

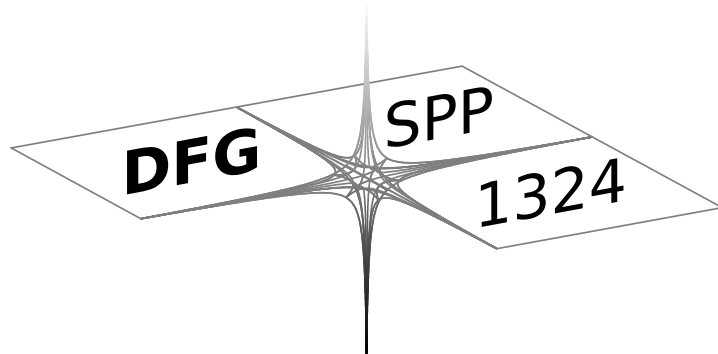
# DFG-Schwerpunktprogramm 1324

„Extraktion quantifizierbarer Information aus komplexen Systemen“

## Tractability of the approximation of high-dimensional rank one tensors

E. Novak, D. Rudolf

Preprint 164



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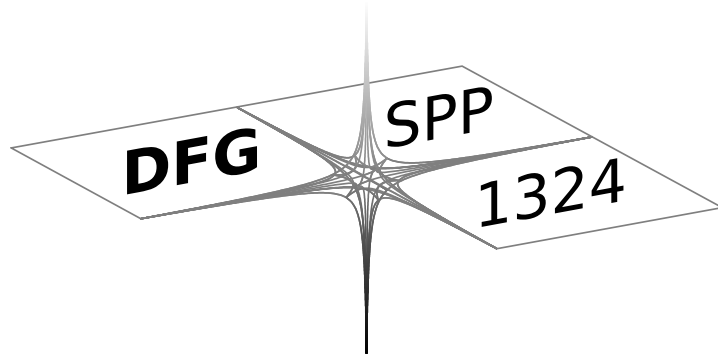
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# Tractability of the approximation of high-dimensional rank one tensors

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## Abstract

We study the approximation of high-dimensional rank one tensors using point evaluations. We prove that for certain parameters (smoothness and norm of the  $r$ th derivative) this problem is intractable while for other parameters the problem is tractable and the complexity is only polynomial in the dimension. We completely characterize the set of parameters that lead to easy or difficult problems, respectively. In the “difficult” case we modify the class to obtain a tractable problem: The problem gets tractable with a polynomial (in the dimension) complexity if the support of the function is not too small.

**Keywords:** High dimensional approximation, rank one tensors, tractability

**Classification:** 65D15, 65Y20

## 1 Introduction and curse of dimensionality

Many real world problems are high-dimensional, they involve functions  $f$  that depend on many variables. It is known that the approximation of functions from certain smoothness classes suffers from the curse of dimensionality, i.e., the complexity (the cost of an optimal

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algorithm) is exponential in the dimension  $d$ . The recent papers [4, 8] contain such results for classical  $C^k$  and also  $C^\infty$  spaces, the known theory is presented in the books [7, 9, 10]. To avoid this curse of dimensionality one studies problems with a structure, see again the monographs just mentioned.

One possibility is to assume that the function  $f$ , say  $f : [0, 1]^d \rightarrow \mathbb{R}$ , is a tensor of rank one, i.e.,  $f$  is of the form

$$f(x_1, x_2, \dots, x_d) = \prod_{i=1}^d f_i(x_i), \quad (1)$$

for  $f_i : [0, 1] \rightarrow \mathbb{R}$ . For short we also write  $f = \bigotimes_{i=1}^d f_i$ . At first glance, the ‘‘complicated’’ function  $f$  is given by  $d$  ‘‘simple’’ functions. One might hope that with this model assumption the curse of dimensionality can be avoided.

In the recent paper [1] the authors investigate how well a rank one function can be captured (approximated in  $L_\infty$ ) from  $n$  point evaluations. They use the function classes

$$F_{M,d}^r = \left\{ f \mid f = \bigotimes_{i=1}^d f_i, \|f_i\|_\infty \leq 1, \|f_i^{(r)}\|_\infty \leq M \right\} \quad (2)$$

and define an algorithm  $A_n$  that uses  $n$  function values. Here, given an integer  $r$  it is assumed that  $f_i \in W_\infty^r[0, 1]$ , where  $W_\infty^r[0, 1]$  is the set of all univariate functions on  $[0, 1]$  which have  $r$  weak derivatives in  $L_\infty$ , and  $f_i^{(r)}$  is the  $r$ th weak derivative.

In [1] the authors consider an algorithm which consists of two phases. For  $f \in F_{M,d}^r$ , the first phase is looking for a  $z^* \in [0, 1]^d$  such that  $f(z^*) \neq 0$ , the second phase takes this  $z^*$  and constructs an approximation of  $f$ . The main error bound [1, Theorem 5.1] distinguishes two cases:

- If in the first phase no  $z^*$  with  $f(z^*) \neq 0$  was found, then  $f$  itself is close to zero, i.e.  $A_n(f) = 0$  satisfies

$$\|f - A_n(f)\|_\infty \leq C_{d,r} M^d n^{-r}. \quad (3)$$

- If such a point  $z^*$  is given in advance, then the second phase returns an approximation  $A_n(f)$  and the bound

$$\|f - A_n(f)\|_\infty \leq C_r M d^{r+1} n^{-r} \quad (4)$$

holds. Here  $C_r > 0$  is independent of  $d, M, n$  and  $n \geq d \max\{(dC_r M)^{1/r}, 2\}$ .

**Remark 1.** The error bounds of (3) and (4) are nice since the order of convergence  $n^{-r}$  is optimal. In this sense, which is the traditional point of view in numerical analysis, the authors of [1] correctly call their algorithm an optimal algorithm. When we study the

tractability of a problem we pose a different problem and want to know whether the number of function evaluations, for given spaces and an error bound  $\varepsilon > 0$ , increases exponentially in the dimension  $d$  or not. The curse of dimensionality may happen even for  $C^\infty$  functions where the order of convergence is excellent, see [8].

Consider again the bound (3). This bound is proved in [1] with the Halton sequence and hence we only have a non-trivial error bound if

$$n > \left(2^d \prod_{i=1}^d p_i\right) (2M)^{d/r} \quad (5)$$

where  $p_1, p_2, \dots, p_d$  are the first  $d$  primes. The number  $n$  of needed function values for given parameters  $(r, M)$  and error bound  $\varepsilon < 1$  increases *always*, i.e., for all  $(r, M, \varepsilon)$ , (super-) exponentially with the dimension  $d$ .  $\square$

We ask whether this *curse of dimensionality* is inherent in the problem or whether it can be avoided by a better algorithm. We shall see that the answer depends on  $r$  and  $M$ , but not on  $\varepsilon$ . The curse of dimensionality is present for the classes  $F_{M,d}^r$  if and only if  $M \geq 2^r r!$ . For smaller  $M$  we construct a randomized algorithm that, for any fixed  $\varepsilon > 0$ , has polynomial (in  $d$ ) cost.

To precisely formulate the results we need some further notation. We want to recover a function  $f$  from a class  $F_d$  of functions defined on  $[0, 1]^d$ . We consider the worst case error of an algorithm  $A_n$  on  $F_d$  and stress that  $F_d$ , in this paper, is not the unit ball with respect to some norm since it is not convex. Hence we can not apply results that are based on this assumption, in particular we allow (and should allow) all adaptive algorithms

$$A_n(f) = \phi(f(x_1), f(x_2), \dots, f(x_n)), \quad (6)$$

with  $\phi: \mathbb{R}^n \rightarrow L_\infty$ , where the  $x_i \in [0, 1]^d$  can be chosen adaptively, depending on the already known function values  $f(x_1), \dots, f(x_{i-1})$ . See, for example, [6, 7]. The worst case error of a deterministic algorithm  $A_n$  is defined as

$$e^{\det}(A_n) = \sup_{f \in F_d} \|f - A_n(f)\|_\infty,$$

whereas the  $n$ th minimal worst case error is

$$e^{\det}(n, F_d) = \inf_{A_n} e^{\det}(A_n), \quad (7)$$

where  $A_n$  runs through the set of all deterministic algorithms that use at most  $n$  function values.

Now we consider  $F_{M,d}^r$  with  $M = 2^r r!$  for  $r \in \mathbb{N}$ . Then there is a function  $g \in W_\infty^r[0, 1]$  with  $\|g\|_\infty = 1$  and  $\|g^{(r)}\|_\infty = M = 2^r r!$  such that the support of  $g$  is  $[0, 1/2]$  or  $[1/2, 1]$ . The  $2^d$  tensor products of such functions show that the initial error 1 of our problem cannot be reduced by less than  $2^d$  function values. We obtain the following result.

**Theorem 2.** Let  $r \in \mathbb{N}$  and  $M = 2^r r!$ . Then

$$e^{\det}(n, F_{M,d}^r) = 1 \quad \text{for} \quad n = 1, 2, \dots, 2^d - 1. \quad (8)$$

In this paper we also allow randomized algorithms  $A_n$ , i.e., the  $x_i$  and also  $\phi$  may be chosen randomly, see Section 4.3.3 of [7]. Then the output  $A_n(f)$  is a random variable and the worst case error of such an algorithm on a class  $F_d$  is defined by

$$e^{\text{ran}}(A_n) = \sup_{f \in F_d} (\mathbb{E}(\|f - A_n(f)\|_\infty)^2)^{1/2}. \quad (9)$$

Similarly to (7) the numbers  $e^{\text{ran}}(n, F_d)$  are again defined by the infimum over all  $e^{\text{ran}}(A_n)$  but now of course we allow randomized algorithms.

Theorem 2 is for deterministic algorithms. Already the authors of [1] suggest that randomized algorithms might be useful for this problem. We will see that this is true if  $M < 2^r r!$  but not for larger  $M$ . This follows from the results of Section 2.2.2 in [5].

**Theorem 3.** Let  $r \in \mathbb{N}$  and  $M = 2^r r!$ . Then

$$e^{\text{ran}}(n, F_{M,d}^r) \geq \frac{1}{2}\sqrt{2} \quad \text{for} \quad n = 1, 2, \dots, 2^{d-1}. \quad (10)$$

What do these lower bounds (8) and (10) mean? We may say that this assumption,  $f$  being a rank one tensor, is not a good assumption to avoid the curse of dimension — at least if we study the approximation problem with standard information (function values) and if  $M \geq 2^r r!$ . If  $M$  is “large” then the classes  $F_{M,d}^r$  are too large and we have the curse of dimensionality. What is the problem? A function  $f \in F_{M,d}^r$  can be non-zero just in a small subset of the cube  $[0, 1]^d$ . Then it might be difficult to find a  $z^*$  with  $f(z^*) \neq 0$ . If such a point  $z^*$  is found then the rest is easy, as analyzed in [1]. For future reference, we restate a result of [1], see (4), as a lemma.

**Lemma 4.** Let  $r \in \mathbb{N}$  and  $M > 0$ . Consider all  $f \in F_{M,d}^r$  with  $f \neq 0$  and assume that a  $z^* \in [0, 1]^d$  is known such that  $f(z^*) \neq 0$ . Then, if  $n > d \max\{(dC_r M)^{1/r}, 2\}$ , there is an algorithm  $A_n$  with

$$\|f - A_n(f)\|_\infty \leq C_r M d^{r+1} n^{-r}. \quad (11)$$



How can we obtain positive results concerning the tractability of the approximation of high-dimensional rank one tensors using function values? We offer two possibilities, both of them are studied in this paper:

- We study the same class  $F_{M,d}^r$  but with “small” values of  $M$ , i.e.,  $M < 2^r r!$ . We do not have the curse of dimension for this class of functions, but of course we need other algorithms than those of [1] to prove tractability. See Section 2.
- We allow an arbitrary  $M > 0$  but study the smaller class

$$F_{M,d}^{r,V} = \{f \in F_{M,d}^r \mid f(x) \neq 0 \text{ for all } x \text{ from a box with volume greater than } V\}. \quad (12)$$

By a box we mean a set of the form  $\prod_{i=1}^d [\alpha_i, \beta_i] \subset [0, 1]^d$ . If  $V$  is not too small then, this is what we will prove, the problem is polynomially tractable and the curse of dimensionality disappears. Again we need new algorithms to prove this result. In Section 3 we study deterministic as well as randomized algorithms.

We end this section with a few more definitions and the proofs of Theorem 2 and Theorem 3. Sometimes it is more convenient to discuss the inverse function of  $e^{\det}(n, F_d)$ ,

$$n^{\det}(\varepsilon, F_d) = \inf\{n \mid e^{\det}(n, F_d) \leq \varepsilon\}, \quad (13)$$

instead of  $e^{\det}(n, F_d)$  itself. The numbers  $n^{\text{ran}}(\varepsilon, F_d)$  are defined similarly. We say that a problem suffers from the curse of dimensionality in the deterministic setting, if  $n^{\det}(\varepsilon, F_d) \geq C \alpha^d$  for some  $\varepsilon > 0$ , where  $C > 0$  and  $\alpha > 1$ . The complexity in the deterministic setting is polynomial in the dimension if  $n^{\det}(\varepsilon, F_d) \leq C d^\alpha$  for each  $\varepsilon > 0$ , where  $C > 0$  and  $\alpha > 1$  may depend on  $\varepsilon$ . By replacing  $n^{\det}(\varepsilon, F_d)$  by  $n^{\text{ran}}(\varepsilon, F_d)$  the curse of dimensionality and polynomial complexity is defined in the randomized setting.

*Proof of Theorem 2.* Assume that  $A_n$  is a deterministic algorithm and  $n \leq 2^d - 1$ . Since  $f_0 = 0$  is in the space  $F_{M,d}^r$  there are function values  $f(x_1) = \dots = f(x_n) = 0$  that are computed for the function  $f = f_0$ . Since  $n \leq 2^d - 1$  there is at least one orthant of  $[0, 1]^d$  which contains no sample point. Without loss of generality we assume that this orthant is  $[0, 1/2]^d$ . The function  $f^+ = \bigotimes_{i=1}^d f_i$  with

$$f_i(y_i) = 2^r \max\{0, (\frac{1}{2} - y_i)^r\}$$

is zero on  $[0, 1]^d \setminus [0, 1/2]^d$ , is an element of  $F_{M,d}^r$  for  $M = 2^r r!$  and  $f^+(0) = 1$ . By construction we have  $f^+(x_1) = \dots = f^+(x_n) = 0$  and, hence,  $A_n(f^+) = A_n(-f^+) = A_n(f_0)$ , since  $A_n$  cannot

distinguish those three inputs. From  $\|f^+ - (-f^+)\|_\infty = 2$  we conclude that  $e^{\det}(A_n) \geq 1$  and hence that

$$e^{\det}(n, F_{M,d}^r) \geq 1.$$

The inequality  $e^{\det}(n, F_{M,d}^r) \leq 1$  is trivial since the zero algorithm has error 1.  $\square$

*Proof of Theorem 3.* Assume again that  $M \geq 2^r r!$  and let  $n = 2^{d-1}$ . Then there are functions  $f_1, \dots, f_{2n}$  such that the  $f_i$  have disjoint supports,  $\|f_i\|_\infty = 1$  and  $\pm f_i \in F_{M,d}^r$ . The  $f_i$  can be taken as in the proof of Theorem 2. Therefore the statement follows (with the technique of Bakhvalov), see Section 2.2.2 in [5] for the details.  $\square$

## 2 Tractability for small values of $M$

Here we study the class  $F_{M,d}^r$  and assume that  $M < 2^r r!$ . We show that we do not have the curse of dimension for this class of functions.

We start with a simple observation which follows by standard error bounds for polynomial interpolation of a smooth function.

**Lemma 5.** Let  $a, b \in \mathbb{R}$  with  $a < b$  and  $g \in W_\infty^r[a, b]$ , further assume that  $g$  has  $r$  distinct zeros. Then

$$\|g\|_\infty \leq \|g^{(r)}\|_\infty \frac{(b-a)^r}{r!}. \quad (14)$$

Now assume that  $f \in F_{M,d}^r$  and at least one of the  $f_i$  has at least  $r$  distinct zeros. Then by Lemma 5  $\|f\|_\infty \leq \|f_i\|_\infty \leq \frac{M}{r!}$  holds. Assume now that  $M, r$  and  $\varepsilon$  are given with  $M \leq r! \varepsilon$ . Then there are only two cases:

1.  $\|f\|_\infty \leq \frac{M}{r!} \leq \varepsilon$ ; in this case we can approximate  $f$  by the zero function and this output is good enough, i.e., the error is bounded by  $\varepsilon$ .
2. All the sets  $\{x \in [0, 1] \mid f_i(x) = 0\}$  have less than  $r$  elements and hence  $\{x \in [0, 1]^d \mid f(x) = 0\}$  has measure zero.

We can consider the algorithm  $S_{1,n}(f)$ :

1. Choose  $x \in [0, 1]^d$  at random uniformly distributed.
2. If  $f(x) \neq 0$  then run the algorithm  $A_n(f)$  of Lemma 4, otherwise return 0.

This leads, by applying the error bound of Lemma 4, to the following result.

**Theorem 6.** Let  $\varepsilon > 0$ ,  $r \in \mathbb{N}$ ,  $M \in (0, r!\varepsilon)$  and  $n \geq d \max\{\varepsilon^{-1/r}(dC_r M)^{1/r}, 2\}$ . Then, for  $f \in F_{M,d}^r$  we have

$$\mathbb{P}(\|f - S_{1,n}(f)\|_\infty \leq \varepsilon) = 1.$$

We give a numerical example: Let  $r = 5$  and  $M = 10$  and  $\varepsilon = 1/10$ . Then the problem is easy, see Theorem 6. A single function evaluation is enough (with probability 1) for the first step of the algorithm. For  $r = 5$  and  $M = 120 \cdot 32$  and all  $\varepsilon$  the problem is difficult, see Theorem 2 and Theorem 3.

Now we assume that  $r$  and  $\varepsilon \in (0, 1)$  are given and  $M$  satisfies  $M \in (\varepsilon r!, 2^r r!)$ . We will construct a randomized algorithm with polynomial in  $d$  cost. For this we define

$$\delta^* = \left( \frac{1}{2^{r+1}} + \frac{r!}{2M} \right)^{1/r} - 1/2 \quad \text{and}$$

$$d^* = \left\lceil \frac{\log \varepsilon^{-1}}{\log \left( \frac{M}{2^{r+1} r!} + \frac{1}{2} \right)^{-1}} \right\rceil$$

and assume that  $d \geq d^*$ . Note that by  $\lambda_d$  we denote the  $d$ -dimensional Lebesgue measure and for  $I \subset \mathbb{N}$  we write  $|I|$  for the cardinality of  $I$ . We need the following lemma.

**Lemma 7.** Let  $M < 2^r r!$  and assume that  $f \in F_{M,d}^r$  with  $\|f\|_\infty \geq \varepsilon$ . Then there exists  $I \subset \{1, \dots, d\}$  with  $|I| \geq d - d^*$  such that for all  $i \in I$  the function  $f_i$  has less than  $r$  zeros on  $[1/2 - \delta^*, 1/2 + \delta^*]$ .

*Proof.* We prove the assertion by contraposition. Assume that there is  $J \subset \{1, \dots, d\}$  with  $|J| > d^*$  such that for all  $i \in J$  the function  $f_i$  has at least  $r$  zeros. Then by Lemma 5

$$\|f_i\|_\infty \leq (1/2 + \delta^*)^r \frac{M}{r!} \tag{15}$$

for all  $i \in J$ . Because of  $M < 2^r r!$ , the choice of  $\delta^*$  and the choice of  $d^*$  we have  $[(1/2 + \delta^*)^r \frac{M}{r!}]^{d^*} < \varepsilon$ , which finally leads to  $\|f\|_\infty < \varepsilon$ .  $\square$

This motivates the following algorithm denoted by  $S_{n_1, n_2}$ :

1. For  $1 \leq i \leq n_1$  choose  $d^*$  different coordinates  $(k_1, \dots, k_{d^*})$  uniformly distributed and set  $\bar{I} = \{k_1, \dots, k_{d^*}\}$ .
  - (a) For  $1 \leq j \leq d$  check whether  $j \in \bar{I}$ . If  $j \in \bar{I}$  then choose  $x_j \in [0, 1]$  uniformly distributed. Otherwise choose  $x_j \in [1/2 - \delta^*, 1/2 + \delta^*]$  uniformly distributed.
  - (b) Check whether  $f(x_1, \dots, x_d) \neq 0$ . If this is the case go to 2. If  $i = n_1$  and we did not find  $f(x) \neq 0$  then return  $S_{n_1, n_2}(f) = 0$ .

2. Run the algorithm of Lemma 4 and return  $S_{n_1, n_2}(f) = A_{n_2}(f)$ .

We obtain the following error bound for this algorithm.

**Theorem 8.** Let  $\varepsilon > 0$ ,  $r \in \mathbb{N}$ ,  $M \in (r!\varepsilon, 2^r r!)$  and  $n_2 \geq d \max\{\varepsilon^{-1/r} (dC_r M)^{1/r}, 2\}$ . Then, for  $f \in F_{M,d}^r$  we have

$$\mathbb{P}(\|f - S_{n_1, n_2}(f)\|_\infty \leq \varepsilon) \geq 1 - \left[ 1 - \left( \frac{1}{3d} \left( \frac{r!\varepsilon}{M} \right)^{1/r} \right)^{1 + \frac{2^{r+1} r! \log \varepsilon^{-1}}{(2^r r! - M)}} \right]^{n_1}.$$

*Proof.* We assume that  $\|f\|_\infty \geq \varepsilon$ , otherwise the zero output is fine. Then, by virtue of Lemma 5 we have for any  $f_i$  that  $\lambda_1(\{f_i \neq 0\}) \geq \left(\frac{r!\varepsilon}{M}\right)^{1/r}$ . Let us denote the probability that we found  $f(x) \neq 0$  in a single iteration of the first step of the algorithm  $S_{n_1, n_2}$  by  $\theta$ . Further note that for every  $1 \leq i \leq n_1$  there are  $\binom{d}{d^*}$  many choices of the  $d^*$  different coordinates in  $\bar{I}$ . Thus, by  $\binom{d}{d^*} \leq \left(\frac{3d}{d^*}\right)^{d^*}$ , Lemma 7 and the fact that  $\lambda_1(\{f_i \neq 0\}) \geq \left(\frac{r!\varepsilon}{M}\right)^{1/r}$  for any  $i \in \{1, \dots, d\}$  it follows

$$\theta \geq \frac{\left(\frac{r!\varepsilon}{M}\right)^{d^*/r}}{\binom{d}{d^*}} \geq \left[ \left(\frac{r!\varepsilon}{M}\right)^{1/r} \frac{d^*}{3d} \right]^{d^*}. \quad (16)$$

Further by  $1 - y < \log y^{-1}$  for  $y \in (0, 1)$  we obtain

$$1 \leq d^* \leq 1 + \frac{2^{r+1} r! \log \varepsilon^{-1}}{(2^r r! - M)}.$$

Now by the choice of  $n_2$ , Lemma 4 and the previous consideration it follows that

$$\mathbb{P}(\|f - S_{n_1, n_2}(f)\|_\infty \leq \varepsilon) = 1 - (1 - \theta)^{n_1} \geq 1 - \left[ 1 - \left( \frac{1}{3d} \left( \frac{r!\varepsilon}{M} \right)^{1/r} \right)^{1 + \frac{2^{r+1} r! \log \varepsilon^{-1}}{(2^r r! - M)}} \right]^{n_1}.$$

□

The error bound tells us that we find  $z^*$  with  $f(z^*) \neq 0$  with probability  $1 - p$  with

$$n_1 = \lceil C_{r, \varepsilon, M} \cdot d^{\alpha_{r, \varepsilon, M}} \log p^{-1} \rceil$$

function evaluations. This holds for

$$\alpha_{r,\varepsilon,M} = 1 + \frac{2^r r! \log \varepsilon^{-1}}{(2^r r! - M)} \quad \text{and} \quad C_{r,\varepsilon,M} = \left( \frac{3^r M}{r! \varepsilon} \right)^{\frac{\alpha_{r,\varepsilon,M}}{r}}.$$

Observe that for any fixed  $r \in \mathbb{N}$ ,  $\varepsilon \in (0, 1)$  and  $M \in (0, 2^r r!)$  this number is polynomial in the dimension. Hence the complexity is bounded by

$$C_{r,\varepsilon,M} \cdot d^{\alpha_{r,\varepsilon,M}} \log p^{-1} + d \max\{\varepsilon^{-1/r} (d C_r M)^{1/r}, 2\}.$$

### 3 Tractability for a modified class of functions

For large  $M$  we obtain the curse of dimensionality for the classes  $F_{M,d}^r$  and therefore we study smaller classes  $F_{M,d}^{r,V}$  by assuming that the support of  $f$  is not too small. By

$$R = \{\Pi_{i=1}^d [\alpha_i, \beta_i] \subseteq [0, 1]^d \mid \alpha_i, \beta_i \in [0, 1], \alpha_i \leq \beta_i, i = 1, \dots, d\}$$

the set of all boxes in  $[0, 1]^d$  is given and we denote the Lebesgue measure of  $A \subset \mathbb{R}^d$  by  $\lambda_d(A)$ . Then, let

$$F_{M,d}^{r,V} = \{f \in F_{M,d}^r \mid \exists A \in R \text{ with } \lambda_d(A) > V \text{ and } f(x) \neq 0, \forall x \in A\}. \quad (17)$$

This class contains only functions which are non-zero on a box with Lebesgue measure at least  $V > 0$ . Obviously, the zero function is not in  $F_{M,d}^{r,V}$ . The basic strategy for the approximation of  $f \in F_{M,d}^{r,V}$  is to find a  $z^* \in [0, 1]^d$  with  $f(z^*) \neq 0$  and after that apply Lemma 4.

For finding  $z^*$  the following definition is useful to measure the quality of a point set. Let

$$\text{disp}(x_1, \dots, x_n) = \sup_{A \in R, A \cap \{x_1, \dots, x_n\} = \emptyset} \lambda_d(A)$$

be the dispersion of the set  $\{x_1, \dots, x_n\}$ . The dispersion of a set is the largest volume of a box which does not contain any point of the set. By  $n^{\text{disp}}(V, d)$  we denote the smallest number of points needed to have at least one point in every box with volume  $V$ , i.e.

$$n^{\text{disp}}(V, d) = \inf\{n \in \mathbb{N} \mid \exists x_1, \dots, x_n \in [0, 1]^d \text{ with } \text{disp}(x_1, \dots, x_n) \leq V\}.$$

The authors of [1] consider as a point set the Halton sequence and use the following result of [3, 11] proved with this sequence.

**Proposition 9.** Let  $p_1, \dots, p_d$  be the first  $d$  prime numbers then

$$n^{\text{disp}}(V, d) \leq \frac{2^d \prod_{i=1}^d p_i}{V}. \quad (18)$$

Let us comment on this result. The nice thing is the dependence on  $V^{-1}$  which is of course optimal, already for  $d = 1$ . The involved constant is, however, super-exponential in the dimension, even for a point set with  $2^d \prod_{i=1}^d p_i$  elements one only obtains the trivial bound 1 of the dispersion. Observe that already for  $d = 20$  we have

$$2^d \prod_{i=1}^d p_i \geq 5.85 \cdot 10^{32}, \quad (19)$$

hence this analysis is useful only if the dimension  $d$  is very small.

The quantity  $n^{\text{disp}}(V, d)$  is well studied. The following result is due to Blumer, Ehrenfeucht, Haussler and Warmuth, see [2, Lemma A2.4]. For this note that the test set of boxes has Vapnik-Chervonenkis dimension  $2d$ . By  $(\Omega, \mathcal{F}, \mathbb{P})$  we denote the common probability space of all considered random variables.

**Proposition 10.** Let  $(X_i)_{1 \leq i \leq n}$  be an i.i.d. sequence of uniformly distributed random variables mapping in  $[0, 1]^d$ . Then for any  $0 < V < 1$  and  $n \in \mathbb{N}$

$$\mathbb{P}(\text{disp}(X_1, \dots, X_n) \leq V) \geq 1 - (en/d)^{2d} 2^{-Vn/2}.$$

Thus

$$n^{\text{disp}}(V, d) \leq 16dV^{-1} \log_2(13V^{-1}). \quad (20)$$

This shows that the number of function values needed to find  $z^*$  with  $f(z^*) \neq 0$  depends only linearly on the dimension  $d$ . The previous result leads to the following theorem.

**Theorem 11.** Let  $r \in \mathbb{N}$ ,  $M \in (0, \infty)$  and  $\varepsilon, V \in (0, 1)$ . Then

$$n^{\text{det}}(\varepsilon, F_{M,d}^{r,V}) \leq 2d [8V^{-1} \log_2(13V^{-1}) + d \max\{\varepsilon^{-1/r} (dC_r M)^{1/r}, 2\}], \quad (21)$$

where  $C_r$  comes from Lemma 4 and does not depend on  $d, V, M$  and  $\varepsilon$ .

Therefore the information complexity of our problem in the deterministic setting is at most quadratic in the dimension, in particular, the problem is polynomially tractable in the worst case setting.

Theorem 11 has a drawback since it is only a result on the *existence* of a fast algorithm. It is based on Proposition 10, which tells us that a uniformly distributed random point set

satisfies the bound on  $n^{\text{disp}}(V, d)$  with high probability. We are not able to construct such point sets.

Because of this, we also present a randomized algorithm  $S_{n_1, n_2}$  which consists of two steps. Here  $n_1 \in \mathbb{N}$  indicates the number of function evaluations for the first step and  $n_2 \in \mathbb{N}$  the function evaluations for the second one. This randomized algorithm has typical advantages and disadvantages compared to deterministic algorithms:

- The advantage is that the randomized algorithm is even faster. For the first phase of the algorithm (search of an  $z^*$  such that  $f(z^*) \neq 0$ ) the number of roughly  $dV^{-1} \log(V^{-1})$  function evaluations is replaced by roughly  $V^{-1}$ .
- The disadvantage is that this algorithm can output a wrong result, even if this probability can be made arbitrarily small.

The method  $S_{n_1, n_2}(f)$  works as follows:

1. For  $1 \leq i \leq n_1$  generate  $x_i \in [0, 1]^d$  uniformly distributed and check whether  $f(x_i) = 0$ . If this is not the case then we set  $z^* = x_i$  and go to 2. If  $i = n_1$  and we did not find  $z^*$  with  $f(z^*) \neq 0$  then return  $S_{n_1, n_2}(f) = 0$ .
2. Run the algorithm of Lemma 4 and return  $S_{n_1, n_2}(f) = A_{n_2}(f)$ .

We have the following error bound.

**Theorem 12.** Let  $n_2 \in \mathbb{N}$  with  $n_2 \geq d \max\{\varepsilon^{-1/r}(C_r d M)^{1/r}, 2\}$ . Then

$$\mathbb{P}(\|f - S_{n_1, n_2}(f)\|_\infty \leq \varepsilon) \geq 1 - (1 - V)^{n_1} \quad (22)$$

for  $f \in F_{M, d}^{r, V}$  and  $n_1 \in \mathbb{N}$ .

*Proof.* Let  $(X_i)_{1 \leq i \leq n_1}$  be an i.i.d. sequence of uniformly distributed random variables with values in  $[0, 1]^d$  and let

$$T = \min\{i \in \mathbb{N} \mid f(X_i) \neq 0\}.$$

Because of the choice of  $n_2$  we have by Lemma 4 that the error is smaller than  $\varepsilon$  if we found  $z^*$  in the first step of  $S_{n_1, n_2}$ . Thus

$$\mathbb{P}(\|f - S_{n_1, n_2}(f)\|_\infty \leq \varepsilon) \geq \mathbb{P}(T \leq n_1) = 1 - (1 - V)^{n_1}.$$

□

Actually we can use a single sequence of uniformly i.i.d random variables  $(X_i)_{1 \leq i \leq n_1}$  for any function  $f \in F_{M,d}^{r,V}$  and, still, the probability that a point  $z^*$  is found decreases exponentially fast for increasing  $n_1$ . More exactly, for  $n_1 \in \mathbb{N}$  we obtain

**Proposition 13.** Let  $n_2 \in \mathbb{N}$  with  $n_2 \geq d \max\{\varepsilon^{-1/r}(C_r d M)^{1/r}, 2\}$ . Then

$$\mathbb{P}\left(\sup_{f \in F_{M,d}^{r,V}} \|f - S_{n_1, n_2}(f)\|_\infty \leq \varepsilon\right) \geq 1 - (en_1/d)^{2d} 2^{-V n_1/2}. \quad (23)$$

*Proof.* Again  $(X_i)_{1 \leq i \leq n_1}$  is an i.i.d. sequence of uniformly distributed random variables in  $[0, 1]^d$  and  $n_2$  is chosen such that the error bound of Lemma 4 is smaller  $\varepsilon$  if we found  $z^*$  in the first step. Let us denote  $S_{n_1, n_2}(f, \omega)$  for  $S_{n_1, n_2}(f)$  which uses the points  $x_i = X_i(\omega)$  for  $1 \leq i \leq n_1$ . Then

$$\begin{aligned} & \{\omega \in \Omega \mid \text{disp}(X_1(\omega), \dots, X_n(\omega)) \leq V\} \\ &= \{\omega \in \Omega \mid \sup_{A \in R, A \cap \{X_1(\omega), \dots, X_{n_1}(\omega)\} = \emptyset} \lambda_d(A) \leq V\} \\ &= \{\omega \in \Omega \mid \forall A \in R \text{ with } \lambda_d(A) > V \exists j \in \{1, \dots, n_1\} \text{ with } X_j(\omega) \in A\} \\ &= \{\omega \in \Omega \mid \forall f \in F_{M,d}^{r,V} \exists j \in \{1, \dots, n_1\} \text{ with } f(X_j(\omega)) \neq 0\} \\ &\subseteq \{\omega \in \Omega \mid \forall f \in F_{M,d}^{r,V} \text{ holds } \|f - S_{n_1, n_2}(f, \omega)\|_\infty \leq \varepsilon\} \\ &= \{\omega \in \Omega \mid \sup_{f \in F_{M,d}^{r,V}} \|f - S_{n_1, n_2}(f, \omega)\|_\infty \leq \varepsilon\} \end{aligned}$$

Finally, for  $n_1 \in \mathbb{N}$  we obtain by Proposition 10

$$\mathbb{P}\left(\sup_{f \in F_{M,d}^{r,V}} \|f - S_{n_1, n_2}(f)\|_\infty \leq \varepsilon\right) \geq \mathbb{P}(\text{disp}(X_1, \dots, X_{n_1}) \leq V) \geq 1 - (en_1/d)^{2d} 2^{-V n_1/2}.$$

□

By a simple probabilistic argument we can also derive from (23) the existence of a “good” deterministic algorithm. Namely, for  $n_1 \geq 16dV^{-1} \log_2(13V^{-1})$  the right-hand-side of (23) is strictly larger than zero, which implies that there exists a realization of  $(X_i)_{1 \leq i \leq n_1}$ , say  $(x_i)_{1 \leq i \leq n_1}$ , such that

$$\sup_{f \in F_{M,d}^{r,V}} \|f - S_{n_1, n_2}(f)\|_\infty \leq \varepsilon.$$

This leads to Theorem 11.

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