L(t_{n-1})) := L^2(\Omega; S_h)\text{-}\lim_{K \rightarrow \infty} \sum_{k=1}^K (L_k(t_n) - L_k(t_{n-1})) P_h e_k,$$

with an orthonormal basis $(e_k)_{k \in \mathbb{N}}$ of U_0 and real-valued uncorrelated Lévy processes $L_k = (L_k(t))_{t \geq 0}$, $k \in \mathbb{N}$, as in Remark 2.2. The limit exists since, by the finite-dimensionality of S_h , one has $P_h \in \mathcal{L}_2(H, S_h) = \mathcal{L}_2(U, S_h) \subset \mathcal{L}_2(U_0, S_h)$. On the other hand, we can extend the orthogonal projection $P_h : H \rightarrow S_h$ to a generalized L^2 -projection $P_h : \dot{H}^{-1} \rightarrow S_h$ defined by

$$\langle P_h v, w \rangle_H = \langle v, w \rangle_{\dot{H}^{-1} \times \dot{H}^1}, \quad v \in \dot{H}^{-1}, \quad w \in S_h.$$

Then, the assumption $\|\Lambda^{-1/2}Q^{1/2}\|_{\mathcal{L}_2(H)} < \infty$ implies that we can take $U_1 := D(\Lambda^{-1/2}) = \dot{H}^{-1}$ as the state space of L , so that the expression $P_h(L(t_n) - L(t_{n-1}))$ makes sense ω -wise. Obviously, both definitions are compatible. In practice, one has to find a suitable way to sample (an approximation of) the discretized noise increment $P_h(L(t_n) - L(t_{n-1}))$. We do not treat this problem in the present paper but refer to [5, 13] and [25, Remark 4] for related considerations.

With $R(\lambda) := 1/(1 + \lambda)$ and $E_{h,\Delta t} := R(\Delta t\Lambda_h) := (I + \Delta t\Lambda_h)^{-1}$ as well as $E_{h,\Delta t}^n := R^n(\Delta t\Lambda_h) := ((I + \Delta t\Lambda_h)^{-1})^n$, the scheme (4.4) can be rewritten as

$$X_{h,\Delta t}^n = E_{h,\Delta t}^n P_h X_0 + \sum_{j=1}^n E_{h,\Delta t}^{n-j+1} P_h (L(t_j) - L(t_{j-1})), \quad n = 0, \dots, N.$$

For $t \in [0, T]$, let $\tilde{E}(t) = \tilde{E}_{h,\Delta t}(t) \in \mathcal{L}(H)$ be defined by

$$\tilde{E}(t) = \tilde{E}_{h,\Delta t}(t) := \mathbf{1}_{\{0\}}(t)P_h + \sum_{n=1}^N \mathbf{1}_{(t_{n-1}, t_n]}(t)E_{h,\Delta t}^n P_h \quad (4.6)$$

and set

$$\tilde{X}(t) = \tilde{X}_{h,\Delta t}(t) := \tilde{E}_{h,\Delta t}(t)X_0 + \int_0^t \tilde{E}_{h,\Delta t}(t-s) dL(s). \quad (4.7)$$

Then $X_{h,\Delta t}^n = \tilde{X}_{h,\Delta t}(t_n)$ \mathbb{P} -almost surely. This follows from the construction of the stochastic integral, using an approximation argument and Itô's isometry (2.13).

The following deterministic estimates will be used in the proof of our weak error result stated in Theorem 4.5 below.

Lemma 4.4. *The operators $E(t)$ and $\tilde{E}(t) = \tilde{E}_{h,\Delta t}(t)$ defined in (4.1) and (4.6) satisfy the error estimates*

$$\|\tilde{E}(s) - E(s)\|_{\mathcal{L}(H)} \leq C(h^2 + \Delta t)s^{-1}, \quad (4.8)$$

$$\|\Lambda^\alpha E(s)\|_{\mathcal{L}(H)} + \|\Lambda^\alpha \tilde{E}(s)\|_{\mathcal{L}(H)} \leq Cs^{-\alpha}, \quad 0 \leq \alpha \leq 1/2, \quad (4.9)$$

$s \in (0, T]$, where $C > 0$ does not depend on h , Δt and s .

Proof. Estimate (4.8) follows from

$$\|E_{h,\Delta t}^n P_h - E(t_n)\|_{\mathcal{L}(H)} \leq C(h^2 + \Delta t)t_n^{-1},$$

see, for example, [42, Theorem 7.7]. We note here that while the latter result is proved under the assumption that \mathcal{O} has smooth boundary, the proof relies on the availability of (4.3) and the analyticity of the heat semigroup and hence it holds for planar convex polygonal domains as well, with the proof carrying over verbatim. For $s \in (t_{n-1}, t_n]$ we have

$$\|(E(t_n) - E(s))v\|_H = \|\Lambda E(s)(E(t_n - s) - \text{id}_H)\Lambda^{-1}v\|_H$$

$$\begin{aligned} &\leq \|\Lambda E(s)\|_{\mathcal{L}(H)} \|(E(t_n - s) - \text{id}_H)\Lambda^{-1}v\|_H \\ &\leq Cs^{-1}\Delta t \|v\|_H, \end{aligned}$$

where we used Theorem 6.13(c),(d) on analytic semigroups in [31, Chapter 2]. Estimate (4.9) is due to Theorem 6.13(c) in [31, Chapter 2], Lemma 7.3 in [42], interpolation, and the fact that $\|A^\alpha v_h\| \leq \|A_h^\alpha v_h\|$ for $v_h \in S_h$, $0 \leq \alpha \leq 1/2$. The latter follows from the basic identity $\|A^{1/2}v_h\| = \|A_h^{1/2}v_h\|$ and interpolation. \square

Here is our result for the weak error of the discretization of the stochastic heat equation.

Theorem 4.5. *Assume that $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathcal{L}_2(H)} < \infty$ for some $\beta \in (0, 1]$. Let $(X(t))_{t \geq 0}$ be the weak solution (1.1) to Eq. (1.2) and let $(X_{h,\Delta t}^n)_{n=0,\dots,N}$ be defined by the scheme (4.4). Given $g \in C_b^2(H, \mathbb{R})$, there exists a constant $C = C(g, T) > 0$ that does not depend on h and Δt , such that*

$$\left| \mathbb{E} \left(g(X_{h,\Delta t}^N) - g(X(T)) \right) \right| \leq C(h^{2\beta} + (\Delta t)^\beta) |\log(h^2 + \Delta t)|$$

for $h^2 + \Delta t \leq 1/e$.

Proof. We are in the setting of Section 2 with $H = U = L^2(\mathcal{O})$, $B = \text{id}_H$, and $(E(t))_{t \geq 0}$, $(\tilde{E}(t))_{t \in [0, T]} = (\tilde{E}_{h,\Delta t}(t))_{t \in [0, T]}$, $(\tilde{X}(t))_{t \in [0, T]} = (\tilde{X}_{h,\Delta t}(t))_{t \in [0, T]}$ being given by (4.1), (4.6), (4.7) respectively. In particular, Assumption 2.6 is fulfilled. Since $X_{h,\Delta t}^N = \tilde{X}(T)$, we can use Corollary 3.5 with $G := g$ to estimate the weak error. Let $F(t) := \tilde{E}(t) - E(t)$ be the deterministic error operator.

We begin with the first term on the right hand side of the formula (3.9) in Corollary 3.5. The mean value theorem and the deterministic estimate (4.8) yield, for $\max(h^2, \Delta t) \leq 1$,

$$\begin{aligned} \left| \mathbb{E} \left\{ u(0, \tilde{E}(T)X_0) - u(0, E(T)X_0) \right\} \right| &\leq \sup_{x \in H} \|u_x(0, x)\|_H \mathbb{E} \left(\|F(T)X_0\|_H \right) \\ &\leq \sup_{x \in H} \|g'(x)\|_H \mathbb{E} \left(\|F(T)X_0\|_H \right) \\ &\leq C \sup_{x \in H} \|g'(x)\|_H (h^2 + \Delta t) T^{-1} \mathbb{E} \left(\|X_0\|_H \right) \\ &\leq C \sup_{x \in H} \|g'(x)\|_H T^{-1} \mathbb{E} \left(\|X_0\|_H \right) (h^{2\beta} + (\Delta t)^\beta). \end{aligned} \tag{4.10}$$

Next, consider the second term on the right hand side of (3.9). We estimate the integrals of the functions Ψ_1 and Ψ_2 separately. Using Lemma 3.1 and Remark 3.2, we obtain

$$\begin{aligned} &\left| \mathbb{E} \int_0^T \int_{U_1} \int_0^1 \Psi_1(t, \theta, y) \, d\theta \, \nu(dy) \, dt \right| \\ &\leq \sup_{x \in H} \|g''(x)\|_{\mathcal{L}(H)} \int_0^T \int_{U_1} \|F(T-t)y\|_H^2 \, \nu(dy) \, dt \\ &= \sup_{x \in H} \|g''(x)\|_{\mathcal{L}(H)} \int_0^T \|F(T-t)\|_{\mathcal{L}_2(U_0, H)}^2 \, dt \\ &\leq C \sup_{x \in H} \|g''(x)\|_{\mathcal{L}(H)} (h^{2\beta} + (\Delta t)^\beta). \end{aligned} \tag{4.11}$$

The last step is due to the fact that, by Itô's isometry (2.13), the integral in the penultimate line is the square of the strong error $\|X_{h,\Delta t}^N - X(T)\|_{L^2(\Omega;H)}$ for zero initial condition $X_0 = 0$, which can be estimated as in the Gaussian case [45, Theorem 1.2], compare Remark 4.2. Further, by the Cauchy-Schwarz inequality, Lemma 3.1, and the fact that $U_0 = Q^{1/2}(U)$,

$$\begin{aligned}
& \left| \mathbb{E} \int_0^T \int_{U_1} \int_0^1 \Psi_2(t, \theta, y) \, d\theta \, \nu(dy) \, dt \right| \\
& \leq \sup_{x \in H} \|g''(x)\|_{\mathcal{L}(H)} \int_0^T \int_{U_1} \|E(T-t)y\|_H \|F(T-t)y\|_H \, \nu(dy) \, dt \\
& \leq \sup_{x \in H} \|g''(x)\|_{\mathcal{L}(H)} \int_0^T \|E(T-t)\|_{\mathcal{L}_2(U_0,H)} \|F(T-t)\|_{\mathcal{L}_2(U_0,H)} \, dt \\
& \leq \sup_{x \in H} \|g''(x)\|_{\mathcal{L}(H)} \|\Lambda^{\frac{\beta-1}{2}} Q^{1/2}\|_{\mathcal{L}_2(H)}^2 \int_0^T \|E(t)\Lambda^{\frac{1-\beta}{2}}\|_{\mathcal{L}(H)} \|F(t)\Lambda^{\frac{1-\beta}{2}}\|_{\mathcal{L}(H)} \, dt
\end{aligned} \tag{4.12}$$

By (4.9) we have

$$\|E(t)\Lambda^{\frac{1-\beta}{2}}\|_{\mathcal{L}(H)} = \|\Lambda^{\frac{1-\beta}{2}} E(t)\|_{\mathcal{L}(H)} \leq Ct^{-\frac{1-\beta}{2}} \tag{4.13}$$

and

$$\|\Lambda^\alpha F(t)\|_{\mathcal{L}(H)} \leq Ct^{-\alpha}, \quad 0 \leq \alpha \leq 1/2. \tag{4.14}$$

Interpolation between (4.8) and (4.14) with $\alpha = 1/2$ gives

$$\|\Lambda^{\frac{1-\beta}{2}} F(t)\|_{\mathcal{L}(H)} \leq C \|F(t)\|_{\mathcal{L}(H)}^\beta \|\Lambda^{\frac{1}{2}} F(t)\|_{\mathcal{L}(H)}^{1-\beta} \leq C(h^2 + \Delta t)^\beta t^{-\frac{1+\beta}{2}}. \tag{4.15}$$

Note that $\|F(t)\Lambda^\alpha\|_{\mathcal{L}(H)} = \|\Lambda^\alpha F(t)\|_{\mathcal{L}(H)}$ due to the self adjointness of $\tilde{E}(t)$, $E(t)$ and Λ^α . Altogether, using (4.13), (4.14) and (4.15), the integral in the last line of (4.12) can be estimated by

$$\begin{aligned}
& \int_0^T \|E(t)\Lambda^{\frac{1-\beta}{2}}\|_{\mathcal{L}(H)} \|F(t)\Lambda^{\frac{1-\beta}{2}}\|_{\mathcal{L}(H)} \, dt \\
& = \left(\int_0^{h^2+\Delta t} + \int_{h^2+\Delta t}^T \right) \|\Lambda^{\frac{1-\beta}{2}} E(t)\|_{\mathcal{L}(H)} \|\Lambda^{\frac{1-\beta}{2}} F(t)\|_{\mathcal{L}(H)} \, dt \\
& \leq C \int_0^{h^2+\Delta t} t^{-\frac{1-\beta}{2}} t^{-\frac{1-\beta}{2}} \, dt + C \int_{h^2+\Delta t}^T t^{-\frac{1-\beta}{2}} (h^2 + \Delta t)^\beta t^{-\frac{1+\beta}{2}} \, dt \\
& = C(h^2 + \Delta t)^\beta (1 + |\log(h^2 + \Delta t)|) \\
& \leq C(h^{2\beta} + (\Delta t)^\beta) |\log(h^2 + \Delta t)|.
\end{aligned} \tag{4.16}$$

for $h^2 + \Delta t \leq 1/e$, where $C > 0$ depends on T . The combination of (4.10), (4.11), (4.12) and (4.16) finishes the proof. \square

5 Application to the wave equation

Here, we apply the general error representation from Section 3 to a discretization of the stochastic wave equation (1.4) via finite elements in space and a rational single step scheme in time.

Let $\mathcal{O} \subset \mathbb{R}^d$ be a convex bounded domain with a C^∞ -boundary and let the spaces \dot{H}^α , $\alpha \in \mathbb{R}$, be as in Section 4. We use the product spaces

$$\mathcal{H}^\alpha := \dot{H}^\alpha \times \dot{H}^{\alpha-1}, \quad \alpha \in \mathbb{R},$$

with inner product $\langle v, w \rangle_{\mathcal{H}^\alpha} := \langle v_1, w_1 \rangle_\alpha + \langle v_2, w_2 \rangle_{\alpha-1}$, $v = (v_1, v_2)^\top$, $w = (w_1, w_2)^\top$ and norm $\|v\|_{\mathcal{H}^\alpha} = (|v_1|_\alpha^2 + |v_2|_{\alpha-1}^2)^{1/2}$, where $\langle \cdot, \cdot \rangle_\alpha$ and $\langle \cdot, \cdot \rangle_{\alpha-1}$ are the inner products in \dot{H}^α and $\dot{H}^{\alpha-1}$ corresponding to the norms $|\cdot|_\alpha$ and $|\cdot|_{\alpha-1}$ introduced in Section 4. We set

$$H := \mathcal{H}^0 = \dot{H}^0 \times \dot{H}^{-1} = L^2(\mathcal{O}) \times H^{-1}(\mathcal{O}), \quad U := \dot{H}^0 = L^2(\mathcal{O})$$

and define operators $A : D(A) \subset H \rightarrow H$ and $B \in \mathcal{L}(U, H)$ by setting $D(A) := \mathcal{H}^1$ and

$$A := \begin{pmatrix} 0 & -I \\ \Lambda & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 \\ I \end{pmatrix},$$

where the Laplace operator Λ from Section 4 is now considered as an operator from \dot{H}^1 to \dot{H}^{-1} . It is well-known that $-A$ generates a strongly continuous semigroup $(E(t))_{t \geq 0} \subset \mathcal{L}(H)$ given by

$$E(t) = \begin{pmatrix} C(t) & \Lambda^{-1/2}S(t) \\ -\Lambda^{1/2}S(t) & C(t) \end{pmatrix}, \quad (5.1)$$

where $C(t) := \cos(t\Lambda^{1/2})$ and $S(t) := \sin(t\Lambda^{1/2})$ are the cosine and sine operators; compare [32, Example B.1], [11, Section A.5.4] and [2, Section 3.14].

With these definitions the abstract equation (1.2) becomes the stochastic wave equation (1.4) with H -valued solution $(X(t))_{t \geq 0} = ((X_1(t), X_2(t))^\top)_{t \geq 0}$. As in the Gaussian case, cf. [19, Lemma 4.1], one sees that the condition $\|\Lambda^{-1/2}Q^{1/2}\|_{\mathcal{L}_2(\dot{H}^0)} < \infty$ implies (2.11) and hence the existence of a unique weak solution $X = (X(t))_{t \geq 0}$, given that the initial condition $X_0 = (X_{0,1}, X_{0,2})^\top$ is H -valued and \mathcal{F}_0 -measurable.

The discretization of Eq. (1.4) is done via finite elements of order $r = 2, 3$ in space and an I -stable rational single step scheme of order $p = 1, 2, \dots$ in time. (By ‘ I -stable’ we mean what is called ‘ I -acceptable’ in [30].) We use the finite element setting introduced in Section 4, the only difference being that now we also consider higher order elements. That is, the finite element spaces $S_h \subset H_0^1(\mathcal{O})$ consist of continuous piecewise polynomials of degree $r - 1$ w.r.t. the underlying triangulations of \mathcal{O} . Under our assumptions on \mathcal{O} the elliptic finite element error estimate

$$\|\Pi_h v - v\|_{L^2(\mathcal{O})} \leq Ch^\beta |v|_\beta, \quad v \in \dot{H}^\beta, \quad 1 \leq \beta \leq r, \quad (5.2)$$

holds instead of (4.3), see, for example, [42, Lemma 1.1]. Although (5.2) does not appear explicitly in the present paper, it is the key ingredient in the proof of the deterministic error estimate for the finite element approximation of the wave equation and hence we state it for the sake of completeness. We also note that for planar convex polygonal domains, (5.2) only holds with $r = 2$ without further restriction on the interior angles. Therefore, here

we only consider domains \mathcal{O} with smooth boundary for simplicity. Let the discretization $A_h : S_h \times S_h \rightarrow S_h \times S_h$ of the operator $A : D(A) \subset H \rightarrow H$ be defined by

$$A_h := \begin{pmatrix} 0 & -I \\ \Lambda_h & 0 \end{pmatrix},$$

where $\Lambda_h : S_h \rightarrow S_h$ is the discrete Laplacian introduced in (4.2). Then $-A_h$ generates a strongly continuous semigroup $(E_h(t))_{t \geq 0} \subset \mathcal{L}(S_h \times S_h)$. As in Section 4, we consider for $N \in \mathbb{N}$ a uniform grid $t_n = n\Delta t = n(T/N)$, $n = 0, \dots, N$, on a finite time interval $[0, T]$. We approximate the operators $E_h(t_n) \in \mathcal{L}(S_h \times S_h)$ by

$$E_{h,\Delta t}^n := (R(\Delta t A_h))^n,$$

where R is a rational function that satisfies the approximation and stability properties

$$\begin{aligned} |R(iy) - e^{-iy}| &\leq C|y|^{p+1}, \quad |y| \leq b, \\ |R(iy)| &\leq 1, \quad y \in \mathbb{R}, \end{aligned}$$

for some positive integer p and some $b > 0$; see [3, 7] for details. For instance, choosing $R(\lambda) = 1/(1 - \lambda)$ and $R(\lambda) = (2 - \lambda)/(2 + \lambda)$ yields the backward Euler method ($p = 1$) and the Crank-Nicolson method ($p = 2$), respectively.

The numerical scheme for the stochastic wave equation (1.2) can now be formulated as follows: For $h > 0$ and $N \in \mathbb{N}$, the discretization $(X_{h,\Delta t}^n)_{n=0,\dots,N}$ of $(X(t))_{t \in [0,T]}$ in space and time is given as the solution to

$$X_{h,\Delta t}^n = E_{h,\Delta t} \left(X_{h,\Delta t}^{n-1} + P_h B(L(t_n) - L(t_{n-1})) \right), \quad n = 1, \dots, N; \quad X_{h,\Delta t}^0 = P_h X_0. \quad (5.3)$$

By slight abuse of notation, we denote here and in the sequel by P_h both the generalized L^2 -projection from \dot{H}^{-1} onto S_h defined by $\langle P_h v, w \rangle_{L^2(\mathcal{O})} = \langle v, w \rangle_{\dot{H}^{-1} \times \dot{H}^1}$, $v \in \dot{H}^{-1}$, $w \in S_h$, and the corresponding projection from $H = \dot{H}^0 \times \dot{H}^{-1}$ onto $S_h \times S_h$ defined by the action of the former projection on the coordinates of elements in $\dot{H}^0 \times \dot{H}^{-1}$. Moreover, $P^1 : H \rightarrow \dot{H}^0$ is the projection of elements in $H = \dot{H}^0 \times \dot{H}^{-1}$ on the first coordinate.

Remark 5.1 (strong error). As observed for the discretization of the heat equation in Remark 4.2, strong L^2 -error estimates for the scheme (5.3) carry over from the Gaussian case in the Lévy L^2 -martingale case since they only use Itô's isometry (2.13). Arguing as in the proof of [19, Theorem 4.13], we obtain that, if

$$\|\Lambda^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\dot{H}^0)} < \infty \quad \text{and} \quad X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathcal{H}^\beta) \quad (5.4)$$

for some $\beta > 0$, then the scheme (5.3) approximates the first component $X_1 = P^1 X$ of the solution X to (1.2) with strong order $\min(\beta r/(r+1), r)$ in space and $\min(\beta p/(p+1), 1)$ in time:

$$\|X_{h,\Delta t,1}^n - X_1(t_n)\|_{L^2(\Omega; H^0)} \leq C \left(h^{\min(\beta \frac{r}{r+1}, r)} + (\Delta t)^{\min(\beta \frac{p}{p+1}, 1)} \right), \quad n = 0, \dots, N.$$

Here we have set $X_{h,\Delta t,1}^n := P^1 X_{h,\Delta t}^n$. The condition (5.4) implies that the solution $X = (X(t))_{t \geq 0}$ takes values in \mathcal{H}^β , cf. [22, Theorem 3.1].

The solution to the scheme (5.3) is given by

$$X_{h,\Delta t}^n = E_{h,\Delta t}^n P_h X_0 + \sum_{j=1}^n E_{h,\Delta t}^{n-j+1} P_h B(L(t_n) - L(t_{n-1})), \quad n = 0, \dots, N.$$

For $t \in [0, T]$, define operators $\tilde{E}(t) = \tilde{E}_{h,\Delta t}(t) \in \mathcal{L}(H)$ by

$$\tilde{E}(t) = \tilde{E}_{h,\Delta t}(t) := \mathbf{1}_{\{0\}}(t) P_h + \sum_{j=1}^N \mathbf{1}_{(t_{n-1}, t_n]}(t) E_{h,\Delta t}^n P_h, \quad (5.5)$$

where the projection P_h is understood as a mapping from $H = \dot{H}^0 \times \dot{H}^{-1}$ to $S_h \times S_h$. Then, analogously to the corresponding argument in Section 4, one sees that the $S_h \times S_h$ -valued process $(\tilde{X}(t))_{t \in [0, T]} = (\tilde{X}_{h,\Delta t}(t))_{t \in [0, T]}$ defined by

$$\tilde{X}(t) = \tilde{X}_{h,\Delta t}(t) := \tilde{E}_{h,\Delta t}(t) X_0 + \int_0^t \tilde{E}_{h,\Delta t}(t-s) B \, dL(s) \quad (5.6)$$

satisfies $X_{h,\Delta t}^n = \tilde{X}(t_n)$ \mathbb{P} -almost surely.

The proof of the deterministic error estimate in the next lemma is postponed to the end of this section.

Lemma 5.2. *Let $\alpha \geq 0$. The operators $E(t)$ and $\tilde{E}(t) = \tilde{E}_{h,\Delta t}(t)$ defined in (5.1) and (5.5) satisfy the error estimate*

$$\begin{aligned} \sup_{t \in [0, T]} \left(\|P^1(\tilde{E}(t) - E(t))\|_{\mathcal{L}(\mathcal{H}^\alpha, \dot{H}^0)} + \|P^1(\tilde{E}(t) - E(t))B\|_{\mathcal{L}(\dot{H}^{(\alpha/2)-1}, \dot{H}^{-\alpha/2})} \right) \\ \leq C \left(h^{\min(\alpha \frac{r}{r+1}, r)} + (\Delta t)^{\min(\alpha \frac{p}{p+1}, 1)} \right), \end{aligned} \quad (5.7)$$

for $\Delta t \leq 1$, where $C = C(T) > 0$ does not depend on h and Δt .

We are now in the position to prove the following result concerning the weak error of the approximation $X_{h,\Delta t,1}^N := P^1 X_{h,\Delta t}^N$ of the first component $X_1(T) = P^1 X(T)$ of the solution to the stochastic wave equation (1.2) at time T .

Theorem 5.3. *Assume that $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathcal{H}^{2\beta})$ and $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathcal{L}_2(\dot{H}^0)} < \infty$ for some $\beta > 0$. Let $(X(t))_{t \geq 0}$ be the weak solution (1.1) to Eq. (1.2) and let $(X_{h,\Delta t}^n)_{n=0, \dots, N}$ be given by the scheme (5.3). Let $g \in C_b^2(\dot{H}^0, \mathbb{R})$ be such that*

$$\sup_{x \in \dot{H}^0} \|\Lambda^{\frac{\beta}{2}} g''(x) \Lambda^{-\frac{\beta}{2}}\|_{\mathcal{L}(\dot{H}^0)} < \infty. \quad (5.8)$$

Then, there exists a constant $C = C(g, T) > 0$ that does not depend on h and Δt , such that

$$\left| \mathbb{E} \left(g(X_{h,\Delta t,1}^N) - g(X_1(T)) \right) \right| \leq C \left(h^{\min(2\beta \frac{r}{r+1}, r)} + (\Delta t)^{\min(2\beta \frac{p}{p+1}, 1)} \right)$$

for $\Delta t \leq 1$.

Proof. We apply Theorem 3.3 and Corollary 3.5 with $G = g \circ P^1$. Note that $G'(x) = (P^1)^*g'(P^1x) \in H$ and $G''(x) = (P^1)^*g''(P^1x)P^1 \in \mathcal{L}(H)$ for all $x \in H$, where $(P^1)^* \in \mathcal{L}(\dot{H}^0, H)$ is the Hilbert space adjoint of $P^1 \in \mathcal{L}(H, \dot{H}^0)$. Using (3.5) one obtains

$$u_x(t, \xi) = \mathbb{E}\left(\left.(P^1)^*g'(P^1Z(T, t, x))\right)\right|_{x=\xi}, \quad u_{xx}(t, \xi) = \mathbb{E}\left(\left.(P^1)^*g''(P^1Z(T, t, x))P^1\right)\right|_{x=\xi} \quad (5.9)$$

for all H -valued random variables ξ and $t \in [0, T]$.

We combine (5.9) and the deterministic error estimate (5.7) with $\alpha = 2\beta$ in order to estimate the first term on the right hand side of the error representation formula (3.9) in Corollary 3.5:

$$\begin{aligned} & \left| \mathbb{E}\left\{u(0, \tilde{E}(T)X_0) - u(0, E(T)X_0)\right\} \right| \\ &= \left| \mathbb{E}\left\{u(0, \tilde{Y}(0)) - u(0, Y(0))\right\} \right| \\ &= \left| \mathbb{E} \int_0^1 \left\langle u_x(0, Y(0) + \theta(\tilde{Y}(0) - Y(0))), \tilde{Y}(0) - Y(0) \right\rangle_H d\theta \right| \\ &= \left| \mathbb{E} \int_0^1 \left\langle \mathbb{E}\left(g'(P^1Z(T, 0, x))\right)\right|_{x=Y(0)+\theta(\tilde{Y}(0)-Y(0))}, P^1(\tilde{Y}(0) - Y(0)) \right\rangle_{\dot{H}^0} d\theta \right| \quad (5.10) \\ &\leq \sup_{x \in \dot{H}^0} \|g'(x)\|_{\dot{H}^0} \mathbb{E}\left(\|P^1(\tilde{E}(T) - E(T))X_0\|_{\dot{H}^0}\right) \\ &\leq \sup_{x \in \dot{H}^0} \|g'(x)\|_{\dot{H}^0} \|P^1(\tilde{E}(T) - E(T))\|_{\mathcal{L}(\mathcal{H}^{2\beta}, \dot{H}^0)} \|X_0\|_{L^1(\Omega; \mathcal{H}^{2\beta})} \\ &\leq \sup_{x \in \dot{H}^0} \|g'(x)\|_{\dot{H}^0} \|X_0\|_{L^1(\Omega; \mathcal{H}^{2\beta})} C\left(h^{\min(2\beta\frac{\tau}{\tau+1}, r)} + (\Delta t)^{\min(2\beta\frac{p}{p+1}, 1)}\right). \end{aligned}$$

Using (5.9), Lemma 3.1 and Remark 3.2, the integral of the function Ψ_1 in the second term on the right hand side of the formula (3.9) can be treated as follows:

$$\begin{aligned} & \left| \mathbb{E} \int_0^T \int_{U_1} \int_0^1 \Psi_1(t, \theta, y) d\theta \nu(dy) dt \right| \\ &= \left| \mathbb{E} \int_0^T \int_{U_1} \int_0^1 (1 - \theta) \left\langle \mathbb{E}\left(g''(P^1Z(T, t, x + E(T-t)By + \theta F(T-t)y))\right)\right|_{x=\tilde{Y}(t)} \right. \\ &\quad \left. \times P^1F(T-t)y, P^1F(T-t)y \right\rangle_{\dot{H}^0} d\theta \nu(dy) dt \right| \\ &\leq \sup_{x \in \dot{H}^0} \|g''(x)\|_{\mathcal{L}(\dot{H}^0)} \int_0^T \|P^1F(T-t)\|_{\mathcal{L}_2(U_0, \dot{H}^0)}^2 dt \\ &\leq \sup_{x \in \dot{H}^0} \|g''(x)\|_{\mathcal{L}(\dot{H}^0)} C\left(h^{\min(\beta\frac{\tau}{\tau+1}, r)} + (\Delta t)^{\min(\beta\frac{p}{p+1}, 1)}\right)^2 \quad (5.11) \end{aligned}$$

The last step in (5.11) is due to the fact that, by Itô's isometry (2.13), the integral in the penultimate line is the square of the strong error $\|X_{h,k,1}^N - X_1(T)\|_{L^2(\Omega; \dot{H}^0)}$ for zero initial condition $X_0 = 0$; it can be estimated as in the Gaussian case [19, Theorem 4.13], compare Remark 5.1.

Concerning the integral of the function Ψ_2 in the second term on the right hand side of Eq. (3.9), we have by (5.9), Lemma 3.1, (2.15) and since $U_0 = Q^{1/2}(U) = Q^{1/2}(\dot{H}^0)$,

$$\begin{aligned}
& \left| \mathbb{E} \int_0^T \int_{U_1} \int_0^1 \Psi_2(t, \theta, y) \, d\theta \, \nu(dy) \, dt \right| \\
&= \left| \mathbb{E} \int_0^T \int_{U_1} \int_0^1 \left\langle \mathbb{E} \left(g'' \left(P^1 Z(T, t, x + \theta E(T-t)By) \right) \right) \Big|_{x=\tilde{Y}(t)} \right. \right. \\
&\quad \left. \left. \times P^1 E(T-t)By, P^1 F(T-t)y \right\rangle_{\dot{H}^0} \, d\theta \, \nu(dy) \, dt \right| \\
&= \left| \mathbb{E} \int_0^T \int_{U_1} \int_0^1 \left\langle \mathbb{E} \left(\Lambda^{\frac{\beta}{2}} g'' \left(P^1 Z(T, t, x + \theta E(T-t)By) \right) \Lambda^{-\frac{\beta}{2}} \right) \Big|_{x=\tilde{Y}(t)} \right. \right. \\
&\quad \left. \left. \times \Lambda^{\frac{\beta}{2}} P^1 E(T-t)B \Lambda^{\frac{1-\beta}{2}} \Lambda^{\frac{\beta-1}{2}} y, \Lambda^{-\frac{\beta}{2}} P^1 F(T-t) \Lambda^{\frac{1-\beta}{2}} \Lambda^{\frac{\beta-1}{2}} y \right\rangle_{\dot{H}^0} \, d\theta \, \nu(dy) \, dt \right| \\
&\leq \sup_{x \in \dot{H}^0} \left\| \Lambda^{\frac{\beta}{2}} g''(x) \Lambda^{-\frac{\beta}{2}} \right\|_{\mathcal{L}(\dot{H}^0)} \left\| \Lambda^{\frac{\beta-1}{2}} Q^{\frac{1}{2}} \right\|_{\mathcal{L}_2(\dot{H}^0)}^2 \\
&\quad \times \int_0^T \left\| \Lambda^{\frac{\beta}{2}} P^1 E(T-t)B \Lambda^{\frac{1-\beta}{2}} \right\|_{\mathcal{L}(\dot{H}^0)} \left\| \Lambda^{-\frac{\beta}{2}} P^1 F(T-t) \Lambda^{\frac{1-\beta}{2}} \right\|_{\mathcal{L}(\dot{H}^0)} \, dt.
\end{aligned} \tag{5.12}$$

Note that, by the definition of $B = (0, I)^\top$ and $E(t)$ from (5.1) we have

$$\left\| \Lambda^{\frac{\beta}{2}} P^1 E(T-t)B \Lambda^{\frac{1-\beta}{2}} \right\|_{\mathcal{L}(\dot{H}^0)} = \left\| \Lambda^{\frac{\beta-1}{2}} S(T-t) \Lambda^{\frac{1-\beta}{2}} \right\|_{\mathcal{L}(\dot{H}^0)} = \|S(T-t)\|_{\mathcal{L}(\dot{H}^0)} \leq 1;$$

it remains to estimate the integral

$$\begin{aligned}
\int_0^T \left\| \Lambda^{-\frac{\beta}{2}} P^1 F(T-t) \Lambda^{\frac{1-\beta}{2}} \right\|_{\mathcal{L}(\dot{H}^0)} \, dt &= \int_0^T \|P^1 F(t)\|_{\mathcal{L}(\dot{H}^{\beta-1}, \dot{H}^{-\beta})} \, dt \\
&= \int_0^T \|P^1(\tilde{E}(t) - E(t))B\|_{\mathcal{L}(\dot{H}^{\beta-1}, \dot{H}^{-\beta})} \, dt.
\end{aligned}$$

To this end, it suffices to apply the deterministic error estimate (5.7) with $\alpha = 2\beta$. The combination of (5.10), (5.11) and (5.12) finishes the proof. \square

Remark 5.4. In contrast to our result for the stochastic heat equation (Theorem 4.5) we have to assume the additional condition (5.8) on g to obtain that the weak order of convergence for the approximation of the stochastic wave equation in Theorem 5.3 is twice the strong order of convergence. As an example for a test function g satisfying (5.8) consider

$$g(x) := f(\langle \varphi_1, x \rangle_{\dot{H}^0}, \dots, \langle \varphi_n, x \rangle_{\dot{H}^0}), \quad x \in \dot{H}^0,$$

where $f \in C_b^2(\mathbb{R}^n, \mathbb{R})$ and $(\varphi_k)_{k \in \mathbb{N}} \subset D(\Lambda)$ is an orthonormal basis of $\dot{H}^0 = L^2(\mathcal{O})$ consisting of eigenfunctions of Λ with corresponding eigenvalues $(\lambda_k)_{k \in \mathbb{N}} \subset (0, \infty)$. Then, for $x, y \in \dot{H}^0$,

$$\Lambda^{\beta/2} g''(x) \Lambda^{-\beta/2} y = \sum_{j,k=1}^n \lambda_j^{-\beta/2} \lambda_k^{\beta/2} (\partial_j \partial_k f) \left(\langle \varphi_1, x \rangle_{\dot{H}^0}, \dots, \langle \varphi_n, x \rangle_{\dot{H}^0} \right) \langle \varphi_j, y \rangle_{\dot{H}^0} \varphi_k$$

and (5.8) holds. More generally, the condition (5.8) is satisfied by all $g \in C_b^2(\dot{H}^0, \mathbb{R})$ of the form $g = \tilde{g} \circ \Lambda^{-\beta/2}$, $\tilde{g} \in C_b^2(\dot{H}^0, \mathbb{R})$. For such g we have $g''(x) = \Lambda^{-\beta/2} \tilde{g}''(\Lambda^{-\beta/2} x) \Lambda^{-\beta/2}$.

Proof of Lemma 5.2. We use the estimates

$$\sup_{n \in \{0, \dots, N\}} \|P^1(E_{h, \Delta t}^n P_h - E(t_n))\|_{\mathcal{L}(\mathcal{H}^\alpha, \dot{H}^0)} \leq C(T) \left(h^{\min(\alpha \frac{r}{r+1}, r)} + (\Delta t)^{\min(\alpha \frac{p}{p+1}, p)} \right) \quad (5.13)$$

and

$$\|E(t) - E(s)\|_{\mathcal{L}(\mathcal{H}^\delta, H)} \leq C|t - s|^\delta, \quad t, s \geq 0, \delta \in [0, 1]. \quad (5.14)$$

from Corollary 4.11 and Lemma 4.4 in [19]. Corollary 4.11 in [19] is based on an error estimate proved in [3].

Because of the ‘piecewise’ definition of $\tilde{E}(t)$ in (5.5), the combination of (5.13) and (5.14) gives

$$\begin{aligned} & \sup_{t \in [0, T]} \|P^1(\tilde{E}(t) - E(t))\|_{\mathcal{L}(\mathcal{H}^\alpha, \dot{H}^0)} \\ & \leq \sup_{n \in \{0, \dots, N\}} \|P^1(\tilde{E}(t_n) - E(t_n))\|_{\mathcal{L}(\mathcal{H}^\alpha, \dot{H}^0)} + \sup_{n \in \{1, \dots, N\}} \sup_{t \in (t_{n-1}, t_n)} \|E(t_n) - E(t)\|_{\mathcal{L}(\mathcal{H}^\alpha, \mathcal{H})} \\ & \leq C(T) \left(h^{\min(\alpha \frac{r}{r+1}, r)} + (\Delta t)^{\min(\alpha \frac{p}{p+1}, p)} + (\Delta t)^{\min(\alpha, 1)} \right) \\ & = C(T) \left(h^{\min(\alpha \frac{r}{r+1}, r)} + (\Delta t)^{\min(\alpha \frac{p}{p+1}, 1)} \right) \end{aligned} \quad (5.15)$$

for $\Delta t \leq 1$. It remains to show that

$$\sup_{t \in [0, T]} \|P^1(\tilde{E}(t) - E(t))B\|_{\mathcal{L}(\dot{H}^{(\alpha/2)-1}, \dot{H}^{-\alpha/2})} \leq C(T) \left(h^{\min(\alpha \frac{r}{r+1}, r)} + (\Delta t)^{\min(\alpha \frac{p}{p+1}, 1)} \right). \quad (5.16)$$

To this end, we will prove the estimate

$$\sup_{n \in \{0, \dots, N\}} \|P^1(\tilde{E}(t_n) - E(t_n))B\|_{\mathcal{L}(\dot{H}^{(\alpha/2)-1}, \dot{H}^{-\alpha/2})} \leq C(T) \left(h^{\min(\alpha \frac{r}{r+1}, r)} + (\Delta t)^{\min(\alpha \frac{p}{p+1}, p)} \right). \quad (5.17)$$

Then, (5.16) follows from (5.17) and (5.14) by estimating analogously to (5.15) and using the fact that

$$\begin{aligned} \|P^1(E(t_n) - E(t))B\|_{\mathcal{L}(\dot{H}^{(\alpha/2)-1}, \dot{H}^{-\alpha/2})} &= \|\Lambda^{-\frac{\alpha}{4}} P^1(E(t_n) - E(t))B\Lambda^{\frac{1}{2} - \frac{\alpha}{4}}\|_{\mathcal{L}(\dot{H}^0)} \\ &= \|P^1(E(t_n) - E(t))B\Lambda^{\frac{1-\alpha}{2}}\|_{\mathcal{L}(\dot{H}^0)} \\ &\leq \|P^1(E(t_n) - E(t))\|_{\mathcal{L}(\mathcal{H}^\alpha, \dot{H}^0)} \|B\Lambda^{\frac{1-\alpha}{2}}\|_{\mathcal{L}(\dot{H}^0, \mathcal{H}^\alpha)}, \end{aligned}$$

where $\|B\Lambda^{\frac{1-\alpha}{2}}\|_{\mathcal{L}(\dot{H}^0, \mathcal{H}^\alpha)} = \|B\|_{\mathcal{L}(\dot{H}^{\alpha-1}, \mathcal{H}^\alpha)} = 1$.

In order to show (5.17), we distinguish the cases $\alpha > 2$ and $0 \leq \alpha \leq 2$. For $\alpha > 2$ we have by (5.13)

$$\begin{aligned} & \sup_{n \in \{0, \dots, N\}} \|P^1(\tilde{E}(t_n) - E(t_n))B\|_{\mathcal{L}(\dot{H}^{\alpha-1}, \dot{H}^0)} \\ & \leq \sup_{n \in \{0, \dots, N\}} \|P^1(\tilde{E}(t_n) - E(t_n))\|_{\mathcal{L}(\mathcal{H}^\alpha, \dot{H}^0)} \|B\|_{\mathcal{L}(\dot{H}^{\alpha-1}, \mathcal{H}^\alpha)} \\ & \leq C(T) \left(h^{\min(\alpha \frac{r}{r+1}, r)} + (\Delta t)^{\min(\alpha \frac{p}{p+1}, p)} \right) \end{aligned} \quad (5.18)$$

As the operator $P^1(\tilde{E}(t) - E(t))B \in \mathcal{L}(\dot{H}^0)$ is symmetric in \dot{H}^0 and since $\dot{H}^{-\alpha+1}$ can be identified with the dual space of $\dot{H}^{\alpha-1}$, we have

$$\|P^1(\tilde{E}(t) - E(t))B\|_{\mathcal{L}(\dot{H}^{\alpha-1}, \dot{H}^0)} = \|P^1(\tilde{E}(t) - E(t))B\|_{\mathcal{L}(\dot{H}^0, \dot{H}^{-\alpha+1})}$$

and therefore also

$$\sup_{n \in \{0, \dots, N\}} \|P^1(\tilde{E}(t_n) - E(t_n))B\|_{\mathcal{L}(\dot{H}^0, \dot{H}^{-\alpha+1})} \leq C(T) \left(h^{\min(\alpha \frac{r}{r+1}, r)} + (\Delta t)^{\min(\alpha \frac{p}{p+1}, p)} \right). \quad (5.19)$$

Next, we use the fact that $\dot{H}^{(\alpha/2)-1}$ and $\dot{H}^{-\alpha/2}$ can be represented as the real interpolation spaces $(\dot{H}^0, \dot{H}^{\alpha-1})_{\theta, 2}$ and $(\dot{H}^{-\alpha+1}, \dot{H}^0)_{\theta, 2}$, respectively, where $\theta = ((\alpha/2) - 1)/(\alpha - 1) \in (0, 1)$, cf. Remark 4.1. Thus, interpolation between (5.18) and (5.19) yields

$$\begin{aligned} & \sup_{n \in \{0, \dots, N\}} \|P^1(\tilde{E}(t_n) - E(t_n))B\|_{\mathcal{L}(\dot{H}^{(\alpha/2)-1}, \dot{H}^{-\alpha/2})} \\ & \leq \sup_{n \in \{0, \dots, N\}} C(\alpha) \|P^1(\tilde{E}(t_n) - E(t_n))B\|_{\mathcal{L}(\dot{H}^0, \dot{H}^{-\alpha+1})}^{1-\theta} \|P^1(\tilde{E}(t_n) - E(t_n))B\|_{\mathcal{L}(\dot{H}^{\alpha-1}, \dot{H}^0)}^{\theta} \\ & \leq C(T, \alpha) \left(h^{\min(\alpha \frac{r}{r+1}, r)} + (\Delta t)^{\min(\alpha \frac{p}{p+1}, p)} \right), \end{aligned}$$

see, e.g., Definition 1.2.2/2 and Theorem 1.3.3(a) in [43].

For $0 \leq \alpha \leq 2$, we note that

$$\begin{aligned} \|P^1(\tilde{E}(t_n) - E(t_n))B\|_{\mathcal{L}(\dot{H}^0, \dot{H}^{-1})} &= \|P^1(\tilde{E}(t_n) - E(t_n))B\|_{\mathcal{L}(\dot{H}^1, \dot{H}^0)} \\ &\leq \|P^1(\tilde{E}(t_n) - E(t_n))\|_{\mathcal{L}(\mathcal{H}^2, \dot{H}^0)} \|B\|_{\mathcal{L}(\dot{H}^1, \mathcal{H}^2)}, \end{aligned}$$

where we used again the symmetry of $P^1(\tilde{E}(t) - E(t))B \in \mathcal{L}(\dot{H}^0)$. By (5.13) we obtain

$$\sup_{n \in \{0, \dots, N\}} \|P^1(\tilde{E}(t_n) - E(t_n))B\|_{\mathcal{L}(\dot{H}^0, \dot{H}^{-1})} \leq C(T) \left(h^{\min(2 \frac{r}{r+1}, r)} + (\Delta t)^{\min(2 \frac{p}{p+1}, p)} \right), \quad (5.20)$$

which is (5.17) for $\alpha = 0$. Moreover, also by (5.13),

$$\begin{aligned} & \sup_{n \in \{0, \dots, N\}} \|P^1(\tilde{E}(t_n) - E(t_n))B\|_{\mathcal{L}(\dot{H}^{-1}, \dot{H}^0)} \\ & \leq \sup_{n \in \{0, \dots, N\}} \|P^1(\tilde{E}(t_n) - E(t_n))\|_{\mathcal{L}(H, \dot{H}^0)} \|B\|_{\mathcal{L}(\dot{H}^{-1}, H)} \leq C(T), \end{aligned} \quad (5.21)$$

i.e., we have (5.17) for $\alpha = 2$. Finally, if $\alpha \in (0, 2)$, interpolation with $\theta = (\alpha/2) - 1 \in (0, 1)$ between (5.20) and (5.21) gives

$$\begin{aligned} & \sup_{n \in \{0, \dots, N\}} \|P^1(\tilde{E}(t_n) - E(t_n))B\|_{\mathcal{L}(\dot{H}^{(\alpha/2)-1}, \dot{H}^{-\alpha/2})} \\ & \leq \sup_{n \in \{0, \dots, N\}} C(\alpha) \|P^1(\tilde{E}(t_n) - E(t_n))B\|_{\mathcal{L}(\dot{H}^0, \dot{H}^{-1})}^{1-\theta} \|P^1(\tilde{E}(t_n) - E(t_n))B\|_{\mathcal{L}(\dot{H}^{-1}, \dot{H}^0)}^{\theta} \\ & \leq C(T, \alpha) \left(h^{\min(2 \frac{r}{r+1}, r)} + (\Delta t)^{\min(2 \frac{p}{p+1}, p)} \right)^{\frac{\alpha}{2}} \\ & = C(T, \alpha) \left(h^{2 \frac{r}{r+1}} + (\Delta t)^{2 \frac{p}{p+1}} \right)^{\frac{\alpha}{2}} \\ & \leq C(T, \alpha) \left(h^{\min(\alpha \frac{r}{r+1}, r)} + (\Delta t)^{\min(\alpha \frac{p}{p+1}, p)} \right). \end{aligned} \quad \square$$

6 Concluding remarks

We expect that our results can be generalized to unbounded test functions $G \in C^2(H; \mathbb{R})$ with $\sup_{x \in H} \|G''(x)\|_{\mathcal{L}(H)} < \infty$, including in particular $G(x) = \|x\|_H^2$. This is especially important for the stochastic wave equation as for the specific and important test function $g(x) = \|x\|_{H^0}^2$ the extra assumption (5.8) is automatically fulfilled. Generalization of our results to Lévy processes that are not square-integrable is also possible using a suitable stopping argument as in [25, Appendix B]. We will also be looking at extending the results to cover Lévy processes with non-trivial Gaussian part as well as including stochastic Volterra-type evolution equations in the analysis to obtain results corresponding to the ones in the Gaussian case [21].

A Poisson random measures and a comparison of stochastic integrals

Our proof of Theorem 3.3 is based on Itô's formula for Banach space-valued jump processes driven by Poisson random measures as presented in [26]. Alternatively, one could use Itô's formula as proved in [14], but the formula in [26] is more convenient in our setting. In this section, we use Lemma 3.1 to relate our setting to the setting in [26].

It is well-known that the jumps of a Lévy process determine a Poisson random measure on the product space of the underlying time interval and the state space. We refer to [32, Section 6] for a definition and properties of Poisson random measures. For $(\omega, t) \in \Omega \times (0, \infty)$ we denote by $\Delta L(t)(\omega) := L(t)(\omega) - \lim_{s \nearrow t} L(s)(\omega) \in U_1$ the jump of a trajectory of L at time t . Setting

$$N(\omega) := \sum_{\Delta L(t)(\omega) \neq 0} \delta_{(t, \Delta L(t)(\omega))}, \quad \omega \in \Omega,$$

defines a Poisson random measure N on $([0, \infty) \times U_1, \mathcal{B}([0, \infty)) \otimes \mathcal{B}(U_1))$ with intensity measure (or compensator) $\lambda \otimes \nu$, where λ is Lebesgue measure on $[0, \infty)$ and ν is the jump intensity measure of L . This follows, e.g., from Theorem 6.5 in [32] together with Theorems 4.9, 4.15, 4.23 and Lemma 4.25 therein. We denote the compensated Poisson random measure by

$$q := N - \lambda \otimes \nu. \tag{A.1}$$

Let V be a (real and separable) Hilbert space. The stochastic integral with respect to q of functions in $L^2(\Omega_T \times U_1, \mathbb{P}_T \otimes \nu; V) = L^2(\Omega_T \times U_1, \mathcal{P}_T \otimes \mathcal{B}(U_1), \mathbb{P}_T \otimes \nu; V)$ is constructed as a linear isometry

$$L^2(\Omega_T \times U_1, \mathbb{P}_T \otimes \nu; V) \rightarrow \mathcal{M}_T^2(V), \quad f \mapsto \left(\int_0^t \int_{U_1} f(s, x) q(ds, dx) \right)_{t \in [0, T]}.$$

In particular, the V -valued integral processes have càdlàg modifications; we will always work with such a càdlàg modification. Using a standard stopping procedure, the stochastic

integral can be extended to functions $f \in L^0(\Omega_T \times U_1, \mathcal{P}_T \otimes \mathcal{B}(U_1), \mathbb{P}_T \otimes \nu; V)$ such that

$$\mathbb{P}\left(\int_0^T \int_{U_1} \|f(s, x)\|_V^2 \nu(dx) ds < \infty\right) = 1.$$

We refer to [26], [34] and the references therein for details on stochastic integration w.r.t. Poisson random measures, compare also [32, Section 8.7].

Remark A.1. Strictly speaking, in [26] the integrands f do not have to be predictable but only $\mathcal{F}_t \otimes \mathcal{B}(U_1)$ -adapted and $\mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(U_1)$ -measurable. However, it is clear that in the case of predictable, i.e., $\mathcal{P}_T \otimes \mathcal{B}(U_1)$ -measurable, and square integrable Hilbert space-valued integrands f the stochastic integral in [26] coincides with the stochastic integral considered in [32], [34]. See [39] for a detailed comparison of the different spaces of integrands.

Since $\mathbb{E} \int_0^T \int_{U_1} \|x\|_{U_1}^2 \nu(dx) dt$ is finite for all $T < \infty$, the integral process $(\int_0^t x q(ds, dx))_{t \geq 0}$ is uniquely determined (up to indistinguishability) as a U_1 -valued square-integrable càdlàg martingale. Taking into account the assumptions on the Lévy process L , the Lévy-Khinchin decomposition [32, Theorem 4.23], the definition of q , and the construction of the stochastic integral w.r.t. q , it is not difficult to see that the processes L and $(\int_0^t x q(ds, dx))_{t \geq 0}$ are indistinguishable, i.e.,

$$\mathbb{P}\left(L(t) = \int_0^t \int_{U_1} x q(ds, dx) \quad \forall t \geq 0\right) = 1. \quad (\text{A.2})$$

Using Lemma 3.1, we are now able to identify stochastic integrals w.r.t. L and stochastic integrals w.r.t. the compensated Poisson random measure q . Recall from Remark 3.2 that we identify processes $\Phi \in L^2(\Omega_T, \mathbb{P}_T; \mathcal{L}_2(U_0, H))$ with the corresponding functions $\kappa(\Phi) \in L^2(\Omega_T \times U_1, \mathbb{P}_T \otimes \nu; H)$. Thus, for such Φ the integral process $(\int_0^t \int_{U_1} \Phi(s) x q(ds, dx))_{t \in [0, T]}$ is defined.

Lemma A.2. *Given $\Phi \in L^2(\Omega_T, \mathbb{P}_T; \mathcal{L}_2(U_0, H))$, the H -valued càdlàg integral processes $(\int_0^t \Phi(s) dL(s))_{t \in [0, T]}$ and $(\int_0^t \int_{U_1} \Phi(s) x q(ds, dx))_{t \in [0, T]}$ are indistinguishable. That is,*

$$\mathbb{P}\left(\int_0^t \Phi(s) dL(s) = \int_0^t \int_{U_1} \Phi(s) x q(ds, dx) \quad \forall t \in [0, T]\right) = 1.$$

Proof. We first assume that Φ is a simple $\mathcal{L}(U_1, H)$ -valued process of the form

$$\Phi(s) = \sum_{k=0}^{m-1} \mathbf{1}_{F_k} \mathbf{1}_{(t_k, t_{k+1}]}(s) \Phi_k, \quad s \in [0, T],$$

with $0 \leq t_0 < t_1 < \dots < t_m \leq T$, $m \in \mathbb{N}$, $F_k \in \mathcal{F}_{t_k}$ and $\Phi_k \in \mathcal{L}(U_1, H)$. Recall from Section 2.2 that $\mathcal{L}(U_1, H)$ is a subspace of $\mathcal{L}_2(U_0, H)$. Using (A.2) and applying standard arguments for the evaluation of stochastic integrals, we obtain for fixed $t \in [0, T]$, \mathbb{P} -almost surely,

$$\int_0^t \Phi(s) dL(s) = \sum_{k=0}^{m-1} \mathbf{1}_{F_k} \Phi_k (L(t_{k+1} \wedge t) - L(t_k \wedge t))$$

$$\begin{aligned}
&= \sum_{k=0}^{m-1} \mathbb{1}_{F_k} \Phi_k \left(\int_0^T \int_{U_1} \mathbb{1}_{(t_k \wedge t, t_{k+1} \wedge t]}(s) x q(ds, dx) \right) \\
&= \sum_{k=0}^{m-1} \int_0^T \int_{U_1} \mathbb{1}_{F_k} \mathbb{1}_{(t_k \wedge t, t_{k+1} \wedge t]}(s) \Phi_k x q(ds, dx) \\
&= \int_0^t \int_{U_1} \Phi(s) x q(ds, dx).
\end{aligned}$$

Since both processes are right-continuous, we see that the processes $(\int_0^t \Phi(s) dL(s))_{t \in [0, T]}$ and $(\int_0^t \int_{U_1} \Phi(s) x q(ds, dx))_{t \in [0, T]}$ are indistinguishable.

For general $\Phi \in L^2(\Omega_T, \mathbb{P}_T; \mathcal{L}_2(U_0, H))$, we take a sequence $(\Phi_n)_{n \in \mathbb{N}}$ of simple $\mathcal{L}(U_1, H)$ -valued process such that $\Phi_n \rightarrow \Phi$ in $L^2(\Omega_T, \mathbb{P}_T; \mathcal{L}_2(U_0, H))$; see, e.g., [35, Proposition 2.3.8] for a proof of the existence of such a sequence. Then, the processes $\int_0^\cdot \Phi_n(s) dL(s) = (\int_0^t \Phi_n(s) dL(s))_{t \in [0, T]}$ and $\int_0^\cdot \int_{U_1} \Phi_n(s) x q(ds, dx) = (\int_0^t \int_{U_1} \Phi_n(s) x q(ds, dx))_{t \in [0, T]}$ are indistinguishable for all $n \in \mathbb{N}$, and we have the convergence $\int_0^\cdot \Phi_n(s) dL(s) \rightarrow \int_0^\cdot \Phi(s) dL(s)$ in $\mathcal{M}_T^2(H)$ by the construction of the stochastic integral w.r.t. L . According to Lemma 3.1, the convergence $\Phi_n \rightarrow \Phi$ in $L^2(\Omega_T, \mathbb{P}_T; \mathcal{L}_2(U_0, H))$ entails the convergence $\kappa(\Phi_n) \rightarrow \kappa(\Phi)$ in $L^2(\Omega_T \times U_1, \mathbb{P}_T \otimes \nu; H)$, so that we also have $\int_0^\cdot \int_{U_1} \Phi_n(s) x q(ds, dx) \rightarrow \int_0^\cdot \int_{U_1} \Phi(s) x q(ds, dx)$ in $\mathcal{M}_T^2(H)$. Thus, $\int_0^\cdot \Phi(s) dL(s) = \int_0^\cdot \int_{U_1} \Phi(s) x q(ds, dx)$ as an equality in $\mathcal{M}_T^2(H)$, which yields the assertion. \square

References

- [1] D. Applebaum, M. Riedle: Cylindrical Lévy processes in Banach spaces. *Proc. London Math. Soc.* **101**(3) (2010) 697–726.
- [2] W. Arendt, C.J.K. Batty, M. Hieber, F. Neubrander: *Vector-valued Laplace transforms and Cauchy problems*. Birkhäuser, Basel 2011.
- [3] G.A. Baker, J.H. Bramble: Semidiscrete and single step fully discrete approximations for second order hyperbolic equations. *RAIRO Numer. Anal.* **13** (1979) 76–100.
- [4] A. Barth: A finite element method for martingale-driven stochastic partial differential equations. *Commun. Stoch. Anal.* **4**(3) (2010) 355–375.
- [5] A. Barth, A. Lang: Simulation of stochastic partial differential equations using finite element methods. *Stochastics* **84**(2-3) (2012) 217–231.
- [6] A. Barth, A. Lang: Milstein approximation for advection-diffusion equations driven by multiplicative noncontinuous martingale noises. *Appl. Math. Optim.* **66** (2012) 387–413.
- [7] P. Brenner, V. Thomeé: On rational approximations of groups of operators. *SIAM J. Numer. Anal.* **17**(1) (1980) 119–125.

- [8] Z. Brzeźniak, J. Zabczyk: Regularity of Ornstein-Uhlenbeck processes driven by a Lévy white noise. *Potential Anal.* **32** (2010) 153–188.
- [9] D. Conus, A. Jentzen, R. Kurniawan: Weak convergence rates of spectral Galerkin approximations for SPDEs with nonlinear diffusion coefficients. *Preprint*, 2014. arXiv:1408.1108v1
- [10] G. Da Prato, A. Jentzen, M. Röckner: A mild Itô formula for SPDEs. *Preprint*, 2012. arXiv:1009.3526v4
- [11] G. Da Prato, J. Zabczyk: *Stochastic equations in infinite dimensions*. Cambridge University Press, Cambridge 1992.
- [12] A. Debussche and J. Printems: Weak order for the discretization of the stochastic heat equation. *Math. Comp.* **78** (2009) 845–863.
- [13] T. Dunst, E. Hausenblas, A. Prohl: Approximate Euler method for parabolic stochastic partial differential equations driven by space-time Lévy noise. *SIAM J. Numer. Anal.* **50**(6) (2012) 2873–2896.
- [14] J.B. Gravereaux, J. Pellaumail: Formule de Ito pour des processus non continus à valeurs dans des espaces de Banach. *Ann. Inst. Henri Poincaré Probab. Statist.* **10**(4) (1974) 399–422.
- [15] E. Hausenblas: Finite element approximation of stochastic partial differential equations driven by Poisson random measures of jump type. *SIAM J. Numer. Anal.* **46** (2008) 437–471.
- [16] E. Hausenblas, I. Marchis: A numerical approximation of parabolic stochastic partial differential equations driven by a Poisson random measure. *BIT* **46** (2006) 773–811.
- [17] J. Jacod, T.G. Kurtz, S. Méléard, P. Protter: The approximate Euler method for Lévy driven stochastic differential equations. *Ann. Inst. Henri Poincaré Probab. Stat.* **41**(3) (2005) 523–558.
- [18] M. Kovács, S. Larsson, F. Lindgren: Weak convergence of finite element approximations of linear stochastic evolution equations with additive noise. *BIT* **52** (2012) 85–108.
- [19] M. Kovács, S. Larsson, F. Lindgren: Weak convergence of finite element approximations of linear stochastic evolution equations with additive noise II: Fully discrete schemes. *BIT* **53** (2013) 497–525.
- [20] M. Kovács, J. Printems: Strong order of convergence of a fully discrete approximation of a linear stochastic Volterra type evolution equation. *Math. Comp.* **83**(298) (2014) 2325–2346.

- [21] M. Kovács, J. Printems: Weak convergence of a fully discrete approximation of a linear stochastic evolution equation with a positive-type memory term. *J. Math. Anal. Appl.* **413** (2014) 939–952.
- [22] M. Kovács, S. Larsson, F. Saedpanah: Finite element approximation of the linear stochastic wave equation with additive noise. *SIAM J. Numer. Anal.* **48** (2010) 408–427.
- [23] A. Lang: Mean square convergence of a semidiscrete scheme for SPDEs of Zakai type driven by square integrable martingales. *Procedia Computer Science* **1** (2012) 1615–1623.
- [24] S. Larsson, V. Thomée: *Partial differential equations with numerical methods*. Springer, Berlin 2003.
- [25] F. Lindner, R.L. Schilling: Weak order for the discretization of the stochastic heat equation driven by impulsive noise. *Potential Anal.* **38**(2) (2013) 345–379.
- [26] V. Mandrekar, B. Rüdiger, S. Tappe: Itô’s formula for Banach space-valued jump processes driven by Poisson random measures. In: C. Dalang, M. Dozzi, F. Russo (eds.): *Seminar on stochastic analysis, random fields and applications VII. Centro Stefano Franscini, Ascona, May 2011*. Progress in Probability **67**. Birkhäuser, Basel 2013, 171–186.
- [27] R. Mikulevičius, C. Zhang: On the rate of convergence of weak Euler approximation for nondegenerate SDEs driven by Lévy processes. *Stochastic Process. Appl.* **121** (2011) 1720–1748.
- [28] M. Métivier: *Semimartingales. A course on stochastic processes*. de Gruyter, Berlin 1982.
- [29] M. Métivier, J. Pellaumail: *Stochastic integration*. Academic Press, New York 1980.
- [30] S.P. Norsett, G. Wanner: The real pole sandwich for rational approximations and oscillation equations. *BIT* **19** (1979) 79–94.
- [31] A. Pazy: *Semigroups of linear operators and applications to partial differential equations*. Springer, New York 1983.
- [32] S. Peszat, J. Zabczyk: *Stochastic partial differential equations with Lévy noise. An evolution equation approach*. Cambridge University Press, Cambridge 2007.
- [33] E. Platen, N. Bruti-Liberati: *Numerical solutions of stochastic differential equations with jumps in finance*. Springer, Berlin 2010.
- [34] C. Prévôt: Existence, uniqueness and regularity w.r.t. the initial condition of mild solutions of SPDEs driven by Poisson noise. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **13**(1) (2010) 133–163.

- [35] C. Prévôt, M. Röckner: *A concise course on stochastic partial differential equations*. Springer, Lecture Notes in Mathematics **1905**, Berlin 2007.
- [36] P. Protter, D. Talay: The Euler scheme for Lévy driven stochastic differential equations. *Ann. Probab.* **25**(1) (1997) 393–423.
- [37] M. Riedle: Stochastic integration with respect to cylindrical Lévy processes in Hilbert spaces: an L^2 approach. *Preprint*, 2012. arXiv:1207.2634
- [38] M. Riedle: Ornstein-Uhlenbeck processes driven by cylindrical Lévy processes. *Preprint*, 2014. arXiv:1212.3832v3
- [39] B. Rüdiger, S. Tappe: Isomorphisms for spaces of predictable processes and an extension of the Itô integral. *Stoch. Anal. Appl.* **30**(3) (2012) 529–537.
- [40] K. Sato: *Lévy processes and infinitely divisible distributions*. Cambridge University Press, Cambridge 2013.
- [41] R.L. Schilling, R. Song, Z. Vondracek: *Bernstein functions. Theory and applications*. de Gruyter, Berlin 2010.
- [42] V. Thomée: *Galerkin finite element methods for parabolic problems*. (2nd edn) Springer, Springer Series in Computational Mathematics **25**, Berlin 2006.
- [43] H. Triebel: *Interpolation theory, function spaces, differential operators*. (2nd edn) Johann Ambrosius Barth, Heidelberg 1995 .
- [44] J. Weidmann: *Linear operators in Hilbert spaces*. Springer, Graduate texts in mathematics **68**, New York 1980.
- [45] Y. Yan: Galerkin finite element methods for stochastic parabolic partial differential equations. *SIAM J. Numer. Anal.* **43**(4) (2005) 1363–1384.

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