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MULTILEVEL PATH SIMULATION FOR WEAK APPROXIMATION SCHEMES: MYTH OR REALITY [1]

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In this paper we discuss the possibility of using multilevel Monte Carlo (MLMC) methods for weak approximation schemes. It turns out that by means of a simple coupling between consecutive time discretisation levels, one can achieve the same complexity gain as under the presence of a strong convergence. We exemplify this general idea in the case of weak Euler and Milstein schemes, and prove that the complexity of the corresponding "weak" MLMC estimates are of order $\varepsilon^{-2}\log^2(\varepsilon)$ and ε^{-2}, respectively. Thus, we propose a new simple approach how to achieve the complexity ε^{-2} without time-consuming Lévy area simulation. The numerical performance of the new "weak" MLMC methods is illustrated by several numerical examples.

1 INTRODUCTION

The multilevel path simulation method introduced in Giles [4] has gained huge popularity as a complexity reduction tool in recent times. The main advantage of the MLMC methodology is that it can be simply applied to various situations and requires almost no prior knowledge on the path generating process. Any multilevel Monte Carlo (MLMC) algorithm uses a number of levels of resolution, $l = 0, 1, \ldots, L$, with $l = 0$ being the coarsest, and $l = L$ being the finest. In the context of an SDE simulation on the interval $[0, T]$, level 0 corresponds to one timestep $\Delta_0 = T$, whereas the level L has 2^L uniform timesteps $\Delta_L = 2^{-L}T$.

Assume that a filtered probability space $(\Omega, \mathscr{F}, \mathrm{P}, (\mathscr{F}_t))$ and an $\mathbb{R}^m$-valued standard Brownian motion (W_t) are given. Let b be a Lipschitz function from $\mathbb{R}^d$ to $\mathbb{R}^d$, and σ a Lipschitz function from $\mathbb{R}^d$ to $\mathbb{R}^{d\otimes m}$. Consider now a d-dimensional diffusion process (X_t) solving the SDE

$$X_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \quad t \in [0, T] \tag{1.1}$$

and assume that we would like to estimate the expectation $\mathrm{E}[f(X_T)]$, where f is a Lipschitz function from $\mathbb{R}^d$ to $\mathbb{R}$. Furthermore, let X_T^l be an approximation

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for X_T by means of a numerical discretisation with time step Δ_l. The main idea of the multilevel approach pioneered in Giles [4] consists in writing the expectation of the finest approximation $\mathrm{E}[f(X_T^L)]$ as a telescopic sum

$$\mathrm{E}[f(X_T^L)] = \mathrm{E}[f(X_T^0)] + \sum_{l=1}^{L} \mathrm{E}[f(X_T^l) - f(X_T^{l-1})]$$

and then applying Monte Carlo to estimate each expectation in the telescopic sum. One important prerequisite for MLMC to work is that X_T^l and X_T^{l-1} are coupled in some way and this can be achieved by using the same discretised trajectories of the underlying Brownian motions to construct the approximations X_T^l and X_T^{l-1}. The degree of coupling is usually measured in terms of the variance $\mathrm{Var}[f(X_T^l) - f(X_T^{l-1})]$. It is shown in Giles [4], that under the assumptions

$$\left|\mathrm{E}[f(X_T^l)] - \mathrm{E}[f(X_T)]\right| \le c_1 \Delta_l^{\alpha}, \quad \mathrm{Var}\left[f(X_T^l) - f(X_T^{l-1})\right] \le c_2 \Delta_l^{\beta}, \qquad (1.2)$$

for some $\alpha \ge 1/2$, $\beta > 0$, $c_1 > 0$ and $c_2 > 0$, the computational complexity of the resulting multilevel estimate needed to achieve the accuracy ε (in terms of RMSE) is proportional to

$$\mathscr{C} = \begin{cases} \varepsilon^{-2}, & \beta > 1, \\ \varepsilon^{-2} \log^2(\varepsilon), & \beta = 1, \\ \varepsilon^{-2-(1-\beta)/\alpha}, & 0 < \beta < 1. \end{cases}$$

The standard way of checking the assumptions (1.2) is to prove that the underlying approximation scheme has weak convergence of order α and the strong convergence of order $\beta/2$. Indeed, in the latter case we have for any Lipschitz continuous function f

$$\mathrm{Var}\left[f(X_T^l) - f(X_T^{l-1})\right] \le c_f\, \mathrm{E}\left[\left|X_T^l - X_T\right|^2\right] + c_f\, \mathrm{E}\left[\left|X_T^{l-1} - X_T\right|^2\right] \le 2c_f \Delta_l^{\beta}$$

with some constant $c_f > 0$ depending on f.

However, in recent years the so-called weak approximation schemes, i.e., schemes that, in general, fulfil only the first assumption in (1.2) became quite popular. The weak Euler scheme is a first-order scheme with $\alpha = 1$, and has been studied by many researchers. Talay and Tubaro [12] show the first-order convergence of the weak Euler scheme. The fact that the convergence rate of the Euler scheme also holds for certain irregular functions under a Hörmander type condition has been proved by Bally and Talay [1] using Malliavin calculus. The Itô-Taylor (weak-Taylor) high-order scheme is a natural extension of the weak Euler scheme. In the continuous diffusion case, some new discretization schemes (also called Kusuoka type schemes) which are of order $\alpha \ge 2$ without the Romberg extrapolation have been introduced by Kusuoka

[8], Lyons and Victoir [9], Ninomiya and Victoir [11], and Ninomiya and Ninomiya [10]. The main advantage of the weak approximation schemes is that simple discrete random variables can be used instead of the Brownian motion increments, thus allowing to avoid a time-consuming simulation of the multiple Wiener integrals. Unfortunately, due to the absence of the strong convergence, the MLMC methodology can not be used with the weak approximation schemes directly. In this paper we make the first attempt to overcome this difficulty and develop a kind of weak MLMC approach which can be applied to various weak approximation schemes.

The plan of the paper is as follows. First, we present the general idea of our approach. Next, we consider the weak Euler scheme and show how to construct a weak MLMC algorithm which is able to reduce the complexity of the standard MC to order $\varepsilon^{-2}\log^2(\varepsilon)$. Then we turn to a weak version of the multidimensional Milstein scheme, which does not require the Lévy area simulation and present the corresponding MLMC algorithm with the complexity order ε^{-2}. Finally, we discuss the numerical performance of the presented weak MLMC algorithms.

2 General idea

Fix some n, set $\Delta = T/n$ and consider the general weak approximation scheme for (1.1) of order $\alpha > 0$ (see [7])

$$\begin{cases} X_0^{\Delta} = X_0, \\ X_{j\Delta}^{\Delta} = \sum_{\gamma \in \mathscr{A}_\alpha \setminus \{\emptyset\}} c_\gamma \left((j-1)\Delta, X_{(j-1)\Delta}^{\Delta}\right) \xi_{\gamma,j}(\Delta), \end{cases}$$

for $j = 1, \ldots, n$, where $\mathscr{A}_\alpha = \{\gamma : |\gamma| \leq r\}$ and for any multi-index $\gamma = (\gamma_1, \ldots, \gamma_l)$ with $\gamma_i \in \{0, 1, \ldots, m\}$,

$$|\gamma| = l, \quad \|\gamma\| = |\gamma| + \text{ number of zero components of } \gamma.$$

Suppose that the r.v. ξ_γ fulfil

$$\mathrm{E}[\xi_{\gamma^1,j} \cdot \ldots \cdot \xi_{\gamma^k,j}] = \mathrm{E}[I_{\gamma^1} \cdot \ldots \cdot I_{\gamma^k}] \tag{2.1}$$

for all multi-indexes $\gamma^1, \ldots, \gamma^k$ with $\|\gamma^1\| + \ldots + \|\gamma^k\| \leq 2\alpha + 1$, where

$$I_\gamma = \int_0^\Delta \int_0^{s_l} \ldots \int_0^{s_2} dW_{s_1}^{\gamma_1} \ldots dW_{s_{l-1}}^{\gamma_{l-1}} dW_{s_l}^{\gamma_l}$$

is the corresponding multiple Wiener integral. Then under some smoothness assumptions on the coefficients of (1.1) and the output function f, one can derive (see, e.g. [7]) that

$$\left|\mathrm{E}[f(X_T)] - \mathrm{E}[f(X_T^{\Delta})]\right| \leq K\Delta^\alpha$$

for some constant $K > 0$, i.e. the scheme X^Δ has weak convergence order α. Thus, the first assumption in (1.2) is fulfilled. In order to successfully apply the multilevel path simulation approach, we need to couple the approximations on two consecutive levels in such a way, that the second condition in (1.2) is satisfied. Let us fix two natural numbers n_c ("coarse" level) and n_f ("fine" level) with $n_f/n_c \in \mathbb{N} \setminus \{1\}$, and set $\Delta_c = T/n_c$, $\Delta_f = T/n_f$. In order to couple the approximations X^{Δ_c} and X^{Δ_f}, we couple the random vectors $\xi^c_\gamma = (\xi_{\gamma,1}(\Delta_c),\ldots,\xi_{\gamma,n_c}(\Delta_c))$ and $\xi^f_\gamma = (\xi_{\gamma,1}(\Delta_f),\ldots,\xi_{\gamma,n_f}(\Delta_f))$. The main idea of our approach is to define

$$\xi^c_{\gamma,j} = \sum_{\|\gamma^1\|+\ldots+\|\gamma^k\|\leq\|\gamma\|} \beta_{\gamma^1,\ldots,\gamma^k,1}\xi^f_{\gamma^1,(j-1)r+1}\cdot\ldots\cdot\xi^f_{\gamma^k,(j-1)r+1}+\ldots$$
$$\ldots+\sum_{\|\gamma^1\|+\ldots+\|\gamma^k\|\leq\|\gamma\|} \beta_{\gamma^1,\ldots,\gamma^k,r}\xi^f_{\gamma^1,(j-1)r+1}\cdot\ldots\cdot\xi^f_{\gamma^k,jr}$$

for $j = 1,\ldots,n_c$, and $r = n_f/n_c$, where the coefficients $\beta_{\gamma^1,\ldots,\gamma^k,j}$ are chosen to satisfy

$$\mathrm{E}[\xi^c_{\gamma^1,j}\cdot\ldots\cdot\xi^c_{\gamma^k,j}] = \mathrm{E}[I_{\gamma^1}\cdot\ldots\cdot I_{\gamma^k}]$$

for $\|\gamma^1\|+\ldots+\|\gamma^k\| \leq 2\alpha+1$. By choosing the coefficients $\beta_{\gamma^1,\ldots,\gamma^k,j}$ in a proper way, one can get

$$\mathrm{E}\left[|X^{\Delta_c}_T - X^{\Delta_f}_T|^2\right] \leq C\Delta_f^\alpha$$

for some $C > 0$. In the next two sections we illustrate this general construction for the case of weak Euler and Milstein's schemes.

3 Multilevel path simulation for weak Euler scheme

3.1 Weak Euler scheme

Fix some $n \in \mathbb{N}$ and set $\Delta = T/n$. For a r.v. X_0, the weak Euler scheme reads as follows

$$\begin{aligned} X^\Delta_0 &= X_0, \\ X^\Delta_{j\Delta} &= X^\Delta_{(j-1)\Delta} + b(X^\Delta_{(j-1)\Delta})\,\Delta + \sum_{k=1}^m \sigma_k(X^\Delta_{(j-1)\Delta})\,\xi^k_j, \end{aligned} \tag{3.1}$$

where $j = 1,\ldots,n$, and i.i.d. random variables (ξ^k_j) satisfy

$$\left|\mathrm{E}[\xi^k_j]\right|+\left|\mathrm{E}\left[(\xi^k_j)^3\right]\right| + \left|\mathrm{E}\left[(\xi^k_j)^2\right]-\Delta\right| \leq c\Delta^2.$$

Under some additional assumptions on the coefficient functions b and σ, and the output function f spelled out in Talay and Tubaro [12] and Bally and Talay [1], it holds

$$\left|\mathrm{E}[f(X^\Delta_T)] - \mathrm{E}[f(X_T)]\right| \leq c\Delta$$

for some $c > 0$. Hence the standard complexity analysis (see e.g. Duffie and Glynn [3]) shows that the complexity of the standard Monte Carlo algorithm for evaluating $\mathrm{E}[f(X_T)]$ with accuracy ε (in terms of RMSE) is of order ε^{-3}. In the next section, we present a MLMC algorithm based on the weak Euler scheme which reduces the complexity order to $\varepsilon^{-2}\log^2(\varepsilon)$.

3.2 MLMC algorithm

Fix some $L > 0$ and set $\Delta_l = 2^{-l}T$, $l = 0, \ldots, L$. Denote

$$\xi_L = (\xi^i_{L,1}, \ldots, \xi^i_{L,2^L}, \, i = 1, \ldots, m),$$

where $\xi^i_{L,1}, \ldots, \xi^i_{L,2^L}$ are i.i.d. random variables satisfying

$$\mathrm{E}[\xi^i_{L,1}] = \mathrm{E}[(\xi^i_{L,1})^3] = 0, \quad \mathrm{E}[(\xi^i_{L,1})^2] = \Delta_L. \tag{3.2}$$

The simplest way of constructing a r.v. ξ with the property (3.2) is to take

$$\mathrm{P}\left(\xi^i_{L,j} = \pm\sqrt{\Delta_L}\right) = \frac{1}{2}. \tag{3.3}$$

Now we define recursively the independent random vectors $\xi_{L-1}, \ldots, \xi_0$ with $\xi_l \in \mathbb{R}^{m\times 2^l}$ via

$$\begin{aligned} \xi_{l-1} &\sim \varsigma(\xi_l) \qquad (3.4) \\ &\doteq \left(\xi^i_{l,1} + \xi^i_{l,2}, \ldots, \xi^i_{l,(2k-1)} + \xi^i_{l,2k}, \ldots, \xi^i_{l,2^l-1} + \xi^i_{l,2^l}, \, i = 1, \ldots, m\right) \end{aligned}$$

starting with $l = L$ and continuing backwardly to $l = 1$. As can be easily seen, the components of the each vector ξ_l are i.i.d. random variables with

$$\mathrm{E}[\xi^i_{l,j}] = \mathrm{E}[(\xi^i_{l,j})^3] = 0, \quad \mathrm{E}[(\xi^i_{l,j})^2] = \Delta_l, \quad j = 1, \ldots, 2^l.$$

Next, for any $l = 1, \ldots, L$, and any random vector $\xi = (\xi^i_1, \ldots, \xi^i_{2^l}, \, i = 1, \ldots, m) \in \mathbb{R}^{m\times 2^l}$, consider the approximations

$$\begin{aligned} X^l_0(\xi) &= X_0, \\ X^l_{j\Delta_l}(\xi) &= X^l_{(j-1)\Delta_l}(\xi) + b(X^l_{(j-1)\Delta_l}(\xi))\,\Delta_l + \sum_{i=1}^m \sigma_i(X^l_{(j-1)\Delta_l}(\xi))\,\xi^i_j \end{aligned} \tag{3.5}$$

with $j = 1, \ldots, 2^l$, and some r.v. $X_0 \in \mathbb{R}^d$. Finally fix a vector of natural numbers $\mathbf{N} = (N_0, \ldots, N_L)$ and define a weak MLMC estimate for $\mathrm{E}[f(X_T)]$ as follows

$$\begin{aligned} Y_{L,\mathbf{N}} \doteq \frac{1}{N_0}\sum_{n=1}^{N_0}\left[f(X^0_1(\xi^{(n)}_0)\right] + \\ + \sum_{l=1}^L \frac{1}{N_l}\sum_{n=1}^{N_l}\left[f\left(X^l_1\left(\xi^{(n)}_l\right)\right) - f\left(X^{l-1}_1\left(\varsigma(\xi^{(n)}_l)\right)\right)\right], \end{aligned} \tag{3.6}$$

where $\xi^{(n)}_l$, $n = 1, \ldots, N_l$, are i.i.d. copies of ξ_l.

Remark 1. Observe that distribution of the components of the vector ξ_l under (3.3) defined by (3.4) is closely related to the Binomial distribution, namely

$$\frac{\xi^i_{l,j}}{2\sqrt{\Delta_L}} + 2^{L-l-1} \sim \mathrm{Bi}\left(2^{L-l}, \frac{1}{2}\right). \tag{3.7}$$

Hence the generation of variates $\xi^i_{l,j}$ is straightforward when a generator of binomially distributed random variates is available. For a fixed L, the weak MLMC algorithm implies generation of ξ_l for l starting from 0 up to L. Since all probabilities of distributions $\mathrm{Bi}\left(2^{L-l}, \frac{1}{2}\right)$ for $l \in \{0, 1, \ldots, L\}$ are rational numbers, table look-up or alias methods (see [2]) can be used to achieve fast single random number generation. Since the distributions of ξ_l do not change between different runs of the MLMC method, all the preprocessing required can be done only once and the resulting tables can be stored. In problem-specific hardware (FPGA or ASIC) these tables can be kept in permanent shared constant storage, which is often cheap, fast and abundant. With table lookup methods this would provide $O(L)$ worst-case single random variate generation time at the price of storing $O(2^{L+1})$ items of preprocessing data, and with alias methods it is possible to attain $O(1)$ worst-case single random variate generation time at the price of storing $O(2^{L+1})$ items of preprocessing data. Note that the whole procedure of binomial increments generation can be implemented with the use of integer numbers only. Therefore, a good MLMC implementation for binomial increments can possibly outperform its counterpart for Normal increments.

Discussion As we can see, the coupling between consecutive levels in the weak Euler-based MLMC algorithm is achieved by coupling of the corresponding Euler approximations via a linear transformation of increments. In fact, what we are presenting here is only one possible way of inducing the dependence between levels. Another possibility of coupling is the use of antithetics (see Giles and Szpruch [5]). In the next proposition we show that the introduced coupling is strong enough to ensure (1.2) with $\beta = 1$.

Proposition 2. *Suppose that the coefficient functions b, σ are uniformly Lipschitz and have at most linear growth, i.e.,*

$$\|b(x) - b(x')\| \le L_b \|x - x'\|, \quad \|\sigma(x) - \sigma(x')\| \le L_\sigma \|x - x'\|, \tag{3.8}$$

$$\|b(x)\|^2 \le B_b^2 (1 + \|x\|^2), \quad \|\sigma(x)\|^2 \le B_\sigma^2 (1 + \|x\|^2), \tag{3.9}$$

for any $x, x' \in \mathbb{R}^d$ and some constants L_b, L_σ, $B_b > 0$ and $B_\sigma > 0$. Then for any Lipschitz continuous function f, the inequality

$$\mathrm{E}\left[\left|X^l_1\left(\xi_l\right) - X^{l-1}_1\left(\varsigma\left(\xi_l\right)\right)\right|^2\right] \le c\Delta_l \tag{3.10}$$

holds for $l = 1, \ldots, L$, and some $c > 0$, provided $\mathrm{E}[X_0^2] < \infty$.

Proposition 2 together with Theorem 3.1 from Giles [4] immediately implies the following

Corollary 3. *Under a proper choice of* $\mathbf{N}$ *and* L*, the complexity of the estimate* $Y_{L,\mathbf{N}}$ *needed to achieve the accuracy* ε *(as measured by RMSE) is of order* $\varepsilon^{-2}\log^2(\varepsilon)$.

4 Multilevel path simulation for weak Milstein scheme

4.1 Weak Milstein scheme

The (strong) Milstein's scheme has the form

$$\begin{aligned} X_0^\Delta &= X_0, \\ X_{j\Delta}^\Delta &= X_{(j-1)\Delta}^\Delta + b(X_{(j-1)\Delta}^\Delta)\Delta + \sum_{k=1}^m \sigma_k(X_{(j-1)\Delta}^\Delta)(W_{j\Delta} - W_{(j-1)\Delta}) \\ &\quad + \sum_{i,k=1}^m \partial\sigma_i(X_{(j-1)\Delta}^\Delta)\sigma_k(X_{(j-1)\Delta}^\Delta)\int_{(j-1)\Delta}^{j\Delta}(W_s^k - W_{(j-1)\Delta}^k)\,dW_s^i. \end{aligned}$$

This scheme has been introduced by Milstein to improve the strong convergence rate of the Euler scheme. Unfortunately, if $m > 1$, the numerical complexity of the Milstein scheme is substantially larger than the one of the Euler scheme. Indeed, the Milstein scheme requires the simulation of m^2 stochastic integrals

$$I_{k,i} \doteq \int_{(j-1)\Delta}^{j\Delta}(W_s^k - W_{(j-1)\Delta}^k)\,dW_s^i$$

and if $k \neq i$ the exact simulation of $I_{k,i}$ is impossible. One way to overcome this difficulty in the framework of multilevel Monte Carlo approach has recently been suggested in Giles and Szpruch [5]. Here we propose to use a weak Milstein scheme with $I_{k,j}$ replaced by some simple random variables with the same moments up to order 3. In order to perform multilevel path simulation in this case, we couple these random variables on consecutive levels in a proper way. The weak Milstein scheme can be defined as follows. Fix some natural number $n > 0$ and define with $\Delta = T/n$

$$\begin{aligned} X_0^\Delta &= X_0, \\ X_{j\Delta}^\Delta &= X_{(j-1)\Delta}^\Delta + b\left(X_{(j-1)\Delta}^\Delta\right)\Delta + \sum_{k=1}^m \sigma_k\left(X_{(j-1)\Delta}^\Delta\right)\xi_j^k \\ &\quad + \frac{1}{2}\sum_{i,k=1}^m \partial\sigma_i\left(X_{(j-1)\Delta}^\Delta\right)\sigma_k\left(X_{(j-1)\Delta}^\Delta\right)\left(\xi_j^i\xi_j^k + \eta_j^{i,k}\right), \end{aligned} \tag{4.1}$$

for $j = 1, \ldots, n$, where each random vector $\xi = (\xi_1^i, \ldots, \xi_n^i,\ i = 1, \ldots, m)$ has i.i.d. components satisfying

$$\mathrm{E}[\xi_j^1] = \mathrm{E}\left[(\xi_j^1)^3\right] = 0, \quad \mathrm{E}\left[(\xi_j^1)^2\right] = \Delta, \quad \mathrm{E}\left[(\xi_j^1)^4\right] \leq \Delta^2$$

and the independent random variables $\eta_j^{i,k}$ fulfil

$$\mathrm{E}[\eta_j^{i,k}] = 0, \quad \mathrm{E}\left[(\eta_j^{i,k})^2\right] \leq \Delta^2, \quad \mathrm{E}\left[(\eta_j^{i,k})^3\right] = 0 \tag{4.2}$$

for $k < i$ and

$$\eta_j^{i,i} = -\Delta, \quad \eta^{i,k} = -\eta^{k,i}.$$

Discussion The weak Milstein scheme (4.1) has the same weak convergence order as the weak Euler scheme (3.1). This can be seen from the fact (see [12]) that the matrices

$$\mathrm{E}\left[\left(X_{j\Delta}^{\Delta} - X_{(j-1)\Delta}^{\Delta}\right)\left(X_{j\Delta}^{\Delta} - X_{(j-1)\Delta}^{\Delta}\right)^{\top}\right], \quad j = 1, \ldots, n,$$

coincide for both schemes. In fact, the strong convergence orders of the schemes (4.1) and (3.1) are also equal. So at first sight it may seem that the weak Milstein scheme has no advantages over the weak Euler scheme. However, as we will see later, in the context of the multilevel path simulation approach, the additional term in (4.1) becomes crucial.

4.2 MLMC algorithm

Fix some $L > 0$ and set $\Delta_l = 2^{-l}T$, $l = 0, \ldots, L$. Denote

$$\boldsymbol{\xi}_L = \left(\xi_{L,1}^i, \ldots, \xi_{L,2^L}^i,\, i = 1, \ldots, m\right) \in \mathbb{R}^{m \times 2^L},$$

where $\xi_{L,1}^i, \ldots, \xi_{L,2^L}^i$ are i.i.d. random variables satisfying

$$\mathrm{E}[\xi_{L,1}^i] = \mathrm{E}[(\xi_{L,1}^i)^3] = 0, \quad \mathrm{E}[(\xi_{L,1}^i)^2] = \Delta_L, \quad \mathrm{E}[(\xi_{L,1}^i)^{2p}] \leq \Delta_L^p \tag{4.3}$$

for $p = 2, 4$. Moreover, let $\boldsymbol{\eta}_L$ be a random vector of the length $m^2 \times 2^L$ with i.i.d. components defined as

$$\boldsymbol{\eta}_L = \left(\eta_{L,1}^{i,k}, \ldots, \eta_{L,2^L}^{i,k},\, i, k = 1, \ldots, m\right),$$

where each r.v. $\eta_{L,1}^{i,k}$ fulfils

$$\mathrm{E}[\eta_{L,1}^{i,k}] = \mathrm{E}[(\eta_{L,1}^{i,k})^3] = 0, \quad \mathrm{E}[(\eta_{L,1}^{i,k})^p] \leq \Delta_L^p \tag{4.4}$$

for $p = 2, 4$. Now we define recursively the independent random vectors $\boldsymbol{\xi}_{L-1}, \ldots, \boldsymbol{\xi}_0$ and $\boldsymbol{\eta}_{L-1}, \ldots, \boldsymbol{\eta}_0$ via $\boldsymbol{\xi}_{l-1} \sim \boldsymbol{\varsigma}(\boldsymbol{\xi}_l) \in \mathbb{R}^{m \times 2^{l-1}}$ with

$$\varsigma_j(\boldsymbol{\xi}_l) \doteq \left(\xi_{l,(2j-1)}^i + \xi_{l,2j}^i,\, i = 1, \ldots, m\right), \quad j = 0, \ldots, 2^{l-1},$$

and $\boldsymbol{\eta}_{l-1} \sim \boldsymbol{\varsigma}(\boldsymbol{\eta}_l) \in \mathbb{R}^{m^2 \times 2^{l-1}}$ with

$$\varsigma_j(\boldsymbol{\eta}_l) \doteq \left(\eta_{l,(2j-1)}^{i,k} + \eta_{l,2j}^{i,k} - \xi_{l,(2j-1)}^i \xi_{l,2j}^k + \xi_{l,(2j-1)}^k \xi_{l,2j}^i,\, i, k = 1, \ldots, m\right)$$

for $j=0,\dots,2^{l-1}$. As can be easily seen, the components of the vectors $\boldsymbol{\xi}_l$ and $\boldsymbol{\eta}_l$ are again i.i.d. random variables satisfying the moment conditions

$$\mathrm{E}[\xi_{l,1}^i]=\mathrm{E}[(\xi_{l,1}^i)^3]=0,\quad \mathrm{E}[(\xi_{l,1}^i)^2]=\Delta_l,\quad \mathrm{E}[(\xi_{l,1}^i)^{2p}]\le\Delta_l^p$$

and

$$\mathrm{E}[\eta_{l,1}^{i,k}]=\mathrm{E}[(\eta_{l,1}^{i,k})^3]=0,\quad \mathrm{E}[(\eta_{l,1}^{i,k})^p]\le\Delta_l^p$$

for $p=2,4$. Next for any $l=1,\dots,L$, and any random pair $(\boldsymbol{\xi},\boldsymbol{\eta})\in\mathbb{R}^{m\times 2^l}\times\mathbb{R}^{m^2\times 2^l}$, consider the weak Milstein scheme

$$\begin{aligned}
X_0^l(\boldsymbol{\xi},\boldsymbol{\eta}) &= X_0,\\
X_{j\Delta_l}^l(\boldsymbol{\xi},\boldsymbol{\eta}) &= X_{(j-1)\Delta_l}^l(\boldsymbol{\xi},\boldsymbol{\eta})+b\left(X_{(j-1)\Delta_l}^l(\boldsymbol{\xi},\boldsymbol{\eta})\right)\Delta+\sum_{k=1}^m\sigma_k\left(X_{(j-1)\Delta_l}^l(\boldsymbol{\xi},\boldsymbol{\eta})\right)\xi_{l,j}^k\\
&\quad+\frac{1}{2}\sum_{i,k=1}^m\partial\sigma_i\left(X_{(j-1)\Delta_l}^l(\boldsymbol{\xi},\boldsymbol{\eta})\right)\sigma_k\left(X_{(j-1)\Delta}^l(\boldsymbol{\xi},\boldsymbol{\eta})\right)\left(\xi_{l,j}^i\xi_{l,j}^k+\eta_{l,j}^{i,k}\right),
\end{aligned}\tag{4.5}$$

with $j=1,\dots,2^l$, and some r.v. $X_0\in\mathbb{R}^d$. Finally fix a vector of natural numbers $\mathbf{N}=(N_0,\dots,N_L)$ and define the weak MLMC estimate for $\mathrm{E}[f(X_T)]$ as follows

$$\begin{aligned}
Y_{L,\mathbf{N}} &\doteq \frac{1}{N_0}\sum_{n=1}^{N_0}\left[f\left(X_1^0(\boldsymbol{\xi}_0^{(n)},\boldsymbol{\eta}_0^{(n)})\right)\right]+\\
&\quad+\sum_{l=1}^L\frac{1}{N_l}\sum_{n=1}^{N_l}\left[f\left(X_1^l\left(\boldsymbol{\xi}_l^{(n)},\boldsymbol{\eta}_l^{(n)}\right)\right)\right.\\
&\quad\left.-f\left(X_1^{l-1}\left(\varsigma(\boldsymbol{\xi}_l^{(n)}),\varsigma(\boldsymbol{\eta}_l^{(n)})\right)\right)\right],
\end{aligned}\tag{4.6}$$

where $(\boldsymbol{\xi}_l^{(n)},\boldsymbol{\eta}_l^{(n)})$, $n=1,\dots,N_l$, are i.i.d. copies of $(\boldsymbol{\xi}_l,\boldsymbol{\eta}_l)$. The next proposition is analogous to Proposition 2.

Proposition 4. *Suppose that the coefficient functions b, σ are uniformly Lipschitz and have at most linear growth, i.e., they satisfy the conditions* (3.8) *and* (3.9) *Moreover, suppose that each function $\partial_{x^p}\sigma=(\partial_{x^p}\sigma_{ij})$, $p=1,\dots,d$, is uniformly Lipschitz and bounded on $\mathbb{R}^d$, i.e.,*

$$\|\partial_{x^p}\sigma(x)\|\le B_{\partial\sigma},\ \|\partial_{x^p}\sigma(x)-\partial_{x^p}\sigma(x')\|=L_{\partial\sigma}\|x-x'\|,\quad x,x'\in\mathbb{R}^d$$

for $B_{\partial\sigma}>0$ and $L_{\partial\sigma}>0$. Then

$$\mathrm{E}\left[\left|X_1^l\left(\boldsymbol{\xi}_l,\boldsymbol{\eta}_l\right)-X_1^{l-1}\left(\varsigma(\boldsymbol{\xi}_l),\varsigma(\boldsymbol{\eta}_l)\right)\right|^2\right]\le c\Delta_l^2\tag{4.7}$$

holds for $l=1,\dots,L$, and some $c>0$, provided $\mathrm{E}[X_0^4]<\infty$.

Corollary 5. *Under a proper choice of* $\mathbf{N}$ *and L, the complexity of the estimate $Y_{L,\mathbf{N}}$ needed to achieve the accuracy ε (as measured by RMSE) is of order ε^{-2}.*

4.3 Implementation

First note that in (4.4) we only require $\mathrm{E}\big[(\eta^{i,k}_{L,j})^p\big] \le \Delta^p$ for $i < k$ and $p = 2, 4$. So, in principle, one can take $\eta^{i,k}_{L,j} = 0$ for all j and $i < k$. The reason for this is that we do not need to approximate the Lévy area, but only have to provide coupling between levels. The coupling relation for η

$$\eta^{i,k}_{l-1,j} = (\eta^{i,k}_{l,2j-1} + \eta^{i,k}_{l,2j} - \xi^{i}_{l,2j-1}\xi^{k}_{l,2j} + \xi^{k}_{l,2j-1}\xi^{i}_{l,2j}).$$

can be written in the matrix form as follows

$$H_{l-1,j} = H_{l,2j-1} + H_{l,2j} + (\Xi_{l,j} - \Xi^{\top}_{l,j})$$

with $\Xi_{l,j} = \xi_{l,2j-1}\xi^{\top}_{l,2j}$. Let us consider the case of normal innovations $\xi^k_{l,j}$. Since $\xi^k_{l,2j-1} \cdot \xi^i_{l,2j} \sim \Delta_l \cdot \zeta^k_{l,2j-1} \cdot \zeta^i_{l,2j}$, where $\zeta_{l,2j-1} = (\zeta^1_{l,2j-1}, \ldots, \zeta^m_{l,2j-1})$ and $\zeta_{l,2j} = (\zeta^1_{l,2j}, \ldots, \zeta^m_{l,2j})$ are two independent m-dimensional vectors with independent $\mathcal{N}(0,1)$ distributed components, we have

$$H_{l-1,j} \sim H_{l,2j-1} + H_{l,2j} + \Delta_l(\mathscr{W}_{l,j} - \mathscr{W}^{\top}_{l,j})$$

with $\mathscr{W}_{l,j} = \zeta_{l,2j-1}\zeta^{\top}_{l,2j}$. At the top level we can take

$$H_{L,j} = -\mathrm{diag}(\Delta_L), \quad j = 1, \ldots, 2^L, \tag{4.8}$$

and as a result

$$H_{L-k,j} \sim -2^k \mathrm{diag}(\Delta_L) + \sum_{p=0}^{k-1} \Delta_{L-p} \sum_{i=0}^{2^{k-1-p}-1} \left(\mathscr{W}_{L-p,(2^{k-1-p}j-i)} - \mathscr{W}^{\top}_{L-p,(2^{k-1-p}j-i)}\right).$$

Using the identity

$$\mathscr{W}_{l,j} - \mathscr{W}^{\top}_{l,j} = \mathrm{Im}\Big[(\zeta_{l,2j-1} + \mathrm{i}\zeta_{l,2j})(\zeta_{l,2j-1} + \mathrm{i}\zeta_{l,2j})^{\star}\Big],$$

where $A^{\star}$ stands for the transposed conjugate of the complex valued matrix A, we get for the distribution of the matrix $H_{L-k,j}$

$$H_{L-k,j} \sim -2^k \mathrm{diag}(\Delta_L) + \sum_{p=0}^{k-1} \Delta_{L-p}\, \mathrm{Im}[W_m(2^{k-p-1})], \tag{4.9}$$

where $W_m(2^{k-p})$ is an $m \times m$ random matrix having complex Wishart distribution with 2^{k-p} degrees of freedom and unit covariance matrix (see Goodman [6]). The distribution of the random matrix $W_m(n)$ for $n > m-1$ is given by

$$p_W(A) = \frac{|A|^{n-m}}{C(n,m)} \exp(-\mathrm{tr}(A)), \quad A \in \mathscr{A},$$

where $C(n,m) = \pi^{m(m-1)/2}\Gamma(n)\cdot\ldots\cdot\Gamma(n-m+1)$ and $\mathscr{A}$ is the open convex cone of the Hermitian $m\times m$ matrices, which are positive definite. The efficient generation of random matrices with complex Wishart distribution can be done via the so called Bartlett decomposition

$$A = T^{\star}T,$$

where T is an upper triangle complex matrix with real and positive diagonal elements. In the case of complex Wishart distributed A, the distribution of T has rather simple form (see Goodman [6])

$$p(T) = \frac{1}{C(n,m)} T_{11}^{2n-1} T_{22}^{2n-3}\cdot\ldots\cdot T_{mm}^{2n-(2m-1)} \exp(-\operatorname{tr}(T^{\star}T)/2)$$

with $T^{\star}T = \sum_{j\leq i} |T_{ij}|^2$. In particular, it can be shown that $T_{ii} = \sqrt{V_i}$, $i = 1,\ldots,m$, where V_i is a random variables having chi-squared distribution with the parameter $2(n-i+1)$ and all V_i are independent. Moreover, $T_{ij} = T_{ij,1} + \mathrm{i}T_{ij,2}$, $j < i$, where $T_{ij,1}$ and $T_{ij,2}$ are independent random variables with the standard normal distribution. This allows to simulate a single matrix A with complex Wishart distribution in $O(m^2)$ operations.

Example 6. *If $m = 2$, we have, due to (4.9), the following representation for the distribution of $\eta_{L-k,j}^{1,2}$*

$$\eta_{L-k,j}^{1,2} \sim \frac{1}{2}\sum_{p=0}^{k-1} \Delta_{L-p}(\chi_{2^{k-p}} - \widetilde{\chi}_{2^{k-p}}), \tag{4.10}$$

where $\chi_{2^{k-p}}$ and $\widetilde{\chi}_{2^{k-p}}$ are two independent random variables having the chi-squared distributions with 2^{k-p} degrees of freedom. Furthermore, it can be shown that the difference of two chi-square distributed random variables with ν degrees of freedom has the so-called variance-gamma distribution with density of the form

$$f(z) = \frac{1}{2^{\nu/2}\sqrt{\pi}}\frac{1}{\Gamma(\nu/2)}|z|^{(\nu-1)/2}K_{(\nu-1)/2}(|z|),$$

where K is the modified Bessel function.

The formula (4.9) gives the joint distribution of η's. Next, in each level $l-1$ and for any $k \in \{1,\ldots,m\}$, we need to generate a vector

$$(\eta_{l-1,j}^{1,k},\ldots,\eta_{l-1,j}^{k-1,k},\xi_{l-1,j}^{1},\ldots,\xi_{l-1,j}^{k}), \quad j = 1,\ldots,2^{l-1},$$

via the relations

$$\begin{aligned}
\eta_{l-1,j}^{i,k} &\sim \eta_{l,2j-1}^{i,k} + \eta_{l,2j}^{i,k} - \xi_{l,2j-1}^{i}\xi_{l,2j}^{k} + \xi_{l,2j-1}^{k}\xi_{l,2j}^{i},\\
\xi_{l-1,j}^{i} &\sim \xi_{l,2j-1}^{i} + \xi_{l,2j}^{i}
\end{aligned}$$

for $i = 1,\ldots,k$, where $(\eta_{l,2j}^{i,k})$ and $(\eta_{l,2j-1}^{i,k})$ are two independent random matrices distributed according to (4.9) with $L-k = l$, and $\xi_{l,2j-1}$,$\xi_{l,2j}$, are m-dimensional independent random vectors with $\mathscr{N}(0,\Delta_l)$ components, which are both independent of $(\eta_{l,2j}^{i,k})$ and $(\eta_{l,2j-1}^{i,k})$.

5 Numerical experiments

It is important to state, that the top level L can be chosen in advance to be as large, as it is convenient. This choice simply bounds from below the bias, that one can get with this method. We would also like to point out, that the range of L is actually bounded from above by the hardware resources one has, so our approach to fix the final L is not very restrictive, as from the algorithmic point of view, it can be chosen as large, as it is necessary. In all examples below, the parameters of the weak ML algorithm are chosen as suggested in the original paper of Giles [4].

5.1 One-dimensional example with the weak Euler approximation

Our numerical experiments aim to demonstrate that the MLMC algorithm preserves its convergence properties when standard Brownian paths increments in the Euler approximation scheme are replaced by the increments defined by (3.7), which satisfy conditions (3.2)–(3.3), and the coupling is done according to (3.4). We start with a one-dimensional example. Let X be a geometric Brownian motion, i.e. X solves (1.1) with $b(X) = r \cdot X$ and $\sigma(X) = \sigma \cdot X$. Our aim is to compute $\mathrm{E}[f(X_T)]$ with

$$f(x) = e^{-r \cdot T} \max(x - K, 0).$$

The well known Black-Scholes formula gives

$$\mathrm{E}[f(X_T)] = N(d_1)X_0 - N(d_2)Ke^{-rT},$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}} \left(\ln\left(\frac{X_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T \right), \quad d_2 = d_1 - \sigma\sqrt{T}.$$

We chose the following parameters set:

$$r = 0.05, \quad \sigma = 0.2, \quad T = 1, \quad K = 1, \quad X_0 = 1.$$

Let X^l, $l = 1, \ldots, L$, be the approximations given by (3.5) with the top level $L = 8$. Sample trajectories of X^l for $l \in \{0, 2, 4, 6, 8\}$, are presented on Figure 5.1. Let us first look at the variances of r.v.

$$Y_l = \frac{1}{N_l} \sum_{n=1}^{N_l} \left[f\left(X_1^l \left(\xi_l^{(n)}, \eta_l^{(n)} \right) \right) - f\left(X_1^{l-1} \left(\varsigma(\xi_l^{(n)}), \varsigma(\eta_l^{(n)}) \right) \right) \right].$$

In Figure 5.2 the estimates for $N_l \operatorname{Var}(Y_l)$ are presented for different values of l. We observe that the version with the binomial increments has the same convergence of variances as one with the normal increments. We also estimate the variance decay in l by fitting the line $\alpha_1 - \alpha_2 \cdot l$ to the estimated log-variances

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