DFG-Schwerpunktprogramm 1324

„Extraktion quantifizierbarer Information aus komplexen Systemen”

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Preprint 159

Edited by

AG Numerik/Optimierung
Fachbereich 12 - Mathematik und Informatik
Philipps-Universität Marburg
Hans-Meerwein-Str.
35032 Marburg

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Approximation of multivariate periodic functions by trigonometric polynomials based on sampling along rank-1 lattice with generating vector of Korobov form

Lutz Kämmerer* Daniel Potts* Toni Volkmer*

In this paper, we present error estimates for the approximation of multivariate periodic functions in periodic Hilbert spaces of isotropic and dominating mixed smoothness by trigonometric polynomials. The approximation is based on sampling of the multivariate functions on rank-1 lattices. We use reconstructing rank-1 lattices with generating vectors of Korobov form for the sampling and generalize the technique from [24], in order to show that the aliasing error of that approximation is of the same order as the error of the approximation using the partial sum of the Fourier series. The main advantage of our method is that the computation of the Fourier coefficients of such a trigonometric polynomial, which we use as approximant, is based mainly on a one-dimensional fast Fourier transform, i.e., the arithmetic complexity of the computation depends only on the cardinality of the support of the trigonometric polynomial in the frequency domain. Numerical results are presented up to dimension $d = 10$.

Keywords and phrases : approximation of multivariate functions, trigonometric polynomials, hyperbolic cross, lattice rule, rank-1 lattice, fast Fourier transform

2000 AMS Mathematics Subject Classification : 65T40, 42A10,

*Technische Universität Chemnitz, Faculty of Mathematics, 09107 Chemnitz, Germany
{lutz.kaemmerer, daniel.potts, toni.volkmer}@mathematik.tu-chemnitz.de

1 Introduction

We approximate functions $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ from the Hilbert space

$$\mathcal{H}^\omega(\mathbb{T}^d) := \left\{ f \in C(\mathbb{T}^d) \colon \|f|\mathcal{H}^\omega(\mathbb{T}^d)\| := \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \omega(\boldsymbol{k})^2 |\hat{f}_{\boldsymbol{k}}|^2} < \infty \right\},$$

where $\omega\colon \mathbb{Z}^d \to (0,\infty]$ is a weight function, by trigonometric polynomials p with frequencies supported on an index set $I \subset \mathbb{Z}^d$ of finite cardinality, $p(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$. Thereby, we are especially interested in the higher-dimensional cases, i.e., $d \geq 4$. As usual, we denote the Fourier coefficients of the function $f \in L^2(\mathbb{T}^d)$ by

$$\hat{f}_{\boldsymbol{k}} := \int_{\mathbb{T}^d} f(\boldsymbol{x}) \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}} \mathrm{d}\boldsymbol{x}, \quad \boldsymbol{k} \in \mathbb{Z}^d.$$

We remark that for the special choice $\omega \equiv 1$, we have $\mathcal{H}^\omega(\mathbb{T}^d) = C(\mathbb{T}^d) \cap L^2(\mathbb{T}^d)$. One theoretical possibility to obtain such a trigonometric polynomial p is to formally approximate the function f by the Fourier partial sum

$$S_I f := \sum_{\boldsymbol{k}\in I} \hat{f}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\circ},$$

where $I \subset \mathbb{Z}^d$ is a frequency index set of finite cardinality. Since $S_I f$ is the truncated Fourier partial sum of the function f, this approximation causes a truncation error $\|f - S_I f\|$, where $\|\cdot\|$ is an arbitrarily chosen norm. For a function $f \in L^2(\mathbb{T}^d) \cap \mathcal{H}^\omega(\mathbb{T}^d)$ we choose a frequency index set $I = I_N := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \omega(\boldsymbol{k})^{1/\nu} \leq N\}$ of refinement $N \in \mathbb{R}$, $N \geq 1$, $\nu > 0$, and obtain

$$\|f - S_{I_N} f | L^2(\mathbb{T}^d)\| \leq N^{-\nu} \|f|\mathcal{H}^\omega(\mathbb{T}^d)\|,$$

see Lemma 3.3. We stress the fact that $S_{I_N} f$ is the best approximation of the function f with respect to the $L^2(\mathbb{T}^d)$ norm in the space $\Pi_{I_N} := \mathrm{span}\{\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\circ} \colon \boldsymbol{k} \in I_N\}$ of trigonometric polynomials with frequencies supported on the index set I_N and that the operator $S_{I_N} \colon L^1(\mathbb{T}^d) \to \Pi_{I_N}$ only depends on the frequency index set I_N.

Since, in general, we do not know the Fourier coefficients $\hat{f}_{\boldsymbol{k}}$, we are going to approximate the function f from samples using the approximated Fourier partial sum

$$\tilde{S}_{I_N} f := \sum_{\boldsymbol{k}\in I_N} \tilde{\hat{f}}_{\boldsymbol{k}} \, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\circ}.$$

We compute the approximated Fourier coefficients $\tilde{\hat{f}}_{\boldsymbol{k}} \in \mathbb{C}$, $\boldsymbol{k} \in I_N$, of the function f using a lattice rule by

$$\tilde{\hat{f}}_{\boldsymbol{k}} := \frac{1}{M} \sum_{j=0}^{M-1} f\left(\boldsymbol{x}_j\right) \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j} \quad \text{for } \boldsymbol{k} \in I_N, \tag{1.1}$$

where the sampling nodes $\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod 1$ are the nodes of a so-called reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N)$ with generating vector $\boldsymbol{z} \in \mathbb{Z}^d$ and rank-1 lattice size $M \in \mathbb{N}$ for the frequency index set I_N, see Section 2.2 for the definition. Lattice rules have extensively been investigated for the integration of functions of many variables for a long time, cf. e.g., [22, 4, 5] and the extensive reference list therein. Especially, rank-1 lattice rules have also been studied for the approximation of multivariate functions of suitable smoothness, cf. [24, 18, 17, 19]. Furthermore, there exist already comprehensive tractability results for numerical integration and approximation using rank-1 lattices, see [20, 17].

Since we consider the partial sum $\tilde{S}_{I_N} f$ of the approximated Fourier coefficients $\tilde{\hat{f}}_{\boldsymbol{k}}$ instead of the Fourier partial sum $S_{I_N} f$ of Fourier coefficients $\hat{f}_{\boldsymbol{k}}$, we obtain an additional error. We estimate the approximation error $\|f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\|$ using the triangle inequality $\|f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\| \leq \|f - S_{I_N} f | L^2(\mathbb{T}^d)\| + \|S_{I_N} f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\|$, where $\|f - S_{I_N} f | L^2(\mathbb{T}^d)\|$ is called the truncation error and $\|S_{I_N} f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\|$ is called the aliasing error.

In this paper, we consider frequency index sets I_N of special structure and show that there exists a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N)$ of reasonable size M, see Section 2.2 for the definition, such that the order of the aliasing error $\|S_{I_N} f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\|$ is bounded by the order of the truncation error $\|f - S_{I_N} f | L^2(\mathbb{T}^d)\|$. To this end, we use the highly structured rank-1 lattice rules with generating vector of Korobov form. This allows us to generalize the ideas of V. N. Temlyakov, see [24], in order to estimate the aliasing error. We consider, similar to [7] and as in [14], functions f from the Hilbert space

$$\mathcal{H}^{\omega}(\mathbb{T}^d) = \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d) := \left\{ f \in C(\mathbb{T}^d) \colon \|f | \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\| := \sqrt{\sum\nolimits_{\boldsymbol{k} \in \mathbb{Z}^d} \omega^{\alpha,\beta}(\boldsymbol{k})^2 |\hat{f}_{\boldsymbol{k}}|^2} < \infty \right\},$$

where the parameter $\beta \in \mathbb{R}$, $\beta \geq 0$, characterizes the dominating mixed smoothness, the parameter $\alpha \in \mathbb{R}$, $\alpha > -\beta$, characterizes the isotropic smoothness, and the weights $\omega(\boldsymbol{k}) = \omega^{\alpha,\beta}(\boldsymbol{k})$ are given by

$$\omega^{\alpha,\beta}(\boldsymbol{k}) := \max(1, \|\boldsymbol{k}\|_1)^{\alpha} \prod_{s=1}^{d} \max(1, |k_s|)^{\beta}, \; \boldsymbol{k} := (k_1, \ldots, k_d)^{\top}.$$

We remark that one can use various equivalent weights $\omega(\boldsymbol{k})$ which have different approximation properties for large dimensions d, cf. [16]. Furthermore, we define the corresponding frequency index sets $I_N = I_N^{d,T}$, $N \in \mathbb{R}$, $N \geq 1$, $T \in \mathbb{R}$, $-\infty < T < 1$, by

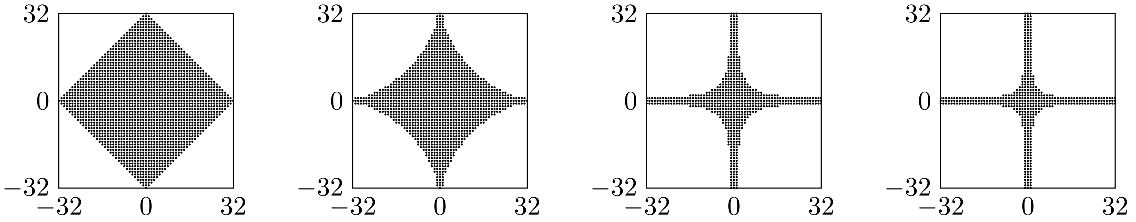
$$I_N^{d,T} := \left\{ \boldsymbol{k} \in \mathbb{Z}^d \colon \omega^{-\frac{T}{1-T}, \frac{1}{1-T}}(\boldsymbol{k}) = \max(1, \|\boldsymbol{k}\|_1)^{-\frac{T}{1-T}} \prod_{s=1}^{d} \max(1, |k_s|)^{\frac{1}{1-T}} \leq N \right\}.$$

In the cases $0 < T < 1$, the frequency index sets $I_N^{d,T}$ are called energy-norm based hyperbolic crosses, see [2, 3], and in the case $T = 0$ symmetric hyperbolic crosses. As a natural extension for $T = -\infty$, we define the frequency index set $I_N^{d,-\infty}$ as the d-dimensional ℓ_1-ball of size N,

$$I_N^{d,-\infty} := \left\{ \boldsymbol{k} \in \mathbb{Z}^d \colon \max(1, \|\boldsymbol{k}\|_1) \leq N \right\}.$$

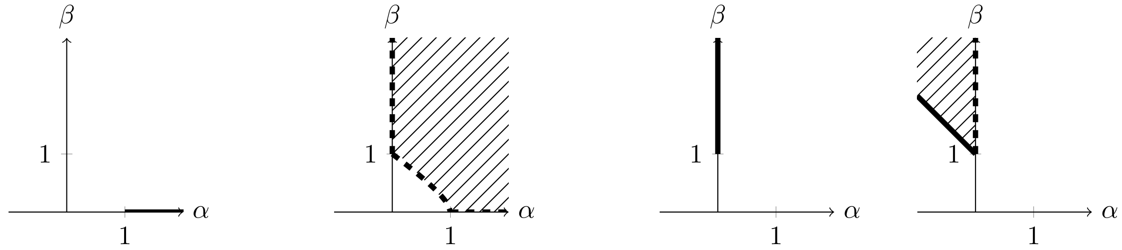
Figure 1.1 illustrates the frequency index sets $I_N^{d,T}$ in the two-dimensional case for different choices $-\infty \leq T < 1$ of the parameter T. The cardinalities of the frequency index sets $I_N^{d,T}$ are given in Lemma 4.1, which reads for fixed $d \in \mathbb{N}$ and $T := -\alpha/\beta$ as follows

$$\left| I_N^{d,-\alpha/\beta} \right| = \begin{cases} \Theta(N^d) & \text{for } \alpha > 0 \text{ and } \beta = 0 \quad (\Longleftrightarrow T = -\infty), \\ \Theta(N^{d\frac{\beta+\alpha}{d\beta+\alpha}}) & \text{for } \alpha > 0 \text{ and } \beta > 0 \quad (\Longleftrightarrow -\infty < T < 0), \\ \Theta(N \log^{d-1} N) & \text{for } \alpha = 0 \text{ and } \beta > 0 \quad (\Longleftrightarrow T = 0), \\ \Theta(N) & \text{for } \alpha < 0 \text{ and } \beta > -\alpha \quad (\Longleftrightarrow 0 < T < 1). \end{cases} \tag{1.2}$$



l_1-ball $I_{32}^{2,-\infty}$ | index set $I_{32}^{2,-5}$ | symmetric hyperbolic cross $I_{32}^{2,0}$ | energy-norm based hyperbolic cross $I_{32}^{2,1/2}$

Figure 1.1: Two-dimensional frequency index sets $I_{32}^{2,T}$ for $T \in \{-\infty, -5, 0, \frac{1}{2}\}$.



$\alpha > \frac{d}{2}$ and $\beta = 0$ ($T = -\infty$) | $\beta > 0$ and $\alpha > \max\left(0, (\frac{1}{4} - \beta + \frac{1}{4}\sqrt{8\beta+1})d\right)$ ($-\infty < T < 0$) | $\alpha = 0$ and $\beta > 1$ ($T = 0$) | $\alpha < 0$ and $\beta > 1 - \alpha$ ($0 < T < 1$)

Figure 1.2: Visualization of the admissible values of α and β in the case $d = 2$, such that (1.3) and (1.4) are valid. We set the corresponding values $T := -\alpha/\beta$.

In this setting, we obtain that the $L^2(\mathbb{T}^d)$ truncation error is bounded by

$$\|f - S_{I_N^{d,-\alpha/\beta}} f | L^2(\mathbb{T}^d)\| \leq N^{-(\alpha+\beta)} \, \|f | \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|,$$

see Lemma 4.4. The main result of this paper is, that for fixed dimension $d \in \mathbb{N}$, $d \geq 2$ there exists a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N^{d,-\alpha/\beta})$ with generating vector $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top \in \mathbb{Z}^d$ of Korobov form and size

$$M = \begin{cases} \mathcal{O}(N^d) & \text{for } \alpha > \frac{d}{2} \text{ and } \beta = 0, \\ \mathcal{O}(N^{d\frac{(2d\beta+\alpha)(\beta+\alpha)}{(d\beta+\alpha)^2}}) & \text{for } \beta > 0 \text{ and } \alpha > \max\left(0, (\frac{1}{4} - \beta + \frac{1}{4}\sqrt{8\beta+1})d\right), \\ \mathcal{O}(N^2 \log^{d-1} N) & \text{for } \alpha = 0 \text{ and } \beta > 1, \\ \mathcal{O}(N^2) & \text{for } \alpha < 0 \text{ and } \beta > 1 - \alpha, \end{cases} \tag{1.3}$$

such that the aliasing error is bounded by

$$\|S_{I_N^{d,-\alpha/\beta}} f - \tilde{S}_{I_N^{d,-\alpha/\beta}} f | L^2(\mathbb{T}^d)\| \leq C(d, \alpha, \beta) \, N^{-(\alpha+\beta)} \, \|f | \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|, \tag{1.4}$$

where $C(d, \alpha, \beta) > 0$ is a constant which only depends on d, α, β. In the cases where $\alpha \geq 0$, we obtain estimate (1.4) from Theorem 4.7 and the lower bound for the size M in (1.3) due to (4.4) and (4.1). For $\alpha < 0$, we infer estimate (1.4) from Theorem 4.10 and the lower bound for the size M in (1.3) due to (4.5) and (4.1). Figure 1.2 visualizes the different cases for the admissible values of the isotropic smoothness α and the dominating mixed smoothness β in (1.3) and (1.4) in the two-dimensional case and gives the corresponding values of the parameter T. In Figure 1.3, the admissible values of α and β are shown for the cases $d = 2, 6, 10$. Comparing the number M of sampling nodes $\boldsymbol{x}_j$ in (1.3) and the

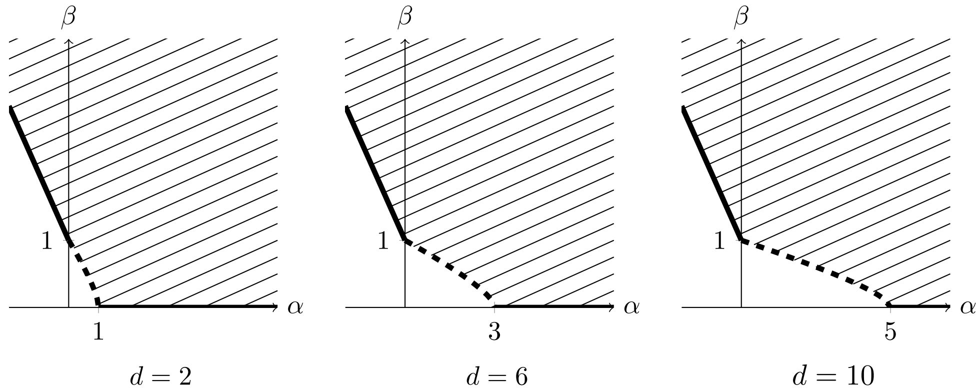


Figure 1.3: Visualiziation of the admissible values of α and β in the cases $d = 2, 6, 10$, such that (1.3) and (1.4) are valid.

number $|I_N^{d,-\alpha/\beta}|$ of frequency indices in (1.2), our results yield in general an oversampling, i.e., $M > |I_N^{d,-\alpha/\beta}|$. In the case $\alpha > \frac{d}{2}$ and $\beta = 0$, where the frequency index sets $I_N^{d,-\infty}$ are l_1-balls, the asymptotic order of M and $|I_N^{d,-\infty}|$ in N is obviously identical. Considering the case $\alpha < 0$ and $\beta > 1 - \alpha$, where the frequency index sets $I_N^{d,-\alpha/\beta}$ are energy-norm based hyperbolic crosses, we obtain a gap between M and $|I_N^{d,-\alpha/\beta}|$ in the asymptotic order in N. However, this gap is necessary in order to obtain an orthogonal Fourier transform as given by (1.1), cf. [10, Lemma 2.1]. Note that in the case $\alpha = 0$, the oversampling factors $M/|I_N^{d,0}|$, i.e., ratios of the rank-1 lattice sizes M and the cardinalities of the symmetric hyperbolic cross index sets $I_N^{d,0}$ are still moderate for reasonable problem sizes compared to the asymptotic statement $\mathcal{O}(N)$ in (1.3) and (1.2), see Table 5.1.

Let us mention that sampling on (generalized) sparse grids, see [25, 1, 30, 8, 26, 2, 23, 3, 6, 21, 9, 7], is another intensively studied approach used to approximate functions of the classes $\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$. One advantage of this method is that only $|I_N^{d,-\alpha/\beta}|$ many samples are required. Furthermore, for $\alpha = 0$, there exists a fast algorithm for computing the approximation of the Fourier partial sum $S_{I_N^{d,0}}$ of a function f in $\mathcal{O}(N \log^d N)$ arithmetic operations. However, the computation may be numerically unstable in this setting, cf. [12]. Known upper bounds for the approximation errors are discussed in Section 4.3. We stress again, that the outstanding property of the sampling method (1.1) discussed in this paper is that the computation of the approximated Fourier coefficients $\tilde{\hat{f}}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I_N^{d,0}$, is perfectly stable and takes $\mathcal{O}(N^2 \log^d N)$ arithmetic operations, since it is mainly based on a one-dimensional fast Fourier transform (FFT), cf. [13] and [14, Algorithm 3].

The paper is organized as follows: We discuss the exact reconstruction of trigonometric polynomials from samples along a rank-1 lattice in Section 2 and prove the existence of a special rank-1 lattice with certain properties. Based on these special properties, we show general estimates for the aliasing error for general frequency index sets I_N in Section 3. Then, in Section 4, we consider the approximation error $\|f - \tilde{S}_{I_N^{d,-\alpha/\beta}} f | L^2(\mathbb{T}^d)\|$. Therefor, we present the estimates for the truncation error in Section 4.1. In Section 4.2, we prove the results (1.3) and (1.4). We compare these results with previously known ones in Section 4.3. Finally, we present numerical tests in Section 5 in order to illustrate the theoretical results and we give some concluding remarks in Section 6.

2 Approximation based on rank-1 lattice sampling

2.1 Reconstruction of trigonometric polynomials from samples

As already discussed in Section 1, we want to approximate a function $f \in \mathcal{H}^\omega(\mathbb{T}^d)$ using a trigonometric polynomial p. Here, we use the following approach. For a given frequency index set $I \subset \mathbb{Z}^d$ of finite cardinality, we want to be able to exactly reconstruct the Fourier coefficients $\hat{p}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, of an arbitrarily chosen trigonometric polynomial $p(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \,\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$ with frequencies supported on I from sampling values $p(\boldsymbol{x}_j)$. As sampling nodes $\boldsymbol{x}_j$, $j = 0, \ldots, M-1$, we use the nodes of a rank-1 lattice $\Lambda(\boldsymbol{z}, M)$ with generating vector $\boldsymbol{z} \in \mathbb{Z}^d$ of size $M \in \mathbb{N}$, i.e., we set $\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod 1$, $j = 0, \ldots, M-1$. Formally, the Fourier coefficients $\hat{p}_{\boldsymbol{k}}$ are given by

$$\hat{p}_{\boldsymbol{k}} := \int_{\mathbb{T}^d} p(\boldsymbol{x})\,\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}\mathrm{d}\boldsymbol{x}$$

and we approximate this integral by the (rank-1) lattice rule

$$\frac{1}{M}\sum_{j=0}^{M-1} p(\boldsymbol{x}_j)\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j} = \frac{1}{M}\sum_{j=0}^{M-1} p\left(\frac{j}{M}\boldsymbol{z}\right)\mathrm{e}^{-2\pi\mathrm{i}j\boldsymbol{k}\boldsymbol{z}/M} =: \hat{\tilde{p}}_{\boldsymbol{k}}.$$

Now, we ask for the exactness of this cubature formula, i.e., when is $\hat{p}_{\boldsymbol{k}} = \hat{\tilde{p}}_{\boldsymbol{k}}\ \forall \boldsymbol{k} \in I$. Since we have

$$\hat{\tilde{p}}_{\boldsymbol{k}} = \frac{1}{M}\sum_{j=0}^{M-1}\sum_{\boldsymbol{k}'\in I}\hat{p}_{\boldsymbol{k}'}\,\mathrm{e}^{2\pi\mathrm{i}j\boldsymbol{k}'\boldsymbol{z}/M}\mathrm{e}^{-2\pi\mathrm{i}j\boldsymbol{k}\boldsymbol{z}/M} = \sum_{\boldsymbol{k}'\in I}\hat{p}_{\boldsymbol{k}'}\frac{1}{M}\sum_{j=0}^{M-1}\mathrm{e}^{2\pi\mathrm{i}j(\boldsymbol{k}'-\boldsymbol{k})\boldsymbol{z}/M},$$

we need the condition

$$\frac{1}{M}\sum_{j=0}^{M-1}\mathrm{e}^{2\pi\mathrm{i}j(\boldsymbol{k}'-\boldsymbol{k})\boldsymbol{z}/M} = \begin{cases}1 & \text{for } \boldsymbol{k}=\boldsymbol{k}'\\ 0 & \text{for } \boldsymbol{k}\neq\boldsymbol{k}', \boldsymbol{k},\boldsymbol{k}'\in I,\end{cases} \tag{2.1}$$

to be fulfilled. This is the case if and only if

$$(\boldsymbol{k}'-\boldsymbol{k})\boldsymbol{z} \not\equiv 0 \pmod{M}\ \forall \boldsymbol{k},\boldsymbol{k}'\in I, \boldsymbol{k}\neq\boldsymbol{k}', \tag{2.2}$$

$$\iff \boldsymbol{k}\cdot\boldsymbol{z} \not\equiv \boldsymbol{k}'\cdot\boldsymbol{z} \pmod{M}\ \forall \boldsymbol{k},\boldsymbol{k}'\in I, \boldsymbol{k}\neq\boldsymbol{k}', \tag{2.3}$$

see [11, Section 2]. Introducing the difference set $\mathcal{D}(I)$ for the index set I, $\mathcal{D}(I) := \{\boldsymbol{k}-\boldsymbol{k}' \colon \boldsymbol{k},\boldsymbol{k}'\in I\}$, we can rewrite the above conditions to

$$\boldsymbol{m}\boldsymbol{z} \not\equiv 0 \pmod{M}\ \forall \boldsymbol{m}\in\mathcal{D}(I)\setminus\{\boldsymbol{0}\}. \tag{2.4}$$

2.2 Reconstructing rank-1 lattices

A rank-1 lattice $\Lambda(\boldsymbol{z}, M)$ which fulfills one of the (equivalent) conditions (2.1),(2.2),(2.3),(2.4) for a given frequency index set I will be called reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I)$. Under mild assumptions, e.g., $I \subset \mathbb{Z}^d \cap (-M/2, M/2)^d$, there always exists a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I)$ of size $\frac{|\mathcal{D}(I)|}{2} \leq M \leq |\mathcal{D}(I)|$ due to [11, Corollary 1] and Bertrand's postulate. We remark that we can compute the approximated Fourier coefficients $\tilde{\hat{f}}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, from (1.1) in $\mathcal{O}(M \log M + d|I|)$ arithmetic operations using a single one-dimensional fast Fourier transform of length M and by computing the scalar products $\boldsymbol{k}\boldsymbol{z}$ for $\boldsymbol{k} \in I$. The generating vector $\boldsymbol{z}$ can be constructed using a component-by-component approach, see [11].

2.3 Existence of a special reconstructing rank-1 lattice

In this section, we proof a generalisation of [24, Lemma 2] and [27, Lemma 4.1]. Conceptually, $I = I_N \subset \mathbb{Z}^d$ represent frequency index sets used as support for the approximation of a function f by a trigonometric polynomial p based on sampling values $f(\boldsymbol{x}_j)$. The index sets $\mathcal{I} = \mathcal{I}_N \subset \mathbb{Z}^d$ may be considered as a superset of the difference set $\mathcal{D}(I) := \{\boldsymbol{k}-\boldsymbol{k}' \colon \boldsymbol{k}, \boldsymbol{k}' \in I\}$. As sampling nodes $\boldsymbol{x}_j$, we use the nodes $\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod 1$, $j = 0, \ldots, M-1$, of a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I)$ with generating vector $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top \in \mathbb{Z}^d$ of Korobov form, i.e., the condition $\boldsymbol{m}\boldsymbol{z} \not\equiv 0 \pmod{M}$ has to be fulfilled for all $\boldsymbol{m} \in \mathcal{D}(I) \setminus \{\boldsymbol{0}\}$ or one of the other (equivalent) conditions (2.1),(2.2),(2.3).

Lemma 2.1. *Let a sequence of frequency index sets $\mathcal{I}_N \subset \mathbb{Z}^d$, $d \in \mathbb{N}$, of finite cardinality $|\mathcal{I}_N|$ be given, which may depend on the refinement $N \in \mathbb{R}$, $N \geq 1$. For fixed refinement $N \in \mathbb{R}$, $N \geq 1$, and arbitrarily chosen parameter $\kappa \in \mathbb{R}$, $\kappa > 0$, let $M \in \mathbb{N}$ be a prime such that*

$$M > \frac{d\,|\mathcal{I}_N|}{1 - 2^{-\kappa}} + 1 \tag{2.5}$$

and

$$\mathcal{I}_N \cap M\mathbb{Z}^d = \{\boldsymbol{0}\}. \tag{2.6}$$

For an arbitrarily chosen monotonic increasing function $\varphi \colon \mathbb{N} \cup \{0\} \to [1, \infty)$ with $\varphi(0) = 1$, we define the shells $F_l(N) := \mathcal{I}_{N\,\varphi(l)} \setminus \mathcal{I}_{N\,\varphi(l-1)}$, $N \in \mathbb{R}$, $N \geq 1$, $l \in \mathbb{N}$, and for each $a \in \{1, \ldots, M-1\}$ the sets

$$M_a^l := \{\boldsymbol{m} \in F_l(N) \colon m_1 + m_2\, a + \ldots + m_d\, a^{d-1} \equiv 0 \pmod{M} \text{ and } \boldsymbol{m} \neq M\boldsymbol{m}' \;\forall \boldsymbol{m}' \in \mathbb{Z}^d\}.$$

Then, there exists a number $a \in \{1, \ldots, M-1\}$, such that

$$m_1 + m_2\, a + \ldots + m_d\, a^{d-1} \not\equiv 0 \pmod{M} \text{ for all } \boldsymbol{m} \in \mathcal{I}_N \setminus \{\boldsymbol{0}\} \tag{2.7}$$

and

$$|M_a^l| \leq A_l^N := |F_l(N)| d 2^{(l+1)\kappa} (2^\kappa - 1)^{-1} (M-1)^{-1}, \quad l \in \mathbb{N}. \tag{2.8}$$

Proof. This proof is a generalisation of the proofs of [24, Lemma 2] and [27, Lemma 4.1]. We remark that $F_l(N) = \emptyset$ may occur for some or all $l \in \mathbb{N}$ and then also $M_a^l = \emptyset$ follows. The idea is to prove that the number of integers $a \in \{1, \ldots, M-1\}$ for which the statement of the lemma is not valid is less than $M-1$ and consequently, at least one $a \in \{1, \ldots, M-1\}$ fulfills the statement. We consider the congruence

$$m_1 + m_2\, a + \ldots + m_d\, a^{d-1} \equiv 0 \pmod{M}. \tag{2.9}$$

For a fixed frequency $\boldsymbol{m} \in \mathbb{Z}^d$, we denote the set of natural numbers $a \in \{1, \ldots, M-1\}$ which are solutions of congruence (2.9) by $A_M(\boldsymbol{m})$, i.e.,

$$A_M(\boldsymbol{m}) := \{a \in \{1, \ldots, M-1\} \colon m_1 + m_2\, a + \ldots + m_d\, a^{d-1} \equiv 0 \pmod{M}\}.$$

Let a frequency $\boldsymbol{m} \in \mathcal{I}_N \setminus \{\boldsymbol{0}\}$ be given. Due to condition (2.6), at least one component fulfills $m_{s'} \not\equiv 0 \pmod{M}$ and we can apply Lagrange's Theorem. This yields that the congruence (2.9) has at most $d-1$ roots modulo M. Therefore, we have

$$|A_M(\boldsymbol{m})| \leq d - 1 < d \tag{2.10}$$

for all $\boldsymbol{m} \in \mathcal{I}_N \setminus \{\boldsymbol{0}\}$. Next, we estimate the number of integers $a \in \{1, \ldots, M-1\}$ for which the relation (2.7) is not valid for at least one $\boldsymbol{m} \in \mathcal{I}_N \setminus \{\boldsymbol{0}\}$. Therefore, we denote by G_0 the set of numbers $a \in \{1, \ldots, M-1\}$ which are solutions of congruence (2.9) for at least one frequency $\boldsymbol{m} \in \mathcal{I}_N \setminus \{\boldsymbol{0}\}$,

$$G_0 = \bigcup_{\boldsymbol{m} \in \mathcal{I}_N \setminus \{\boldsymbol{0}\}} A_M(\boldsymbol{m}).$$

Since $|A_M(\boldsymbol{m})| < d$ by (2.10) and due to (2.5), we obtain

$$|G_0| \leq \sum_{\boldsymbol{m} \in \mathcal{I}_N \setminus \{\boldsymbol{0}\}} |A_M(\boldsymbol{m})| < d\, |\mathcal{I}_N| < (M-1)(1 - 2^{-\kappa}). \tag{2.11}$$

This means, for any $a \in \{1, \ldots, M-1\} \setminus G_0$, the relations (2.7) are valid and
$|\{1, \ldots, M-1\} \setminus G_0| > M - 1 - (M-1)(1-2^{-\kappa}) = (M-1)2^{-\kappa} > 0$.
Next, we consider the inequalities (2.8). For each $l \in \mathbb{N}$, we estimate the number of integers $a \in \{1, \ldots, M-1\}$ for which $|M_a^l| > A_l^N$, i.e., for which the inequalities (2.8) are not fulfilled. Therefor, we define the sets $G_l := \{a \in \{1, \ldots, M-1\} \colon |M_a^l| > A_l^N\}$, $l \in \mathbb{N}$. If $F_l(N) = \emptyset$, then obviously $|G_l| = 0$. Otherwise for $F_l(N) \neq \emptyset$, we have

$$\sum_{a \in G_l} |M_a^l| > \sum_{a \in G_l} A_l^N = |G_l|\, A_l^N. \tag{2.12}$$

We note that estimate (2.10) is also true for all $\boldsymbol{m} \in M_a^l$ due to Lagrange's Theorem, i.e., there exist at most $d-1$ many numbers $a \in \{1, \ldots, M-1\}$ satisfying (2.9) for fixed $\boldsymbol{m} \in M_a^l$. Consequently, for fixed $\boldsymbol{m} \in M_a^l$, there exist at most $d-1$ sets M_a^l which contain $\boldsymbol{m}$. Thus, each $\boldsymbol{m} \in F_l(N)$ can belong to at most $d-1$ different sets M_a^l and therefore

$$\sum_{a \in G_l} |M_a^l| \leq (d-1)\, |F_l(N)| < d\, |F_l(N)|. \tag{2.13}$$

Comparing (2.12) and (2.13), we obtain $|G_l|\, A_l^N < d\, |F_l(N)|$ and by inserting the definition of A_l^N from (2.8), we infer

$$|G_l| < d\, |F_l(N)|/A_l^N = 2^{-(l+1)\kappa}(2^\kappa - 1)(M-1) = 2^{-l\kappa}(M-1)(1-2^{-\kappa}), \quad l \in \mathbb{N}, \tag{2.14}$$

if $F_l(N) \neq \emptyset$. Alltogether, relation (2.11) as well as relation (2.14) if $F_l(N) \neq \emptyset$ and $|G_l| = 0$ if $F_l(N) = \emptyset$ yield

$$\begin{aligned} \sum_{l=0}^{\infty} |G_l| < \sum_{l=0}^{\infty} 2^{-l\kappa}(M-1)(1-2^{-\kappa}) &= (M-1)(1-2^{-\kappa}) \sum_{l=0}^{\infty} (2^{-\kappa})^l \\ &= (M-1)(1-2^{-\kappa}) \frac{1}{1-2^{-\kappa}} = M - 1. \end{aligned}$$

This means that the number of integers $a \in \{1, \dots, M-1\}$ for which the statement of the lemma is not valid is less than $M-1$. Since the cardinality $|\{1, \dots, M-1\}| = M-1$, there exists at least one $a \in \{1, \dots, M-1\}$ for which relations (2.7) and (2.8) are valid. ∎

3 Aliasing error for rank-1 lattice sampling and general frequency index sets

Based on Lemma 2.1, we proof general statements for the aliasing error for arbitrary frequency index sets $I \subset \mathbb{Z}^d$ of finite cardinality. We are going to use the results of this section extensively in Section 4. The following lemma was proven in [24], see [24, Property 2°].

Lemma 3.1. *Let the dimensionality $d \in \mathbb{N}$, $d \geq 2$, a frequency index set $I \subset \mathbb{Z}^d$ of finite cardinality and a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I)$ with the nodes $\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod 1$, $j = 0, \dots, M-1$, be given. We denote the Dirichlet kernel with frequencies supported on the index set I by $D_I(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$. For an arbitrary vector $\boldsymbol{b} := (b_0, \dots, b_{M-1})^\top \in \mathbb{C}^M$, we have*

$$\left\| \frac{1}{M} \sum_{j=0}^{M-1} b_j D_I(\circ - \boldsymbol{x}_j) \middle| L_2(T^d) \right\| \leq \left(\frac{1}{M} \sum_{j=0}^{M-1} |b_j|^2 \right)^{1/2} = \|\boldsymbol{b}/\sqrt{M}\|_2. \tag{3.1}$$

Additionally, for an arbitrary trigonometric polynomial $p\colon \mathbb{T}^d \to \mathbb{C}$ with frequencies supported on the index set I, $p(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$, $\hat{p}_{\boldsymbol{k}} \in \mathbb{C}$, we have $\tilde{S}_I p = p$.

Proof. Due to $\int_{\mathbb{T}^d} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}} \mathrm{d}\boldsymbol{x} = \begin{cases} 1 & \text{for } \boldsymbol{k} = \boldsymbol{0}, \\ 0 & \text{for } \boldsymbol{k} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}, \end{cases}$ we obtain

$$\begin{aligned} \left\| \frac{1}{M} \sum_{j=0}^{M-1} b_j D_I(\circ - \boldsymbol{x}_j) \middle| L_2(\mathbb{T}^d) \right\|^2 &= \frac{1}{M^2} \int_{\mathbb{T}^d} \left| \sum_{j=0}^{M-1} b_j D_I(\boldsymbol{x} - \boldsymbol{x}_j) \right|^2 \mathrm{d}\boldsymbol{x} \\ = \quad \frac{1}{M^2} \sum_{0\leq j,j'\leq M-1} b_j D_I(\boldsymbol{x}_{j'} - \boldsymbol{x}_j) \overline{b_{j'}}. \end{aligned}$$

We can rewrite this as a quadratic form of the vector $\boldsymbol{b}/\sqrt{M}$ and the matrix $\boldsymbol{\mathcal{D}} = \left(\mathcal{D}_{j',j}\right)_{j',j=0}^{M-1}$ with elements $\mathcal{D}_{j',j} := \frac{1}{M} D_I(\boldsymbol{x}_{j'} - \boldsymbol{x}_j)$, $\left\| \frac{1}{M} \sum_{j=0}^{M-1} b_j D_I(\circ - \boldsymbol{x}_j) \middle| L_2(T^d) \right\|^2 = \left(\frac{\boldsymbol{b}}{\sqrt{M}}\right)^{\mathrm{H}} \boldsymbol{\mathcal{D}} \left(\frac{\boldsymbol{b}}{\sqrt{M}}\right)$.
Next, we consider the matrix $\boldsymbol{\mathcal{D}}^2 = \boldsymbol{\mathcal{D}} \cdot \boldsymbol{\mathcal{D}} := \left((\mathcal{D}^2)_{j',j}\right)_{j',j=0}^{M-1}$ with the elements $(\mathcal{D}^2)_{j',j}$. We obtain by using the reconstructing property (2.1) of the reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I)$ that $(\mathcal{D}^2)_{j',j} = \frac{1}{M^2} \sum_{\rho=0}^{M-1} D_I(\boldsymbol{x}_{j'} - \boldsymbol{x}_\rho) D_I(\boldsymbol{x}_\rho - \boldsymbol{x}_j) \overset{(2.1)}{=} \frac{1}{M} \sum_{\boldsymbol{k}\in I} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}(\boldsymbol{x}_{j'} - \boldsymbol{x}_j)} = \mathcal{D}_{j',j}$, i.e., $\boldsymbol{\mathcal{D}}^2 = \boldsymbol{\mathcal{D}}$. Furthermore, we have $\boldsymbol{\mathcal{D}}^{\mathrm{H}} = \boldsymbol{\mathcal{D}}$ and therefore, $\boldsymbol{\mathcal{D}} = \boldsymbol{\mathcal{D}}^2 = \boldsymbol{\mathcal{D}}^{\mathrm{H}}\boldsymbol{\mathcal{D}}$ follows. Consequently, we infer

$$\left\| \frac{1}{M} \sum_{j=0}^{M-1} b_j D_I(\circ - \boldsymbol{x}_j) \middle| L_2(T^d) \right\|^2 \leq \|\boldsymbol{\mathcal{D}}\|_2^2 \left\| \left(\boldsymbol{b}/\sqrt{M}\right) \right\|_2^2 = \sigma_{\max}(\boldsymbol{\mathcal{D}})^2 \left\| \boldsymbol{b}/\sqrt{M} \right\|_2^2,$$

where $\sigma_{\max}(\boldsymbol{\mathcal{D}})$ denotes the largest singular value of the matrix $\boldsymbol{\mathcal{D}}$. Last, we show $\sigma_{\max}(\boldsymbol{\mathcal{D}}) \leq 1$. Let $\boldsymbol{\mathcal{D}} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{H}}$ be a singular value decomposition of the matrix $\boldsymbol{\mathcal{D}}$, where $\boldsymbol{U}$, $\boldsymbol{V}$ are unitary matrices and $\boldsymbol{\Sigma} = \operatorname{diag}((\sigma_1, \ldots, \sigma_M))$ is a diagonal matrix of the singular values $\sigma_j \geq 0$, $j = 1, \ldots, M$, of the matrix $\boldsymbol{\mathcal{D}}$. Then, we infer from $\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\mathrm{H}} = \boldsymbol{\mathcal{D}} = \boldsymbol{\mathcal{D}}^2 = \boldsymbol{\mathcal{D}}^{\mathrm{H}}\boldsymbol{\mathcal{D}} = \boldsymbol{U}\boldsymbol{\Sigma}^2\boldsymbol{V}^{\mathrm{H}}$ that $\sigma_j^2 = \sigma_j$, $j = 1, \ldots, M$. Therefore, each singular value $\sigma_j \in \{0, 1\}$ and we obtain $\sigma_{\max}(\boldsymbol{\mathcal{D}}) \leq 1$.

For $p(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}}$, we infer

$$
\begin{aligned}
\tilde{S}_I\, p(\boldsymbol{x}) &= \sum_{\boldsymbol{k} \in I} \frac{1}{M} \sum_{j=0}^{M-1} p(\boldsymbol{x}_j)\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}} \\
&= \sum_{\boldsymbol{k} \in I} \left(\sum_{\boldsymbol{k}' \in I} \hat{p}_{\boldsymbol{k}'}\, \frac{1}{M} \sum_{j=0}^{M-1} \mathrm{e}^{2\pi\mathrm{i}j(\boldsymbol{k}'-\boldsymbol{k})\boldsymbol{z}/M} \right) \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}} \overset{(2.1)}{=} \sum_{\boldsymbol{k} \in I} \hat{p}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}} = p(\boldsymbol{x})
\end{aligned}
$$

since we use the sampling nodes $\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod 1$, $j = 0, \ldots, M-1$, from a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I)$. ■

Lemma 3.2. *Let the dimensionality $d \in \mathbb{N}$, $d \geq 2$, a function $f \in C(\mathbb{T}^d) \cap L^2(\mathbb{T}^d)$ with point-wise convergent Fourier series, a frequency index set $I \subset \mathbb{Z}^d$ of finite cardinality and a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I)$ with the nodes $\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod 1$, $j = 0, \ldots, M-1$, be given. Additionally, we define shells $U_l \subset \mathbb{Z}^d$, $l \in \mathbb{N} \cup \{0\}$, with the properties $U_{l'} \cap U_{l''} = \emptyset$ for $l' \neq l''$ and $\operatorname{supp}\hat{f} \setminus I \subset \bigcup_{l=0}^{\infty} U_l$, where $\operatorname{supp}\hat{f} := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \hat{f}_{\boldsymbol{k}} \neq 0\}$. Then, we have*

$$
\|\tilde{S}_I\,(f - S_I f)\,|L^2(\mathbb{T}^d)\| \leq \sum_{l=0}^{\infty} \sigma_l, \qquad \sigma_l := \left(\frac{1}{M} \sum_{j=0}^{M-1} |S_{U_l} f(\boldsymbol{x}_j)|^2 \right)^{1/2}.
$$

Proof. By definition, we have

$$
\begin{aligned}
\tilde{S}_I\,(f - S_I f) &= \sum_{\boldsymbol{h} \in I} \frac{1}{M} \sum_{j=0}^{M-1} (f - S_I f)\,(\boldsymbol{x}_j)\, \mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{h}\boldsymbol{x}_j} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{h}\circ} \\
&= \frac{1}{M} \sum_{j=0}^{M-1} S_{\operatorname{supp}\hat{f} \setminus I}\, f\,(\boldsymbol{x}_j)\, D_I(\circ - \boldsymbol{x}_j) \\
&= \frac{1}{M} \sum_{j=0}^{M-1} \left(\sum_{\boldsymbol{k} \in \operatorname{supp}\hat{f} \setminus I} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j} \right) D_I(\circ - \boldsymbol{x}_j) \\
&= \frac{1}{M} \sum_{j=0}^{M-1} \left(\sum_{l=0}^{\infty} \sum_{\boldsymbol{k} \in U_l} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j} \right) D_I(\circ - \boldsymbol{x}_j) \\
&= \sum_{l=0}^{\infty} \frac{1}{M} \sum_{j=0}^{M-1} \sum_{\boldsymbol{k} \in U_l} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}\, D_I(\circ - \boldsymbol{x}_j) \\
&= \sum_{l=0}^{\infty} \frac{1}{M} \sum_{j=0}^{M-1} S_{U_l} f(\boldsymbol{x}_j)\, D_I(\circ - \boldsymbol{x}_j).
\end{aligned}
$$

We apply the Minkowski inequality and Lemma 3.1 with $b_j := S_{U_l} f\,(\boldsymbol{x}_j)$. This yields

$$\begin{aligned}
\|\tilde{S}_I\,(f - S_I f)\,|L^2(\mathbb{T}^d)\| &= \left\| \sum_{l=0}^{\infty} \frac{1}{M} \sum_{j=0}^{M-1} S_{U_l} f\,(\boldsymbol{x}_j)\, D_I(\circ - \boldsymbol{x}_j) \middle| L^2(\mathbb{T}^d) \right\| \\
&\le \sum_{l=0}^{\infty} \left\| \frac{1}{M} \sum_{j=0}^{M-1} S_{U_l} f\,(\boldsymbol{x}_j)\, D_I(\circ - \boldsymbol{x}_j) \middle| L^2(\mathbb{T}^d) \right\| \\
&\overset{(3.1)}{\le} \sum_{l=0}^{\infty} \left(\frac{1}{M} \sum_{j=0}^{M-1} |S_{U_l} f\,(\boldsymbol{x}_j)|^2 \right)^{1/2} = \sum_{l=0}^{\infty} \sigma_l
\end{aligned}$$

and the assertion follows. ■

Lemma 3.3. *Let the dimensionality $d \in \mathbb{N}$, a weight function $\omega : \mathbb{Z}^d \to (0, \infty]$, a smoothness parameter $\nu > 0$, the sequence of frequency index sets $I_N := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \omega(\boldsymbol{k})^{1/\nu} \le N\}$ of refinement $N \in \mathbb{R}$, $N \ge 1$, and a function $f \in L^2(\mathbb{T}^d) \cap \mathcal{H}^\omega(\mathbb{T}^d)$ be given. Then, the truncation error is bounded by*

$$\|f - S_{I_N} f | L^2(\mathbb{T}^d)\|^2 \le N^{-\nu}\ \|f|\mathcal{H}^\omega(\mathbb{T}^d)\|.$$

Proof. We have

$$\mathbb{Z}^d \setminus I_N = \{\boldsymbol{k} \in \mathbb{Z}^d \colon \omega(\boldsymbol{k})^{1/\nu} > N\} = \{\boldsymbol{k} \in \mathbb{Z}^d \colon \frac{1}{\omega(\boldsymbol{k})^{1/\nu}} < \frac{1}{N}\} = \{\boldsymbol{k} \in \mathbb{Z}^d \colon \frac{1}{\omega(\boldsymbol{k})^2} < N^{-2\nu}\}$$

and this yields the assertion since

$$\begin{aligned}
\|f - S_{I_N} f | L^2(\mathbb{T}^d)\|^2 &= \sum_{\boldsymbol{k} \in \mathbb{Z}^d \setminus\ I_N} \frac{\omega(\boldsymbol{k})^2}{\omega(\boldsymbol{k})^2} |\hat{f}_{\boldsymbol{k}}|^2 \le \max_{\boldsymbol{k} \in \mathbb{Z}^d \setminus\ I_N} \frac{1}{\omega(\boldsymbol{k})^2} \sum_{\boldsymbol{k} \in \mathbb{Z}^d \setminus\ I_N} \omega(\boldsymbol{k})^2 |\hat{f}_{\boldsymbol{k}}|^2 \\
&\le N^{-2\nu} \sum_{\boldsymbol{k} \in \mathbb{Z}^d \setminus\ I_N} \omega(\boldsymbol{k})^2 |\hat{f}_{\boldsymbol{k}}|^2 \le N^{-2\nu}\ \|f|\mathcal{H}^\omega(\mathbb{T}^d)\|^2.
\end{aligned}$$

■

Theorem 3.4. *Let the dimensionality $d \in \mathbb{N}$, $d \ge 2$, a function $f \in L^2(\mathbb{T}^d) \cap \mathcal{H}^\omega(\mathbb{T}^d)$ with point-wise convergent Fourier series, a smoothness parameter $\nu > 0$ and the sequence of frequency index sets $I_N := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \omega(\boldsymbol{k})^{1/\nu} \le N\}$ with refinement $N \in \mathbb{R}$, $N \ge 1$, be given, where $\omega : \mathbb{Z}^d \to (0, \infty]$ is a weight function. Furthermore, let $\mathcal{I}_N \subset \mathbb{Z}^d$ be a nested sequence of frequency index sets with refinement $N \in \mathbb{R}$, $N \ge 1$,*

$$\mathcal{I}_{N'} \subset \mathcal{I}_{N''} \text{ for } N' \le N'', \tag{3.2}$$

such that the inclusion $\mathcal{I}_N \supset \mathcal{D}(I_N) := \{\boldsymbol{k} - \boldsymbol{k}' \colon \boldsymbol{k}, \boldsymbol{k}' \in I_N\}$ is valid for all $N \in \mathbb{R}$, $N \ge 1$. For each fixed $N \in \mathbb{R}$, $N \ge 1$, let a parameter $\kappa > 0$ and a prime number $M \in \mathbb{N}$,

$$M > \frac{d\,|\mathcal{I}_N|}{1 - 2^{-\kappa}} + 1, \tag{3.3}$$

be given. Additionally, let the inequality

$$|\{\boldsymbol{m} \in \mathcal{I}_{N2^l} \colon \exists \boldsymbol{m}' \in \mathbb{Z}^d \text{ such that } \boldsymbol{m} = M\boldsymbol{m}'\}| \leq C\frac{|\mathcal{I}_{N2^l}|}{M}\psi(l) + 1 \quad \forall l \in \mathbb{N} \tag{3.4}$$

be valid, where $\psi : [0, \infty) \to [1, \infty)$ *and* $C > 0$ *is a constant which does not depend on* N *or* M*. Then, there exists a reconstructing rank-1 lattice* $\Lambda(\boldsymbol{z}, M, I_N)$ *with generating vector* $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top \in \mathbb{Z}^d$ *of Korobov form such that the aliasing error is bounded by*

$$\begin{aligned}\|S_{I_N} f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\| \;\; &\leq \;\; 2^\nu\, N^{-\nu}\, \|f|\mathcal{H}^\omega(\mathbb{T}^d)\| \\ &\quad \cdot \sum_{l=0}^{\infty} \sqrt{2\,(2 + (1 - 2^{-\kappa})\, C\, \psi(l+1))}\, 2^{(l+1)(\frac{\kappa}{2} - \nu)} \sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}}.\end{aligned}$$

Proof. This proof is a generalisation of [24, Theorem 2]). Since inequality (3.3) is valid, we apply Lemma 2.1 and obtain that there exists a number $a \in \{1, \ldots, M-1\}$ which fulfills properties (2.7) and (2.8). Since property (2.7) is valid, the rank-1 lattice $\Lambda(\boldsymbol{z}, M)$ with the generating vector $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top$ and the nodes $\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod 1$, $j = 0, \ldots, M-1$, is a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N)$. We use this special rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N)$ for computing the approximated Fourier coefficients $\tilde{\hat{f}}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I_N$, from the sampling values $f(\boldsymbol{x}_j)$. Since the Fourier partial sum $S_{I_N} f$ of the function f is a trigonometric polynomial with frequencies supported on the index set I_N and by applying Lemma 3.1, we obtain $\tilde{S}_{I_N}(S_{I_N} f) = S_{I_N} f$. This yields $S_{I_N} f - \tilde{S}_{I_N} f = \tilde{S}_{I_N}(f - S_{I_N} f)$. Next, we set the shells $U_l := I_{N2^{l+1}} \setminus I_{N2^l}$, $l = 0, 1, \ldots$, and consequently, the property $U_l \cap U_{l'} = \emptyset \;\forall l \neq l'$ is valid. We apply Lemma 3.2 and we obtain $\|\tilde{S}_{I_N}(f - S_{I_N} f)\,|L^2(\mathbb{T}^d)\| \leq \sum_{l=0}^{\infty} \sigma_l$, where

$$\sigma_l := \left(\frac{1}{M}\sum_{j=0}^{M-1} |S_{U_l} f(\boldsymbol{x}_j)|^2\right)^{1/2}, \; l \in \mathbb{N} \cup \{0\}.$$

Next, we want to estimate

$$\sigma_l^2 \leq B_l \sum_{\boldsymbol{k} \in U_l} |\hat{f}_{\boldsymbol{k}}|^2,$$

with numbers $B_l \geq 0$, which have to be determined. We have

$$\sigma_l^2 = \frac{1}{M}\sum_{j=0}^{M-1}\left|\sum_{\boldsymbol{k}\in U_l} \hat{f}_{\boldsymbol{k}}\, \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\boldsymbol{x}_j}\right|^2 = \frac{1}{M}\sum_{j=0}^{M-1}\sum_{\boldsymbol{k},\boldsymbol{h}\in U_l} \hat{f}_{\boldsymbol{k}}\overline{\hat{f}_{\boldsymbol{h}}}\mathrm{e}^{2\pi\mathrm{i}(\boldsymbol{k}-\boldsymbol{h})\boldsymbol{x}_j} = \sum_{\boldsymbol{k},\boldsymbol{h}\in U_l} \hat{f}_{\boldsymbol{k}}\overline{\hat{f}_{\boldsymbol{h}}}\; \Delta_M(\boldsymbol{k}-\boldsymbol{h}),$$

where

$$\Delta_M(\boldsymbol{m}) := \frac{1}{M}\sum_{j=0}^{M-1} \mathrm{e}^{2\pi\mathrm{i}j\boldsymbol{m}\boldsymbol{z}/M} = \begin{cases} 1 & \text{for } m_1 + m_2\, a + \ldots + m_d\, a^{d-1} \equiv 0 \pmod{M}, \\ 0 & \text{for } m_1 + m_2\, a + \ldots + m_d\, a^{d-1} \not\equiv 0 \pmod{M}. \end{cases}$$

For fixed frequency $\boldsymbol{k} \in U_l$, we define the set of frequencies

$$\theta_{\boldsymbol{k}} := \{\boldsymbol{h} \in U_l \colon \Delta_M(\boldsymbol{k}-\boldsymbol{h}) = 1\},$$

and by applying the Cauchy Schwarz inequality twice, we obtain

$$\begin{aligned}\sigma_l^2 = \sum_{\boldsymbol{k}\in U_l} \hat{f}_{\boldsymbol{k}} \sum_{\boldsymbol{h}\in\theta_{\boldsymbol{k}}} \overline{\hat{f}_{\boldsymbol{h}}} &\leq \left(\sum_{\boldsymbol{k}\in U_l} |\hat{f}_{\boldsymbol{k}}|^2\right)^{1/2} \left(\sum_{\boldsymbol{k}\in U_l} \left|\sum_{\boldsymbol{h}\in\theta_{\boldsymbol{k}}} \overline{\hat{f}_{\boldsymbol{h}}}\right|^2\right)^{1/2} \\ &\leq \left(\sum_{\boldsymbol{k}\in U_l} |\hat{f}_{\boldsymbol{k}}|^2\right)^{1/2} \left(\sum_{\boldsymbol{k}\in U_l} \left(\sum_{\boldsymbol{h}\in\theta_{\boldsymbol{k}}} 1\cdot|\hat{f}_{\boldsymbol{h}}|\right)^2\right)^{1/2} \\ &\leq \left(\sum_{\boldsymbol{k}\in U_l} |\hat{f}_{\boldsymbol{k}}|^2\right)^{1/2} \left(\sum_{\boldsymbol{k}\in U_l} |\theta_{\boldsymbol{k}}| \sum_{\boldsymbol{h}\in\theta_{\boldsymbol{k}}} |\hat{f}_{\boldsymbol{h}}|^2\right)^{1/2}.\end{aligned}$$

We have $\boldsymbol{k}-\boldsymbol{h} \in \mathcal{D}(I_{N2^{l+1}}) \subset \mathcal{I}_{N2^{l+1}}$ for $\boldsymbol{k}, \boldsymbol{h} \in U_l$ and this yields

$$|\theta_{\boldsymbol{k}}| \leq |\{\boldsymbol{m} \in \mathcal{I}_{N2^{l+1}} : m_1 + m_2\, a + \ldots + m_d\, a^{d-1} \equiv 0 \pmod{M}\}|.$$

We define the function $\varphi(l) := 2^l$ for $l \in \mathbb{N}\cup\{0\}$. Due to property (2.8) in Lemma 2.1, we obtain

$$\begin{aligned}&\left|\{\boldsymbol{m} \in I^{d,0}_{N\,\varphi(l+1)} : m_1 + m_2\, a + \ldots + m_d\, a^{d-1} \equiv 0 \pmod{M} \text{ and } \boldsymbol{m} \neq M\boldsymbol{m'}\ \forall \boldsymbol{m'} \in \mathbb{Z}^d\}\right| \\ &= \left|\bigcup_{j=1}^{l+1} F_j(N)\right| \leq \sum_{j=1}^{l+1} A_j^N.\end{aligned}$$

Then, we have

$$|\theta_{\boldsymbol{k}}| \leq B_l := \sum_{j=1}^{l+1} A_j^N + C\frac{|\mathcal{I}_{N2^{l+1}}|}{M}\psi(l+1) + 1 \tag{3.5}$$

and

$$\sigma_l^2 \leq \left(\sum_{\boldsymbol{k}\in U_l} |\hat{f}_{\boldsymbol{k}}|^2\right)^{1/2} \left(B_l \sum_{\boldsymbol{k}\in U_l}\sum_{\boldsymbol{h}\in\theta_{\boldsymbol{k}}} |\hat{f}_{\boldsymbol{h}}|^2\right)^{1/2}.$$

For an arbitrarily chosen $\boldsymbol{k} \in U_l$, let $\boldsymbol{h} \in \theta_{\boldsymbol{k}}$. This means, we have $(\boldsymbol{h}-\boldsymbol{k})\boldsymbol{z} \equiv 0 \pmod{M}$. If $\boldsymbol{h} \in \theta_{\boldsymbol{k'}}$ for another $\boldsymbol{k'} \in U_l$, $\boldsymbol{k'} \neq \boldsymbol{k}$, then $(\boldsymbol{h}-\boldsymbol{k'})\boldsymbol{z} \equiv 0 \pmod{M}$ is valid and $(\boldsymbol{k}-\boldsymbol{k'})\boldsymbol{z} \equiv 0 \pmod{M}$ follows. This yields $\boldsymbol{k'} \in \theta_{\boldsymbol{k}}$. Especially, we have $\boldsymbol{k} \in \theta_{\boldsymbol{k}}$. Therefore, each frequency $\boldsymbol{h'} \in U_l$ is element of at most B_l many distinct sets $\theta_{\boldsymbol{k}}$. This means, we obtain

$$\sum_{\boldsymbol{k}\in U_l}\sum_{\boldsymbol{h}\in\theta_{\boldsymbol{k}}} |\hat{f}_{\boldsymbol{h}}|^2 \leq \sum_{\boldsymbol{k}\in U_l} B_l |\hat{f}_{\boldsymbol{k}}|^2$$

and

$$\begin{aligned}\sigma_l^2 &\leq \left(\sum_{\boldsymbol{k}\in U_l} |\hat{f}_{\boldsymbol{k}}|^2\right)^{1/2} \left(B_l^2 \sum_{\boldsymbol{k}\in U_l} |\hat{f}_{\boldsymbol{k}}|^2\right)^{1/2} = B_l \sum_{\boldsymbol{k}\in U_l} |\hat{f}_{\boldsymbol{k}}|^2 \leq B_l \sum_{\boldsymbol{k}\in\mathbb{Z}^d\setminus I_{N2^l}} |\hat{f}_{\boldsymbol{k}}|^2 \\ &= B_l\, \|f - S_{I_{N2^l}} f|L^2(\mathbb{T}^d)\|^2 \leq B_l\, (N2^l)^{-2\nu}\, \|f|\mathcal{H}^\omega(\mathbb{T}^d)\|^2.\end{aligned}$$

Next, we estimate B_l. Using the inequality $\frac{1}{M-1} \leq \frac{2}{M}$ for $M \geq 2$ as well as (3.5) and (2.8), we infer

$$\begin{aligned} B_l &= \sum_{j=1}^{l+1} |\mathcal{I}_{N\varphi(j)} \setminus \mathcal{I}_{N\varphi(j-1)}| d 2^{(j+1)\kappa}(2^\kappa - 1)^{-1}(M-1)^{-1} + C\frac{|\mathcal{I}_{N2^{l+1}}|}{M}\psi(l+1) + 1 \\ &\leq d\frac{2^\kappa}{2^\kappa - 1} 2^{(l+1)\kappa}\frac{2}{M}\sum_{j=1}^{l+1}|\mathcal{I}_{N2^j} \setminus \mathcal{I}_{N2^{j-1}}| + C\ \psi(l+1)\ |\mathcal{I}_{N2^{l+1}}|/M + 1 \\ &\overset{(3.2)}{\leq} d2^{(l+1)\kappa}\frac{|\mathcal{I}_{N2^{l+1}}|}{M}\left(\frac{2}{1-2^{-\kappa}} + C\ \psi(l+1)\right) + 1 \\ &\overset{(3.3)}{\leq} d2^{(l+1)\kappa}\frac{|\mathcal{I}_{N2^{l+1}}|}{\frac{d|\mathcal{I}_N|}{1-2^{-\kappa}} + 1}\left(\frac{2}{1-2^{-\kappa}} + C\ \psi(l+1)\right) + 1 \\ &\leq 2^{(l+1)\kappa+1}\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}\left(2 + (1-2^{-\kappa})\, C\ \psi(l+1)\right) \end{aligned}$$

and this yields

$$\begin{aligned} \sigma_l &\leq (N2^l)^{-\nu}\ \|f|\mathcal{H}^\omega(\mathbb{T}^d)\|\ \sqrt{B_l} \\ &\leq \sqrt{2\left(2 + (1-2^{-\kappa})\, C\ \psi(l+1)\right)}\ 2^\nu\ N^{-\nu}\ \|f|\mathcal{H}^\omega(\mathbb{T}^d)\|\ 2^{(l+1)(\frac{\kappa}{2}-\nu)}\sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}}. \end{aligned}$$

■

4 Approximation error for rank-1 lattice sampling and frequency index sets $I_N^{d,T}$

Next, we apply the general results from Section 2 and Section 3. Therefor, we use the index sets $I = I_N = I_N^{d,T}$. In the case $-\infty \leq T \leq 0$, we set $\mathcal{I}_N := I^{d,T}_{2^{\frac{d-T}{1-T}} N^{1+\frac{d}{d-T}}}$, where $\mathcal{I}_N \supset \mathcal{D}(I_N^{d,T})$, see Lemma 4.2. This means, we cover the difference set $\mathcal{D}(I_N^{d,T})$ with an index set $I^{d,T}_{2^{\frac{d-T}{1-T}} N^{1+\frac{d}{d-T}}}$ of larger refinement $2^{\frac{d-T}{1-T}} N^{1+\frac{d}{d-T}}$. In the case $0 < T < 1$, we set $\mathcal{I}_N := \mathcal{D}(I_N^{d,T})$. Before we estimate the truncation error $\|f - S_{I_N^{d,T}} f|L^2(\mathbb{T}^d)\|$ and the aliasing error $\|S_{I_N^{d,T}} f - \tilde{S}_{I_N^{d,T}} f|L^2(\mathbb{T}^d)\|$, we show preliminary lemmata for the cardinalities and embeddings of the frequency index sets $I_N^{d,T}$.

Lemma 4.1. *Let the dimensionality $d \in \mathbb{N}$, and a parameter T, $-\infty \leq T < 1$, be given. Then, the cardinalities of the frequency index sets $I_N^{d,T}$ are*

$$|I_N^{d,T}| = \begin{cases} \Theta(N^d) & \text{for } T = -\infty, \\ \Theta(N^{\frac{T-1}{T/d-1}}) & \text{for } -\infty < T < 0, \\ \Theta(N \log^{d-1} N) & \text{for } T = 0, \\ \Theta(N) & \text{for } 0 < T < 1, \end{cases} \tag{4.1}$$

for fixed $d \in \mathbb{N}$.

Proof. The upper bounds follow from [14, Lemma 2.6] and [15, Section 3.3 Lemma 2]. Therefore, we still have to discuss the lower bounds.

- Case $T = -\infty$. Since the volume of an l_1 ball with radius N is $\frac{2^d}{d!}N^d$ (e.g. see [29]), the lower bound $|I_N^{d,-\infty}| = \Omega(N^d)$ follows.

- Case $-\infty < T < 0$. First, we show $I^{d,-\infty}_{N^{\frac{1-T}{d-T}}} \subset I_N^{d,T}$. For arbitrary $\boldsymbol{k} \in I^{d,-\infty}_{N^{\frac{1-T}{d-T}}}$, we have
$$N^{\frac{1-T}{d-T}} \geq \max(1, \|\boldsymbol{k}\|_1) = \max(1, \|\boldsymbol{k}\|_1)^{-\frac{T}{d-T}} \ \max(1, \|\boldsymbol{k}\|_1)^{1+\frac{T}{d-T}}.$$
Since $\max(1, \|\boldsymbol{k}\|_1)^d \geq \max(1, \|\boldsymbol{k}\|_\infty) \geq \prod_{s=1}^d \max(1, |k_s|)$, we infer
$$\begin{aligned} N^{\frac{1-T}{d-T}} &\geq \max(1, \|\boldsymbol{k}\|_1)^{-\frac{T}{d-T}} \prod_{s=1}^d \max(1, |k_s|)^{\frac{1}{d}(1+\frac{T}{d-T})} \\ &= \max(1, \|\boldsymbol{k}\|_1)^{-\frac{T}{1-T}\frac{1-T}{d-T}} \prod_{s=1}^d \max(1, |k_s|)^{\frac{1}{1-T}\frac{1-T}{d-T}} \end{aligned}$$
and consequently $\max(1, \|\boldsymbol{k}\|_1)^{-\frac{T}{1-T}} \prod_{s=1}^d \max(1, |k_s|)^{\frac{1}{1-T}} \leq N$. This means, we have $\boldsymbol{k} \in I_N^{d,T}$ and therefore we obtain $I^{d,-\infty}_{N^{\frac{1-T}{d-T}}} \subset I_N^{d,T}$. Since we have $|I^{d,-\infty}_{N^{\frac{1-T}{d-T}}}| \geq c_1(d) N^{\frac{1-T}{d-T}d}$, where $c_1(d)$ is a constant which only depends on the dimensionality d, we obtain $|I_N^{d,T}| = \Omega(N^{\frac{T-1}{T/d-1}})$ due to $|I_N^{d,T}| \geq |I^{d,-\infty}_{N^{\frac{1-T}{d-T}}}| \geq c_1(d) N^{\frac{1-T}{d-T}d} = c_1(d) N^{\frac{T-1}{T/d-1}}$.

- Case $T = 0$. For the lower bound $|I_N^{d,0}| = \Omega(N \log^{d-1} N)$, we refer to [8, Section 5.3].

- Case $0 < T < 1$. Since the frequencies on the coordinate axis from $-\lfloor N \rfloor$ to $\lfloor N \rfloor$ are elements of $I_N^{d,T}$, we obtain $|I_N^{d,T}| \geq 2d\lfloor N \rfloor + 1 \geq 2d(N-1) + 1$.

These lower bounds yield the assertion. ∎

Lemma 4.2. *Let the dimensionality $d \in \mathbb{N}$, and a parameter T, $-\infty \leq T \leq 0$, be given. We consider the difference set $\mathcal{D}(I_N^{d,T}) := \left\{\boldsymbol{k}' - \boldsymbol{k} \colon \boldsymbol{k}, \boldsymbol{k}' \in I_N^{d,T}\right\}$. Then, we have the inclusion*
$$\mathcal{D}(I_N^{d,T}) \subset I^{d,T}_{2^{\frac{d-T}{1-T}} N^{1+\frac{d}{d-T}}}. \tag{4.2}$$

Proof. For $\boldsymbol{k} \in I_N^{d,T}$, we have $\max(1, \|\boldsymbol{k}\|_1)^{-\frac{T}{1-T}} \prod_{s=1}^d \max(1, |k_s|)^{\frac{1}{1-T}} \leq N$ by definition. Consequently, for $\boldsymbol{k}, \boldsymbol{k}' \in I_N^{d,T}$ and $-\infty \leq T < 0$, we infer
$$\begin{aligned} &\max(1, \|\boldsymbol{k} - \boldsymbol{k}'\|_1) \prod_{s=1}^d \max(1, |k_s - k_s'|)^{-\frac{1}{T}} \\ \leq\ & \big(\max(1, \|\boldsymbol{k}\|_1) + \max(1, \|\boldsymbol{k}'\|_1)\big) \prod_{s=1}^d \big(\max(1, |k_s|) + \max(1, |k_s'|)\big)^{-\frac{1}{T}} \\ \leq\ & \big(\max(1, \|\boldsymbol{k}\|_1) + \max(1, \|\boldsymbol{k}'\|_1)\big)\ 2^{-\frac{d}{T}} \prod_{s=1}^d \max(1, |k_s|)^{-\frac{1}{T}} \max(1, |k_s'|)^{-\frac{1}{T}} \\ \leq\ & 2^{-\frac{d}{T}} N^{-\frac{1-T}{T}} \left(\prod_{s=1}^d \max(1, |k_s'|)^{-\frac{1}{T}} + \prod_{s=1}^d \max(1, |k_s|)^{-\frac{1}{T}} \right). \end{aligned}$$

Next, we estimate dominating mixed smoothness by isotropic smoothness. Since we have $\prod_{s=1}^d \max(1,|k_s|) \leq \max(1,\|\boldsymbol{k}\|_\infty)^d \leq \max(1,\|\boldsymbol{k}\|_1)^d$ for $\boldsymbol{k}\in\mathbb{Z}^d$, we obtain

$$\begin{aligned}\prod_{s=1}^d \max(1,|k_s|)^{-\frac{1}{T}} &= \prod_{s=1}^d \max(1,|k_s|)^{\frac{1}{d-T}} \prod_{s=1}^d \max(1,|k_s|)^{-\frac{1}{T}-\frac{1}{d-T}} \\ &\leq \max(1,\|\boldsymbol{k}\|_1)^{\frac{d}{d-T}} \prod_{s=1}^d \max(1,|k_s|)^{-\frac{d}{T(d-T)}} \\ &= \left(\max(1,\|\boldsymbol{k}\|_1)^{-\frac{T}{1-T}} \prod_{s=1}^d \max(1,|k_s|)^{\frac{1}{1-T}}\right)^{-\frac{1-T}{T}\frac{d}{d-T}} \\ &\leq N^{-\frac{1-T}{T}\frac{d}{d-T}}\end{aligned}$$

and analogously $\prod_{s=1}^d \max(1,|k'_s|)^{-\frac{1}{T}} \leq N^{-\frac{1-T}{T}\frac{d}{d-T}}$. For $T=0$, we have

$$\prod_{s=1}^d \max(1,|k_s-k'_s|) \leq 2^d \prod_{s=1}^d \max(1,|k_s|) \prod_{s=1}^d \max(1,|k'_s|) \leq 2^d\, N^2.$$

These results yield

$$\max(1,\|\boldsymbol{k}-\boldsymbol{k}'\|_1)^{-\frac{T}{1-T}} \prod_{s=1}^d \max(1,|k_s-k'_s|)^{\frac{1}{1-T}} \leq 2^{\frac{d-T}{1-T}} N^{1+\frac{d}{d-T}} \text{ for all } \boldsymbol{k},\boldsymbol{k}'\in I_N^{d,T}$$

and inclusion (4.2) follows. ■

4.1 Truncation error

We estimate the truncation error $\|f - S_{I_N^{d,T}} f|L^2(\mathbb{T}^d)\|$, since this error is part of the approximation error $\|f-\tilde{S}_{I_N^{d,T}} f|L^2(\mathbb{T}^d)\|$ and since we also need the result as a prerequisite for Theorem 3.4. First, we show $\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\subset L^2(\mathbb{T}^d)$ for $\beta\geq 0$ and $\alpha>-\beta$.

Lemma 4.3. *Let the parameter $\alpha,\beta\in\mathbb{R}$, $\beta\geq 0$, $\alpha>-\beta$ be given. Then, $\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\subset L^2(\mathbb{T}^d)$.*

Proof. In the case $\alpha\geq 0$, we obviously have $\omega^{\alpha,\beta}(\boldsymbol{k})1$ for all $\boldsymbol{k}\in\mathbb{Z}^d$. In the case $\alpha<0$, due to $\prod_{s=1}^d \max(1,|k_s|) \leq \max(1,\|\boldsymbol{k}\|_1)^d$ for $\boldsymbol{k}\in\mathbb{Z}^d$ and $\beta+\frac{\alpha}{d}>\alpha+\beta>0$, we infer

$$\omega^{\alpha,\beta}(\boldsymbol{k}) := \max(1,\|\boldsymbol{k}\|_1)^\alpha \prod_{s=1}^d \max(1,|k_s|)^\beta \geq \prod_{s=1}^d \max(1,|k_s|)^{\beta+\frac{\alpha}{d}} \geq 1 \text{ for all } \boldsymbol{k}\in\mathbb{Z}^d.$$

Consequently, we obtain

$$\|f|L^2(\mathbb{T}^d)\| = \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}|^2} \leq \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \omega^{\alpha,\beta}(\boldsymbol{k})^2|\hat{f}_{\boldsymbol{k}}|^2} = \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\| < \infty$$

for an arbitrarily chosen function $f\in\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$. ■

Next, we estimate the truncation error $\|f - S_{I_N^{d,T}} f|L^2(\mathbb{T}^d)\|$. The following lemma is a special case of [14, Lemma 3.5].

Lemma 4.4. *Let the dimensionality $d \in \mathbb{N}$, a function $f \in \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$ and the d-dimensional index set $I_N^{d,T}$ of refinement $N \in \mathbb{R}$, $N \geq 1$, be given, where $\beta \geq 0$, $\alpha > -\beta$ and $T := -\alpha/\beta$. Then, the truncation error is bounded by*

$$\|f - S_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\| \leq N^{-(\alpha+\beta)} \, \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|.$$

Proof. We set $\omega(\boldsymbol{k}) := \omega^{\alpha,\beta}(\boldsymbol{k})$, $\nu := \alpha + \beta$ and $I_N := I_N^{d,T}$. Since $T := -\alpha/\beta$, the conditions $\beta \geq 0$ and $\alpha > -\beta$ ensure that $-\infty \leq T < 1$. From Lemma 4.3, we obtain $\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$. Next, we apply Lemma 3.3. Due to $\omega(\boldsymbol{k})^{1/\nu} = \max(1, \|\boldsymbol{k}\|_1)^{\frac{\alpha}{\alpha+\beta}} \prod_{s=1}^d \max(1, |k_s|)^{\frac{\beta}{\alpha+\beta}} = \omega^{-\frac{T}{1-T}, \frac{1}{1-T}}(\boldsymbol{k})$, we obtain $\|f - S_{I_N} f | L^2(\mathbb{T}^d)\| \leq N^{-\nu} \, \|f|\mathcal{H}^\omega(\mathbb{T}^d)\| = N^{-(\alpha+\beta)} \, \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|$. ■

4.2 Aliasing error

We are going to apply Theorem 3.4 for the frequency index sets $I_N = I_N^{d,T}$ in order to estimate the aliasing error $\|S_{I_N} f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\|$. Therefor, we show that condition (3.4) is fulfilled for the frequency index sets $I_N^{d,T}$ of refinements $N \in \mathbb{R}$, $N \geq 2$, and parameters $-\infty \leq T < 1$.

4.2.1 Cases $-\infty \leq T \leq 0$

Lemma 4.5. *Let the dimensionality $d \in \mathbb{N}$, $d \geq 2$, a parameter T, $-\infty \leq T \leq 0$, and $M \in \mathbb{N}$, $M \geq 2$, be given. Then, we have*

$$|\{\boldsymbol{m} \in I^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}} : \exists \boldsymbol{m}' \in \mathbb{Z}^d \text{ such that } \boldsymbol{m} = M\boldsymbol{m}'\}| \leq C_{\mathrm{A}}(d,T)\, |I^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}}|/M + 1$$

for all refinements $N \in \mathbb{R}$, $N \geq 1$, and levels $l \in \mathbb{N} \cup \{0\}$, where $C_{\mathrm{A}}(d,T) \geq 1$ is a constant which only depends on d and T.

Proof. We denote $A^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}} := \{\boldsymbol{m} \in I^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}} : \exists \boldsymbol{m}' \in \mathbb{Z}^d \text{ such that } \boldsymbol{m} = M\boldsymbol{m}'\}$ and we group the indices $\boldsymbol{m} \in A^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}}$, where all components are zero, exactly one component is non-zero, ..., $d-1$ components are non-zero, and all d components are non-zero. For $t = 0, \ldots, d$, we denote

$$A^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})},t} := \left\{\boldsymbol{m} \in A^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}} : \text{exactly } t \text{ components of } \boldsymbol{m} \text{ are non-zero}\right\}.$$

- Case $t = 0$. We have $A^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})},0} = \{\boldsymbol{0}\}$.

- Case $1 \leq t \leq d$. If exactly the components $m_{i_1}, \ldots, m_{i_t}$ of $\boldsymbol{m} \in A^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}}$ are non-zero, $i_1,\ldots,i_t \in \{1,\ldots,d\}$, $i_j \neq i_{j'}$ for $j \neq j'$, we have

$$
\begin{aligned}
&\omega^{-\frac{T}{1-T},\frac{1}{1-T}}(\boldsymbol{m})\\
= \;& \max\left(1, M(|m'_{i_1}|+\ldots+|m'_{i_t}|)\right)^{-\frac{T}{1-T}} \prod_{\tau=1}^{t} \max(1, M|m'_{i_\tau}|)^{\frac{1}{1-T}}\\
= \;& M^{-\frac{T}{1-T}} \max(1, |m'_{i_1}|+\ldots+|m'_{i_t}|)^{-\frac{T}{1-T}}\, M^{\frac{t}{1-T}} \prod_{\tau=1}^{t} \max(1, M|m'_{i_\tau}|)^{\frac{1}{1-T}}\\
= \;& M^{\frac{t-T}{1-T}} \omega^{-\frac{T}{1-T},\frac{1}{1-T}}(\boldsymbol{m}') \leq N\,2^{(l+1)(1+\frac{d}{d-T})} \Longleftrightarrow \omega^{-\frac{T}{1-T},\frac{1}{1-T}}(\boldsymbol{m}') \leq \frac{N\,2^{(l+1)(1+\frac{d}{d-T})}}{M^{\frac{t-T}{1-T}}}.
\end{aligned}
$$

Since there are $\binom{d}{t}$ choices for the non-zero components, we have

$$
\left|A^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})},t}\right| = \binom{d}{t}\cdot
\begin{cases}
\mathcal{O}\left(\left(\frac{N\,2^{(l+1)}}{M}\right)^{t}\right) & \text{for } T=-\infty,\\
\mathcal{O}\left(\frac{\left(N\,2^{(l+1)(1+\frac{d}{d-T})}\right)^{\frac{T-1}{T/t-1}}}{M^t}\right) & \text{for } -\infty<T<0,\\
\mathcal{O}\left(\left(\frac{N\,2^{(l+1)2}}{M^t}\right)\log^{t-1}\left(\frac{N\,2^{(l+1)2}}{M^t}\right)\right) & \text{for } T=0,
\end{cases}
$$

for fixed $d \in \mathbb{N}$.

This means

- for $T=-\infty$

$$
\begin{aligned}
|A^{d,-\infty}_{N\,2^{l+1}}| &\leq 1+\sum_{t=1}^{d}\binom{d}{t} C_1(d)\left(\frac{N\,2^{(l+1)}}{M}\right)^{t} \leq 1+\frac{(N\,2^{(l+1)})^d}{M} C_1(d)\,(2^d-1)\\
&\leq 1+\frac{|I^{d,-\infty}_{N\,2^{l+1}}|}{M}\frac{C_1(d)}{c_1(d)}\,(2^d-1)
\end{aligned}
$$

  due to $|I^{d,-\infty}_{N\,2^{l+1}}| \geq c_1(d)(N\,2^{(l+1)})^d$ as stated in Lemma 4.1,

- for $-\infty<T<0$

$$
\begin{aligned}
\left|A^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}}\right| &\leq 1+\sum_{t=1}^{d}\binom{d}{t} C_2(d)\left(\frac{N\,2^{(l+1)(1+\frac{d}{d-T})}}{M^{\frac{t-T}{1-T}}}\right)^{\frac{T-1}{T/t-1}}\\
&= 1+\sum_{t=1}^{d}\binom{d}{t} C_2(d)\frac{\left(N\,2^{(l+1)(1+\frac{d}{d-T})}\right)^{\frac{t(1-T)}{t-T}}}{M^t}\\
&\leq 1+C_2(d)\frac{\left(N\,2^{(l+1)(1+\frac{d}{d-T})}\right)^{\frac{d(1-T)}{d-T}}}{M}\,(2^d-1)\\
&\leq 1+\frac{C_2(d)}{c_1(d)}\frac{\left|I^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}}\right|}{M}\,(2^d-1)
\end{aligned}
$$

due to $|I^{d,T}_{N\,2^{(l+1)(1+\frac{d}{d-T})}}| \geq c_1(d)\,\left(N\,2^{(l+1)(1+\frac{d}{d-T})}\right)^{\frac{T-1}{T/d-1}} = c_1(d)\,\left(N\,2^{(l+1)(1+\frac{d}{d-T})}\right)^{\frac{d(1-T)}{d-T}}$ as stated in Lemma 4.1,

- for $T = 0$

$$\begin{aligned}
\left|A^{d,0}_{N\,2^{(l+1)2}}\right| &\leq 1 + \sum_{t=1}^{d} \binom{d}{t} C_3(d) \left(\frac{N\,2^{(l+1)2}}{M^t}\right) \log^{t-1}\left(\frac{N\,2^{(l+1)2}}{M^t}\right) \\
&\leq 1 + C_3(d) \frac{N\,2^{(l+1)2}}{M} \log^{d-1}\left(N\,2^{(l+1)2}\right)(2^d-1) \\
&\leq 1 + \frac{\left|I^{d,0}_{N\,2^{(l+1)2}}\right|}{M} \frac{C_3(d)}{c_3(d)} (2^d-1)
\end{aligned}$$

due to $\left|I^{d,0}_{N\,2^{(l+1)2}}\right| \geq c_3(d)\,N\,2^{(l+1)2}\,\log^{d-1}\left(N\,2^{(l+1)2}\right)$ as stated in Lemma 4.1.

We set

$$C_{\mathrm{A}}(d,T) := (2^d-1)\cdot\begin{cases} C_1(d)/c_1(d) & \text{for } T=-\infty, \\ C_2(d)/c_1(d) & \text{for } -\infty < T < 0, \\ C_3(d)/c_3(d) & \text{for } T=0, \end{cases}$$

and this yields the assertion. ■

Lemma 4.6. *Let the dimensionality $d \in \mathbb{N}$, $d \geq 2$ and a function $f \in \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$, where $\alpha, \beta \geq 0$ and $\alpha > d(\frac{1}{2} - \beta)$. Then, the function f has an absolutely converging Fourier series,*

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}| < \infty.$$

Proof. Applying the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}| = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \frac{\omega^{\alpha,\beta}(\boldsymbol{k})}{\omega^{\alpha,\beta}(\boldsymbol{k})} |\hat{f}_{\boldsymbol{k}}| &\leq \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \frac{1}{\omega^{\alpha,\beta}(\boldsymbol{k})^2}} \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \omega^{\alpha,\beta}(\boldsymbol{k})^2 |\hat{f}_{\boldsymbol{k}}|^2} \\
&= \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \frac{1}{\max(1,\|\boldsymbol{k}\|_1)^{2\alpha}\,\prod_{s=1}^d \max(1,|k_s|)^{2\beta}}}\;\|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|.
\end{aligned}$$

Due to $\prod_{s=1}^d \max(1,|k_s|) \leq \max(1,\|\boldsymbol{k}\|_1)^d$ for $\boldsymbol{k} \in \mathbb{Z}^d$, we infer

$$\begin{aligned}
\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}| &\leq \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \prod_{s=1}^{d} \frac{1}{\max(1,|k_s|)^{2(\beta+\frac{\alpha}{d})}}}\;\|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\| \\
&= \left(1 + 2\,\zeta\left(2\left(\beta + \frac{\alpha}{d}\right)\right)\right)^{\frac{d}{2}}\;\|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|,
\end{aligned}$$

where ζ is the Riemann zeta function. Since $\beta \geq 0$ and $\alpha > d(\frac{1}{2} - \beta)$, we obtain $2\left(\beta + \frac{\alpha}{d}\right) > 2\left(\beta + \frac{1}{2} - \beta\right) = 1$. Due to this and since $f \in \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$, we infer $\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}| < \infty$. ■

Theorem 4.7. *Let the dimensionality $d \in \mathbb{N}$, $d \geq 2$, a function $f \in \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$ and a refinement $N \in \mathbb{R}$, $N \geq 2$ be given, where $\beta \geq 0$, $\alpha \geq 0$,*

$$\alpha + \beta > \left(1 + \frac{d}{d-T}\right) \frac{T-1}{T/d-1} \frac{1}{2} \tag{4.3}$$

and the parameter $T := -\alpha/\beta$. Additionally, let a prime number $M \in \mathbb{N}$,

$$M > \frac{d \left| I^{d,T}_{2^{\frac{d-T}{1-T}} N^{1+\frac{d}{d-T}}} \right|}{1 - 2^{-\kappa}} + 1, \tag{4.4}$$

be given, where we set the parameter $\kappa := \alpha + \beta - (1 + \frac{d}{d-T}) \frac{T-1}{T/d-1} \frac{1}{2}$. Then, there exists a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N^{d,T})$ with generating vector $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top \in \mathbb{Z}^d$ of Korobov form and nodes $\boldsymbol{x}_j := \frac{j}{M}\boldsymbol{z} \bmod 1$, $j = 0, \ldots, M-1$, such that the aliasing error is bounded by

$$\|S_{I_N^{d,T}} f - \tilde{S}_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\| \leq C(d, \alpha, \beta)\, N^{-(\alpha+\beta)}\, \|f | \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|,$$

where $C(d, \alpha, \beta) > 0$ is a constant which only depends on d, α, β.

Proof. We are going to apply Theorem 3.4. Therefore, we set $\omega(\boldsymbol{k}) := \omega^{\alpha,\beta}(\boldsymbol{k})$, $\nu := \alpha+\beta$ and $I_N := I_N^{d,T}$. Due to $d \geq 2$, $\alpha \geq 0$ and $\beta \geq 0$, we have $\left(1 + \frac{d}{d-T}\right) \frac{T-1}{T/d-1} \frac{1}{2} = \frac{d}{2} \frac{2d\beta+\alpha}{d\beta+\alpha} \frac{\alpha+\beta}{d\beta+\alpha} > 0$ and consequently, $\nu = \alpha + \beta > 0$ follows from condition (4.3). From Lemma 4.3, we obtain $\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$. Furthermore, we obtain $\mathcal{D}(I_N^{d,T}) \subset I^{d,T}_{2^{\frac{d-T}{1-T}} N^{1+\frac{d}{d-T}}}$ from Lemma 4.2. Thus, we set $\mathcal{I}_N := I^{d,T}_{2^{\frac{d-T}{1-T}} N^{1+\frac{d}{d-T}}}$ for all $N \in \mathbb{R}$, $N \geq 1$. Applying Lemma 4.5, we infer

$$\begin{aligned}
&|\{\boldsymbol{m} \in \mathcal{I}_{N2^l} : \exists \boldsymbol{m}' \in \mathbb{Z}^d \text{ such that } \boldsymbol{m} = M\boldsymbol{m}'\}| \\
= \;& |\{\boldsymbol{m} \in I^{d,T}_{2^{\frac{d-T}{1-T}} N^{1+\frac{d}{d-T}} 2^{l(1+\frac{d}{d-T})}} : \exists \boldsymbol{m}' \in \mathbb{Z}^d \text{ such that } \boldsymbol{m} = M\boldsymbol{m}'\}| \\
\leq \;& C_{\mathrm{A}}(d,T) \frac{\left| I^{d,T}_{2^{\frac{d-T}{1-T}} N^{1+\frac{d}{d-T}} 2^{l(1+\frac{d}{d-T})}} \right|}{M} + 1 \quad \text{for all } l \in \mathbb{N}.
\end{aligned}$$

In order to apply Lemma 4.6, we first show $\alpha > \frac{d}{2} - d\beta$. Due to (4.3), we have $\alpha + \beta > \frac{d}{2} \frac{2d\beta+\alpha}{d\beta+\alpha} \frac{\alpha+\beta}{d\beta+\alpha}$. This is equivalent to the condition $2(d\beta + \alpha)^2 > d(2d\beta + \alpha)$ since $d\beta + \alpha \geq \alpha + \beta > 0$. Due to $2d\beta \geq d\beta$, we obtain $2(d\beta + \alpha)^2 > d(d\beta + \alpha)$. Consequently, we have $\alpha > \frac{d}{2} - d\beta$ such that we can apply Lemma 4.6 and we obtain that f has an absolutely converging Fourier series, whereby f has a point-wise convergent Fourier series.
Next, we apply Theorem 3.4 with $\psi \equiv 1$ and we obtain that there exists a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N)$ with generating vector $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top \in \mathbb{Z}^d$ of Korobov form, such that the aliasing error is bounded by

$$\begin{aligned}
\|S_{I_N} f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\| \;\leq\;& 2^{\alpha+\beta}\, N^{-(\alpha+\beta)}\, \|f | \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\| \\
& \cdot \sum_{l=0}^{\infty} 2^{(l+1)(\frac{\kappa}{2} - (\alpha+\beta))} \sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}} \sqrt{2\,(2 + (1 - 2^{-\kappa})\, C_{\mathrm{A}}(d,T))}.
\end{aligned}$$

- Case $T=-\infty$, i.e., $\beta=0$ and $\alpha>\frac{d}{2}$. Due to $\sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}}=\sqrt{\frac{|I^{d,-\infty}_{2N2^{l+1}}|}{|I^{d,-\infty}_{2N}|}}\leq\sqrt{\frac{C_1(d)}{c_1(d)}}\sqrt{\frac{2^d\,N^d\,2^{(l+1)d}}{2^d\,N^d}}=\sqrt{\frac{C_1(d)}{c_1(d)}}\,2^{(l+1)\frac{d}{2}}$ by Lemma 4.1, where $c_1(d)$ and $C_1(d)$ are constants which only depend on d, we obtain

$$\begin{aligned}\sum_{l=0}^{\infty}2^{(l+1)(\frac{\kappa}{2}-\alpha)}\sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}} &\leq \sqrt{\frac{C_1(d)}{c_1(d)}}\sum_{l=0}^{\infty}2^{(l+1)(-\frac{\alpha}{2}+\frac{d}{4})}\\ &= \sqrt{\frac{C_1(d)}{c_1(d)}}\frac{2^{-\frac{\alpha}{2}+\frac{d}{4}}}{1-2^{-\frac{\alpha}{2}+\frac{d}{4}}}=:\widetilde{C}(d,\alpha,0).\end{aligned}$$

- Case $-\infty<T<0$, i.e., $\beta>0$, $\alpha>d\left(\frac{1}{4}+\frac{1}{4}\sqrt{8\beta+1}-\beta\right)$. Due to

$$\begin{aligned}\sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}}=\sqrt{\frac{\left|I^{d,T}_{2^{\frac{d-T}{1-T}}\left(N2^{l+1}\right)^{1+\frac{d}{d-T}}}\right|}{\left|I^{d,T}_{2^{\frac{d-T}{1-T}}N^{1+\frac{d}{d-T}}}\right|}} &\leq \sqrt{\frac{C_2(d)}{c_1(d)}}\sqrt{\frac{\left(2^{\frac{d-T}{1-T}}N^{1+\frac{d}{d-T}}2^{(l+1)(1+\frac{d}{d-T})}\right)^{\frac{T-1}{T/d-1}}}{\left(2^{\frac{d-T}{1-T}}N^{1+\frac{d}{d-T}}\right)^{\frac{T-1}{T/d-1}}}}\\ &= \sqrt{\frac{C_2(d)}{c_1(d)}}2^{(l+1)(1+\frac{d}{d-T})\frac{T-1}{T/d-1}\frac{1}{2}}\end{aligned}$$

by Lemma 4.1, where $c_1(d)$ and $C_2(d)$ are constants which only depend on d, and since we have $(-\frac{\alpha+\beta}{2}+\frac{1}{4}(1+\frac{d}{d-T})\frac{T-1}{T/d-1})<0$ by property (4.3), we obtain

$$\begin{aligned}\sum_{l=0}^{\infty}2^{(l+1)(\frac{\kappa}{2}-(\alpha+\beta))}\sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}} &\leq \sqrt{\frac{C_2(d)}{c_1(d)}}\sum_{l=0}^{\infty}2^{(l+1)(-\frac{\alpha+\beta}{2}+\frac{1}{4}(1+\frac{d}{d-T})\frac{T-1}{T/d-1})}\\ &= \sqrt{\frac{C_2(d)}{c_1(d)}}\frac{2^{-\frac{\alpha+\beta}{2}+\frac{1}{4}(1+\frac{d}{d-T})\frac{T-1}{T/d-1}}}{1-2^{-\frac{\alpha+\beta}{2}+\frac{1}{4}(1+\frac{d}{d-T})\frac{T-1}{T/d-1}}}=:\widetilde{C}(d,\alpha,\beta).\end{aligned}$$

- Case $T=0$, i.e., $\beta>1$ and $\alpha=0$. Due to

$$\begin{aligned}\sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}} &\leq \sqrt{\frac{C_3(d)}{c_3(d)}}\sqrt{\frac{2^d\,N^2\,2^{2(l+1)}\left(\log(2^d\,N^2\,2^{2(l+1)})\right)^{d-1}}{2^d\,N^2\left(\log(2^d\,N^2)\right)^{d-1}}}\\ &= \sqrt{\frac{C_3(d)}{c_3(d)}}\,2^{l+1}\left(\frac{\log(2^d\,N^2)+\log(2^{2(l+1)})}{\log(2^d\,N^2)}\right)^{\frac{d-1}{2}}\\ &\leq \sqrt{\frac{C_3(d)}{c_3(d)}}\,2^{l+1}\left(2\log(2^{2(l+1)})\right)^{\frac{d-1}{2}}=\sqrt{\frac{C_3(d)}{c_3(d)}}\,(2\log 2)^{\frac{d-1}{2}}\,2^{l+1}\,(2l+2)^{\frac{d-1}{2}}\end{aligned}$$

by Lemma 4.1, where $c_3(d)$ and $C_3(d)$ are constants which only depend on d, we have

$$\sum_{l=0}^{\infty}2^{(l+1)(\frac{\kappa}{2}-(\alpha+\beta))}\sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}}\leq\sqrt{\frac{C_3(d)}{c_3(d)}}\,(2\log 2)^{\frac{d-1}{2}}\sum_{l=0}^{\infty}\frac{(2l+2)^{\frac{d-1}{2}}}{2^{(l+1)\frac{\beta-1}{2}}}.$$

Since $\beta > 1$, the term $\sum_{l=0}^{\infty} \frac{(2l+2)^{\frac{d-1}{2}}}{2^{(l+1)\frac{\beta-1}{2}}} < \infty$ and we are going to estimate this sum. The function $g : [0,\infty) \to \mathbb{R}$, $g(l) := \frac{(2l+2)^{\frac{d-1}{2}}}{2^{(l+1)\frac{\beta-1}{2}}}$, has its only maximum at

$$l_{\max} := \max(0, \frac{d-1}{(\beta-1)\log_{\mathrm{e}} 2} - 1)$$

and we estimate

$$\begin{aligned}
&\sum_{l=0}^{\infty} \frac{(2l+2)^{\frac{d-1}{2}}}{2^{(l+1)\frac{\beta-1}{2}}} = \sum_{l=0}^{\infty} g(l) \leq \sum_{l=0}^{\lfloor l_{\max}\rfloor} g(l) + \sum_{l=\lceil l_{\max}\rceil}^{\infty} g(l) \\
\leq\ & g(l_{\max}) + \int_0^{\lfloor l_{\max}\rfloor} g(l)\mathrm{d}l + g(l_{\max}) + \int_{\lceil l_{\max}\rceil}^{\infty} g(l)\mathrm{d}l \leq 2\, g(l_{\max}) + \int_0^{\infty} g(l)\mathrm{d}l \\
\leq\ & 2\max\left(\frac{2^{\frac{d-1}{2}}}{2^{\frac{\beta-1}{2}}}, \left(\frac{2(d-1)}{(\beta-1)\,\mathrm{e}\log_{\mathrm{e}} 2}\right)^{\frac{d-1}{2}} \right) + \frac{(d-1)\,(\frac{4}{(\beta-1)\log_{\mathrm{e}} 2})^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2}) + 2^{\frac{d+2-\beta}{2}}}{(\beta-1)\log_{\mathrm{e}} 2} \\
=:\ & \widetilde{C}(d,0,\beta).
\end{aligned}$$

These estimates yield

$$\|S_{I_N} f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\| \leq \underbrace{\sqrt{2\,(2 + (1-2^{-\kappa})\, C_{\mathrm{A}}(d,T))}\, \widetilde{C}(d,\alpha,\beta)\, 2^{\alpha+\beta}}_{:=C(d,\alpha,\beta)}\ N^{-(\alpha+\beta)}\ \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|.$$

■

4.2.2 Cases $0 < T < 1$

Lemma 4.8. *Let the dimensionality $d \in \mathbb{N}$, $d \geq 2$, a parameter T, $0 < T < 1$, a parameter $\kappa > 0$, and a number $M \in \mathbb{N}$, $M > \frac{d\,|\mathcal{D}(I_N^{d,T})|}{1-2^{-\kappa}} + 1$ be given. Then, we have*

$$|\mathcal{D}(I_{N2^{l+1}}^{d,T}) \cap M\mathbb{Z}^d| \leq C_{\mathrm{A}}(d,T)\, \frac{|\mathcal{D}(I_{N2^{l+1}}^{d,T})|}{M} (l+1)^{d-1} + 1$$

for all refinements $N \in \mathbb{R}$, $N \geq 1$, and levels $l \in \mathbb{N} \cup \{0\}$, where $C_{\mathrm{A}}(d,T) \geq 1$ is a constant which only depends on d and T.

Proof. For $0 \leq T < 1$, we denote

$$\mathcal{A}_{N,t}^{d,T} := \{\boldsymbol{m} \in \mathcal{D}(I_N^{d,T}) \cap M\mathbb{Z}^d \colon \text{exactly } t \text{ components of } \boldsymbol{m} \text{ are non-zero}\}, \quad t = 0,\ldots,d.$$

Then, we have $\mathcal{D}(I_N^{d,T}) \cap M\mathbb{Z}^d = \bigcup_{t=0}^d \mathcal{A}_{N,t}^{d,T}$ and $|\mathcal{D}(I_N^{d,T}) \cap M\mathbb{Z}^d| = \sum_{t=0}^d |\mathcal{A}_{N,t}^{d,T}|$. Next, we estimate $|\mathcal{A}_{N,t}^{d,T}|$ for $t = 0,\ldots,d$.

- Case $t = 0$. Obviously, we have $\mathcal{A}_{N2^{l+1},0}^{d,T} = \{\mathbf{0}\}$ and $|\mathcal{A}_{N2^{l+1},0}^{d,T}| = 1$.

- Case $t=1$. $|\mathcal{A}^{d,T}_{N2^{l+1},1}| \leq d\,\frac{2N2^{l+1}}{M} < 2d\frac{|I^{d,T}_{N2^{l+1}}|}{M} < 2d|\mathcal{D}(I^{d,T}_{N2^{l+1}})|/M$.
- Case $2 \leq t \leq d$. Due to [14, Lemma 2.3] with $\widetilde{T} := 0$, we have $I^{d,T}_N \subset I^{d,0}_{d^{\frac{T}{1-T}}N}$, $N \in \mathbb{R}$, $N \geq 1$, and consequently, we infer

$$\left(\mathcal{D}\left(I^{d,T}_{N2^{l+1}}\right) \cap M\mathbb{Z}^d\right) \subset \left(\mathcal{D}\left(I^{d,0}_{d^{\frac{T}{1-T}}N2^{l+1}}\right) \cap M\mathbb{Z}^d\right)$$

as well as

$$\mathcal{A}^{d,T}_{N2^{l+1},t} \subset \mathcal{A}^{d,0}_{d^{\frac{T}{1-T}}N2^{l+1},t} \subset A^{d,0}_{2^d(d^{\frac{T}{1-T}}N2^{l+1})^2,t} = A^{d,0}_{2^d d^{\frac{2T}{1-T}}N^2\,2^{2(l+1)},t},$$

where $A^{d,0}_{N,t} := \left\{\boldsymbol{m} \in I^{d,0}_N \cap M\mathbb{Z}^d\colon \text{exactly } t \text{ components of } \boldsymbol{m} \text{ are non-zero}\right\}$.
From the proof of Lemma 4.5 and since $|\mathcal{D}(I^{d,T}_N)| \geq (2N+1)^2 > N^2$, we obtain

$$\begin{aligned}
&\left|\mathcal{A}^{d,T}_{N2^{l+1},t}\right| \leq \left|A^{d,0}_{2^d d^{\frac{2T}{1-T}}N^2\,2^{2(l+1)},t}\right| \\
\leq\;& C_3(d)\binom{d}{t}\frac{2^d d^{\frac{2T}{1-T}}N^2\,2^{(l+1)2}}{M^t}\log_2^{t-1}\left(\frac{2^d d^{\frac{2T}{1-T}}N^2\,2^{(l+1)2}}{M^t}\right) \\
\leq\;& C_3(d)\binom{d}{t}2^d d^{\frac{2T}{1-T}}\frac{|\mathcal{D}(I^{d,T}_{N2^{l+1}})|}{M\,M^{t-1}}\log_2^{t-1}\left(2^d d^{\frac{2T}{1-T}}\,\frac{2^{(l+1)2}}{M^{t-1}}\right) \\
\leq\;& C_3(d)\binom{d}{t}2^d d^{\frac{2T}{1-T}}\frac{|\mathcal{D}(I^{d,T}_{N2^{l+1}})|}{M\,M^{t-1}}\left(\log_2\left(2^d d^{\frac{2T}{1-T}}\right)+\log_2 2^{(l+1)2}\right)^{t-1} \\
\leq\;& C_3(d)\binom{d}{t}2^{d+t-1}d^{\frac{2T}{1-T}}\;\log_2^{t-1}\left(2^d d^{\frac{2T}{1-T}}\right)\;\frac{|\mathcal{D}(I^{d,T}_{N2^{l+1}})|}{M}\left(\frac{2(l+1)}{M}\right)^{t-1} \\
\leq\;& C_3(d)\,2^{2d-1}d^{\frac{2T}{1-T}}\log_2^{d-1}\left(2^d d^{\frac{2T}{1-T}}\right)\;\frac{|\mathcal{D}(I^{d,T}_{N2^{l+1}})|}{M}\,(l+1)^{d-1}\;\binom{d}{t}.
\end{aligned}$$

Consequently, this yields

$$|\mathcal{D}(I^{d,T}_{N2^{l+1}}) \cap M\mathbb{Z}^d| \leq C_3(d)\,2^{2d-1}d^{\frac{2T}{1-T}}\log_2^{d-1}\left(2^d d^{\frac{2T}{1-T}}\right)\;\frac{|\mathcal{D}(I^{d,T}_{N2^{l+1}})|}{M}(l+1)^{d-1}+1.$$

■

Lemma 4.9. *Let the dimensionality $d \in \mathbb{N}$, $d \geq 2$ and a function $f \in \mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$, where $0 > \alpha > \frac{1}{2}-\beta$. Then, the function f has an absolutely converging Fourier series,*

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d}|\hat{f}_{\boldsymbol{k}}| < \infty.$$

Proof. As in the proof of Lemma 4.6, we apply the Cauchy-Schwarz inequality and obtain

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d}|\hat{f}_{\boldsymbol{k}}| \;\leq\; \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\frac{1}{\max(1,\|\boldsymbol{k}\|_1)^{2\alpha}\;\prod_{s=1}^d\max(1,|k_s|)^{2\beta}}}\;\|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|.$$

Due to $\max(1,\|\boldsymbol{k}\|_1) \leq 2^d \prod_{s=1}^d \max(1,|k_s|)$ for $\boldsymbol{k}\in\mathbb{Z}^d$, we infer

$$\begin{aligned}\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{f}_{\boldsymbol{k}}| &\leq \sqrt{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} 2^{-d\alpha} \prod_{s=1}^d \frac{1}{\max(1,|k_s|)^{2(\beta+\alpha)}}}\ \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\| \\ &= 2^{-\frac{d\alpha}{2}}\ (1+2\,\zeta\,(2(\alpha+\beta)))^{\frac{d}{2}}\ \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|.\end{aligned}$$

Since we have $2(\alpha+\beta)>1$ and $f\in\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$, we obtain $\sum_{\boldsymbol{k}\in\mathbb{Z}^d}|\hat{f}_{\boldsymbol{k}}|<\infty$. ■

Theorem 4.10. *Let the dimensionality $d\in\mathbb{N}$, $d\geq 2$, a function $f\in\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$ and a refinement $N\in\mathbb{R}$, $N\geq 2$ be given, where $\alpha<0$ and $\beta>1-\alpha$. Additionally, let a prime number $M\in\mathbb{N}$,*

$$M > \frac{d\,|\mathcal{D}(I_N^{d,T})|}{1-2^{-\kappa}} + 1, \tag{4.5}$$

be given, where the parameter $T:=-\alpha/\beta$ and the parameter $\kappa:=\alpha+\beta-1$. Then, there exists a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z},M,I_N^{d,T})$ with generating vector $\boldsymbol{z}:=(1,a,\ldots,a^{d-1})^\top\in\mathbb{Z}^d$ of Korobov form and nodes $\boldsymbol{x}_j:=\frac{j}{M}\boldsymbol{z}$ mod 1, $j=0,\ldots,M-1$, such that the aliasing error is bounded by

$$\|S_{I_N^{d,T}}f - \tilde{S}_{I_N^{d,T}}f|L^2(\mathbb{T}^d)\| \leq C(d,\alpha,\beta)\ N^{-(\alpha+\beta)}\ \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|,$$

where $C(d,\alpha,\beta)>0$ is a constant which only depends on d, α, β.

Proof. We are going to apply Theorem 3.4. Therefore, we set $\omega(\boldsymbol{k}):=\omega^{\alpha,\beta}(\boldsymbol{k})$, $\nu:=\alpha+\beta$, $I_N:=I_N^{d,T}$ and $\mathcal{I}_N:=\mathcal{D}(I_N^{d,T})$. From Lemma 4.3, we obtain $\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\subset L^2(\mathbb{T}^d)$. We apply Lemma 4.8 and this yields

$$\begin{aligned}&|\{\boldsymbol{m}\in\mathcal{I}_{N2^l}\colon \exists \boldsymbol{m}'\in\mathbb{Z}^d \text{ such that } \boldsymbol{m}=M\boldsymbol{m}'\}| = |\mathcal{D}(I_{N2^l}^{d,T})\cap M\mathbb{Z}^d| \\ \leq\ & C_{\mathrm{A}}(d,T)\frac{|\mathcal{D}(I_{N2^l}^{d,T})|}{M} l^{d-1}+1 \quad \text{for all } l\in\mathbb{N}.\end{aligned}$$

Furthermore, we need the property that f has a point-wise convergent Fourier series. Since $\alpha>1-\beta>\frac{1}{2}-\beta$, we can apply Lemma 4.9 and obtain that f has an absolutely converging Fourier series, whereby f has a point-wise convergent Fourier series.
Next, we apply Theorem 3.4 with $\psi(l):=l^d$ and we obtain

$$\begin{aligned}&\|S_{I_N}f-\tilde{S}_{I_N}f|L^2(\mathbb{T}^d)\| \\ \leq\ & 2^{\alpha+\beta}\,N^{-(\alpha+\beta)}\ \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\| \\ &\cdot\sum_{l=0}^{\infty} 2^{(l+1)(\frac{\kappa}{2}-(\alpha+\beta))}\ \sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}}\sqrt{2\,(2+(1-2^{-\kappa})\,C_{\mathrm{A}}(d,T)\,(l+1)^{d-1})}.\end{aligned}$$

Due to $|\mathcal{I}_{N2^{l+1}}|=|\mathcal{D}(I_{N2^{l+1}}^{d,T})|\leq\left(C_4(d)N\,2^{l+1}\right)^2$ and $|\mathcal{I}_N|=|\mathcal{D}(I_N^{d,T})|\geq(2N)^2>N^2$, where

$C_4(d)$ is a constant which only depends on d, we infer $\sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}} \leq C_4(d)\, 2^{l+1}$. Then, we obtain

$$\begin{aligned}
& \sum_{l=0}^{\infty} 2^{(l+1)(\frac{\kappa}{2}-(\alpha+\beta))} \sqrt{\frac{|\mathcal{I}_{N2^{l+1}}|}{|\mathcal{I}_N|}} \sqrt{2\,(2+(1-2^{-\kappa})\,C_{\mathrm{A}}(d,T)\,(l+1)^{d-1})} \\
< \; & \sum_{l=0}^{\infty} 2^{(l+1)(\frac{\kappa}{2}-(\alpha+\beta))}\, C_4(d)\, 2^{l+1} \sqrt{8\,C_{\mathrm{A}}(d,T)\,(l+1)^{d-1}} \\
= \; & C_4(d) \sqrt{8\,C_{\mathrm{A}}(d,T)} \sum_{l=0}^{\infty} 2^{(l+1)(\frac{\alpha+\beta-1}{2}-(\alpha+\beta)+1)}\, (l+1)^{\frac{d-1}{2}} \\
= \; & C_4(d) \sqrt{8\,C_{\mathrm{A}}(d,T)} 2^{-\frac{d-1}{2}} \sum_{l=0}^{\infty} \frac{(2l+2)^{\frac{d-1}{2}}}{2^{(l+1)(\frac{\alpha+\beta-1}{2})}}
\end{aligned}$$

and the term $\sum_{l=0}^{\infty} \frac{(2l+2)^{\frac{d-1}{2}}}{2^{(l+1)(\frac{\alpha+\beta-1}{2})}} < \infty$ since $\alpha+\beta>1$. As in the proof of Theorem 4.7 for the case $T=0$ replacing β by $\alpha+\beta$, we infer

$$\begin{aligned}
& \frac{C_4(d)\sqrt{8\,C_{\mathrm{A}}(d,T)}\, 2^{-\frac{d-1}{2}}}{\sqrt{2\,(2+(1-2^{-\kappa})\,C_{\mathrm{A}}(d,T))}} \sum_{l=0}^{\infty} \frac{(2l+2)^{\frac{d-1}{2}}}{2^{(l+1)\frac{\alpha+\beta-1}{2}}} \\
\leq \; & \frac{C_4(d)\sqrt{8\,C_{\mathrm{A}}(d,T)}\, 2^{-\frac{d-1}{2}}}{\sqrt{2\,(2+(1-2^{-\kappa})\,C_{\mathrm{A}}(d,T))}} 2 \left[\max\left(\frac{2^{\frac{d-1}{2}}}{2^{\frac{\alpha+\beta-1}{2}}}, \left(\frac{2(d-1)}{(\alpha+\beta-1)\,\mathrm{e}\log_{\mathrm{e}} 2}\right)^{\frac{d-1}{2}} \right) \right. \\
& \left. + \frac{(d-1)\,(\frac{4}{(\alpha+\beta-1)\log_{\mathrm{e}} 2})^{\frac{d-1}{2}}\,\Gamma(\frac{d-1}{2}) + 2^{\frac{d+2-\alpha+\beta}{2}}}{(\alpha+\beta-1)\log_{\mathrm{e}} 2} \right] \\
=: \; & \widetilde{C}(d,\alpha,\beta).
\end{aligned}$$

These estimates yield

$$\|S_{I_N} f - \tilde{S}_{I_N} f | L^2(\mathbb{T}^d)\| \leq \underbrace{\sqrt{2\,(2+(1-2^{-\kappa})\,C_{\mathrm{A}}(d,T))}\, \widetilde{C}(d,\alpha,\beta)\, 2^{\alpha+\beta}}_{:=C(d,\alpha,\beta)} \; N^{-(\alpha+\beta)}\; \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|.$$

■

4.3 Comparison with previous results

In [14], the truncation error $\|f - S_{I_N^{d,T}} f | \mathcal{H}^{r,t}(\mathbb{T}^d)\|$ and aliasing error $\|S_{I_N^{d,T}} f - \tilde{S}_{I_N^{d,T}} f | \mathcal{H}^{r,t}(\mathbb{T}^d)\|$ were considered for arbitrarily chosen reconstructing rank-1 lattices $\Lambda(\boldsymbol{z}, M, I_N^{d,T})$ and functions $f \in \mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)$, where $r,t \in \mathbb{R}$, $t \geq 0$, $r > -t$, $\beta \geq 0$, $\alpha > -\beta$, $r+t < \alpha+\beta$, $\lambda > 1/2$, and $T := -\frac{\alpha-r}{\beta-t}$. The truncation error was estimated by

$$\|f - S_{I_N^{d,T}} f | \mathcal{H}^{r,t}(\mathbb{T}^d)\| \leq N^{-(\alpha-r+\beta-t)}\; \|f|\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)\|$$

in [14, Lemma 3.5] and for functions f with point-wise convergent Fourier series, the aliasing error was estimated by

$$\|S_{I_N^{d,T}} f - \tilde{S}_{I_N^{d,T}} f | \mathcal{H}^{r,t}(\mathbb{T}^d)\| \leq (1+2\zeta(2\lambda))^{\frac{d}{2}}\; N^{-(\alpha-r+\beta-t)}\; \|f|\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)\| \qquad (4.6)$$

in [14], which yields

$$\|f - \tilde{S}_{I_N^{d,T}} f|\mathcal{H}^{r,t}(\mathbb{T}^d)\| \leq \left(1 + (1+2\zeta(2\lambda))^{\frac{d}{2}}\right) N^{-(\alpha-r+\beta-t)} \, \|f|\mathcal{H}^{\alpha,\beta+\lambda}(\mathbb{T}^d)\| \tag{4.7}$$

for the approximation error, cf. [14, Corollary 3.8]. We remark that a constructive method for obtaining a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I)$ for a given frequency $I \subset \mathbb{Z}^d$ of finite cardinality is described in [10]. In the present paper, we were able to improve the estimates (4.6) and (4.7). We showed that there exists a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N^{d,T})$ with generating vector $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top$ of Korobov form such that we do not have the dependence on λ for the special cases $r = t = 0$, $\alpha + \beta > (1 + \frac{d}{d-(T)_-})\frac{(T)_- -1}{(T)_-/d-1}\frac{1}{2}$, where $(T)_- := \min(0, T)$, see Theorem 4.7 and 4.10. However, we do not know a constructive method for obtaining such a reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N^{d,T})$.

In [7], functions from the spaces of generalized mixed Sobolev smoothness

$$\mathcal{H}^{t,r}_{\mathrm{mix}}(\mathbb{T}^d) := \left\{ f \colon \sqrt{\sum\nolimits_{\boldsymbol{k}\in\mathbb{Z}^d} \prod\nolimits_{s=1}^d (1+|k_s|)^{2t}(1+\|\boldsymbol{k}|_\infty)^{2r} |\hat{f}_{\boldsymbol{k}}|^2} < \infty \right\}.$$

and generalized hyperbolic cross frequency index sets $I = \Gamma_N^T := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \prod_{s=1}^d (1+|k_d|) \cdot (1+\|\boldsymbol{k}\|_\infty)^{-T} \leq N^{1-T}\}$ were considered. As sampling nodes $\boldsymbol{x}_j$, the nodes of a (generalized) sparse grid with size $M = |\Gamma_N^T|$ were used. We remark that the inclusions $\mathcal{I}^{d,T}_{(N+1)2^{(T-d)/(1-T)}} \subset \Gamma_N^T \subset \mathcal{I}^{d,T}_{(N+1)d^{-T/(1-T)}}$ are valid in the cases $-\infty \leq T \leq 0$ and $\mathcal{I}^{d,T}_{(N+1)d^{-T/(1-T)}2^{-d/(1-T)}} \subset \Gamma_N^T \subset \mathcal{I}^{d,T}_{(N+1)2^{T/(1-T)}}$ in the cases $0 < T < 1$ for $d \in \mathbb{N}$ and arbitrary refinement $N \in \mathbb{R}$, $N \geq 1$, cf. [14, Lemma 2.5]. Furthermore, we obtain from the proof of [14, Lemma 2.5] that $c(d,r,t)\|f|\mathcal{H}^{r,t}(\mathbb{T}^d)\| \leq \|f|\mathcal{H}^{t,r}_{\mathrm{mix}}(\mathbb{T}^d)\| \leq C(d,r,t)\|f|\mathcal{H}^{r,t}(\mathbb{T}^d)\|$, where

$$c(d,r,t) := \begin{cases} d^{-r} & \text{for } r \geq 0, t \geq 0, \\ 2^r & \text{for } 0 > r > -t, t > 0, \end{cases} \qquad C(d,r,t) := \begin{cases} 2^r 2^{dt} & \text{for } r \geq 0, t \geq 0, \\ d^{-r} 2^{dt} & \text{for } 0 > r > -t, t > 0. \end{cases}$$

For the approximation error (and the aliasing error), it was shown, cf. [7, Lemma 8], that

$$\|f - \mathcal{L}_{\Gamma_N^T} f|\mathcal{H}^{0,r}_{\mathrm{mix}}(\mathbb{T}^d)\| \lesssim N^{-(t-r)} \, (\log N)^{d-1} \, \|f|\mathcal{H}^{t,0}_{\mathrm{mix}}(\mathbb{T}^d)\|,$$

where $\mathcal{L}_{\Gamma_N^T}$ is the interpolation operator on the (generalized) sparse grid, $0 \leq r < t$, $t > \frac{1}{2}$, $f \in \mathcal{H}^{t,0}_{\mathrm{mix}}(\mathbb{T}^d)$ and $T := \frac{r}{t}$. In particular in the case $r = 0$, the frequency index sets Γ_N^0 are hyperbolic crosses and the above estimate yields

$$\|f - \mathcal{L}_{\Gamma_N^0} f|L^2(\mathbb{T}^d)\| \lesssim N^{-t} \, (\log N)^{d-1} \, \|f|\mathcal{H}^{t,0}_{\mathrm{mix}}(\mathbb{T}^d)\|,$$

i.e., there is an additional factor of $(\log N)^{d-1}$ compared to [24, Theorem 2] and (1.4). Similarly in [28, 21], where the case $r = 0$ and sparse grids sampling nodes were considered, it was proven that the approximation error

$$\|f - \mathcal{L}_{\Gamma_N^0} f|L^2(\mathbb{T}^d)\| \leq C(d) \, N^{-\beta} \, (\log N)^{\frac{d-1}{2}} \, \|f|\mathcal{H}^{0,\beta}(\mathbb{T}^d)\|,$$

where $C(d) > 0$ is a constant which only depends on d, see [21, Theorem 1]. This means, there is an additional factor of $(\log N)^{\frac{d-1}{2}}$ compared to [24, Theorem 2] and (1.4). However, the sampling schemes in [7, 28, 21] only use $M = |I| = \Theta(N \log^{d-1} N)$ many samples, whereas we require $M = \Theta(N^2 \log^{d-1} N)$ many samples, see (1.3). The advantage of our approach is that the computation of the approximated Fourier coefficients $\tilde{\hat{f}}_{\boldsymbol{k}}$, $\boldsymbol{k} \in I$, using the sampling method (1.1) is numerically perfectly stable whereas the computation using the sampling schemes from [7, 28, 21] may be numerically unstable, cf. [12].

5 Numerical results

In practice, we do not know a method for verifying if a generating vector $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top \in \mathbb{Z}^d$ of Korobov form fulfills property (2.8) in Lemma 2.1 for a given reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I)$. Furthermore, we also do not know how to construct a generating vector $\boldsymbol{z}$ fulfilling property (2.8). However, this special property is crucial for obtaining the estimate (1.4) by Theorem 4.7 and Theorem 4.10. Consequently, we have only the upper bounds from Section 4.3 available. Nevertheless, numerical tests performed in [14, Section 6], which use reconstructing rank-1 lattices $\Lambda(\boldsymbol{z}, M, I)$ obtained from a constructive method described in [10], showed that the approximation error $\|f - \tilde{S}_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ is in $\mathcal{O}(N^{-\beta})\ \|f|\mathcal{H}^{0,\beta}(\mathbb{T}^d)\|$ for the functions considered there, which is of optimal order. This suggests that the aliasing error can also be

$$\|S_{I_N^{d,0}} f - \tilde{S}_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\| \lesssim N^{-\beta}\ \|f|\mathcal{H}^{0,\beta}(\mathbb{T}^d)\|$$

for reconstructing rank-1 lattices $\Lambda(\boldsymbol{z}, M, I)$ with generating vectors $\boldsymbol{z}$ which are not necessarily of Korobov form.

Here, we investigate the approximation error more closely and consider the truncation error and the aliasing error. As in [7] and in [14, Example 6.1], we consider the function

$$f(\boldsymbol{x}) = \prod_{s=1}^{d} \frac{8\sqrt{6}\sqrt{\pi}}{\sqrt{6369\pi - 4096}} \left[4 + \operatorname{sgn}(x_s - \frac{1}{2}) \left(\sin(2\pi x_s)^3 + \sin(2\pi x_s)^4\right)\right], \tag{5.1}$$

where $\|f|L^2(\mathbb{T}^d)\| = 1$, $f \in \mathcal{H}^{0,\frac{7}{2}-\epsilon}(\mathbb{T}^d)$, $\epsilon > 0$, $f \notin \mathcal{H}^{0,\frac{7}{2}}(\mathbb{T}^d)$, and the Fourier coefficients

$$\hat{f}_{\boldsymbol{k}} = \prod_{s=1}^{d} \frac{8\sqrt{6}\sqrt{\pi}}{\sqrt{6369\pi - 4096}} \begin{cases} \frac{-12}{(k_s-3)(k_s-1)(k_s+1)(k_s+3)\pi} & \text{for } k_s \in 2\mathbb{Z} \setminus \{0\}, \\ \frac{48\mathrm{i}}{(k_s-4)(k_s-2)k_s(k_s+2)(k_s+4)\pi} & \text{for } k_s \text{ odd}, \\ 4 - \frac{4}{3\pi} & \text{for } k_s = 0. \end{cases}$$

As frequency index sets I, we use the symmetric hyperbolic cross index sets $I = I_N^{d,0}$ with different refinements N and as sampling nodes $\boldsymbol{x}_j$, we use the nodes of the reconstructing rank-1 lattices $\Lambda(\boldsymbol{z}, M, I_N^{d,0})$ from [14, Table 6.2] for each index set $I_N^{d,0}$. We remark that these reconstructing rank-1 lattices do not fulfill the requirements (2.7) and (2.8) of Lemma (2.1). Nevertheless, we observe for the lattices from [14, Table 6.2] that the truncation errors dominate the aliasing errors, i.e., $\|S_{I_N^{d,0}} f - \tilde{S}_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\| \leq \|f - S_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$. Plots of the $L^2(\mathbb{T}^d)$ approximation error $\|f - \tilde{S}_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ are depicted in [14, Figure 6.1a and 6.2a], where it was observed that the approximation error decreases like $\sim N^{-3.45}$ in the one-dimensional case and slightly slower in the multi-dimensional cases. In Figure 5.1 truncation errors $\|f - S_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ and aliasing errors $\|S_{I_N^{d,0}} f - \tilde{S}_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ are shown for the cases $d = 2, \ldots, 10$. Here, we observe that the aliasing errors $\|S_{I_N^{d,0}} f - \tilde{S}_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ are smaller than the truncation error $\|f - S_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ and that the aliasing errors $\|S_{I_N^{d,0}} f - \tilde{S}_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ also decrease faster. We stress the fact that the truncation errors only depend on the frequency index set $I_N^{d,0}$ and do not depend on the sampling sets. The truncation errors are asymptotically of optimal order. Consequently, the aliasing errors are also of optimal order — assuming that the trends of the plots is maintained.

In Table 5.1, oversampling factors $M/|I_N^{d,0}|$, i.e., ratios of the rank-1 lattice sizes M and the cardinalities of the symmetric hyperbolic cross index sets $I_N^{d,0}$, are shown. These oversampling factors are less than 100 and still moderate compared to the asymptotic statement $\mathcal{O}(N)$ in (1.3) and (1.2), which is $M \sim |I_N^{d,0}|^2 / \log^{d-1} N$. However, we observe that these oversampling factors $M/|I_N^{d,0}|$ grow for increasing refinements N and fixed dimensionality d.

6 Conclusion

In this paper, we generalized the ideas from [24] in order to improve the estimates for the aliasing error $\|S_{I_N^{d,T}} f - \tilde{S}_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\|$ from [14] for functions f from the Hilbert spaces $\mathcal{H}^{\alpha,\beta}(\mathbb{T}^d)$ of isotropic and dominating mixed smoothness when using the lattice rule (1.1). We proved the existence of special reconstructing rank-1 lattices $\Lambda(\boldsymbol{z}, M, I_N^{d,T})$ with generating vectors $\boldsymbol{z} := (1, a, \ldots, a^{d-1})^\top \in \mathbb{Z}^d$ of Korobov form which yield that the order of the aliasing error $\|S_{I_N^{d,T}} f - \tilde{S}_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\|$ is bounded by the order of the truncation error $\|f - S_{I_N^{d,T}} f | L^2(\mathbb{T}^d)\|$.

The central statement of this paper is Theorem 3.4, which is a generalisation of the ideas of V. N. Temlyakov, see [24]. We stress the fact that our theorem is quite general and applicable to a wide range of frequency index sets I_N. In order to apply Theorem 3.4 to a given sequence of frequency index sets I_N, $N \in \mathbb{R}$, $N \geq 1$, we need to choose a nested sequence of index sets $\mathcal{I}_N$, see (3.2), such that the inclusion $\mathcal{I}_N \supset \mathcal{D}(I_N)$ is valid, where $\mathcal{D}(I_N)$ is the difference set of I_N, cf. Section 2.1. Thereby, $\mathcal{I}_N$ has to fulfill the following properties:

- The cardinalities $|\mathcal{I}_N|$ should be close to the cardinalities $|\mathcal{D}(I_N)|$. This is crucial for a small size M of the reconstructing rank-1 lattice $\Lambda(\boldsymbol{z}, M, I_N)$ used as sampling set, see (3.3).
- The upper and lower bound of the cardinalities $|\mathcal{I}_N|$ need to be known and should be almost of the same order, e.g., gaps of logarithmic order between the upper and lower bound are managable as demonstrated in Section 4.2.2.

Then, the strategy to bound the aliasing error is analog to the approach in Section 4.2. We remark that we dealt with the difference sets themselves in Section 4.2.2 and set $\mathcal{I}_N := \mathcal{D}(I_N^{d,T})$, whereas we covered the difference sets $\mathcal{D}(I_N^{d,T})$ with larger index sets $\mathcal{I}_N := I^{d,T}_{2^{\frac{d-T}{1-T}} N^{1+\frac{d}{d-T}}}$ in Section 4.2.1.

Acknowledgements

DP acknowledges the fruitful discussions and contributions during the Oberwolfach Workshop "Uniform Distribution Theory and Applications". Especially we thank V. N. Temlyakov for pointing out the valuable paper [24]. We gratefully acknowledge support by the German Research Foundation (DFG) within the Priority Program 1324, project PO 711/10-2.

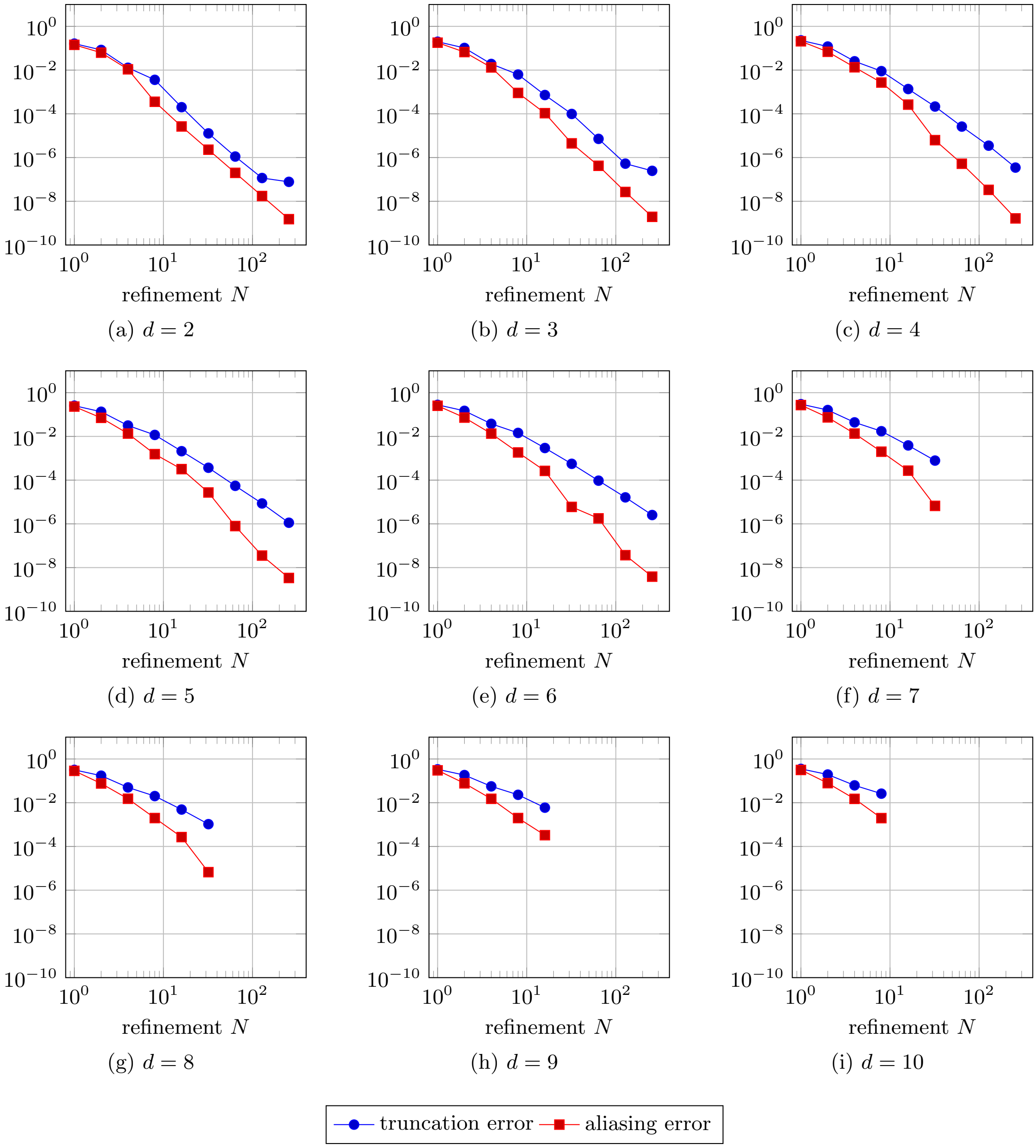


Figure 5.1: Truncation errors $\|f - S_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ and aliasing errors $\|S_{I_N^{d,0}} f - \tilde{S}_{I_N^{d,0}} f | L^2(\mathbb{T}^d)\|$ of the function f from (5.1) as a function of the refinement N.

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d	N	$\lvert I_N^{d,0}\rvert$	M	$\frac{M}{\lvert I_N^{d,0}\rvert}$	$\lVert f - \tilde{S}_{I_N^{d,0}} f \vert L^2(\mathbb{T}^d)\rVert$
2	64	1 377	8 451	6.1	1.1e-06
2	128	3 093	33 283	10.8	1.0e-07
2	256	6 889	132 099	19.2	9.0e-09
3	64	10 113	47 463	4.7	7.2e-06
3	128	24 869	176 603	7.1	5.0e-07
3	256	60 217	753 249	12.5	4.8e-08
4	64	61 889	475 829	7.7	2.6e-05
4	128	164 137	2 244 100	13.7	3.5e-06
4	256	426 193	10 561 497	24.8	2.5e-07
5	64	338 305	3 752 318	11.1	5.5e-05
5	128	958 345	20 645 268	21.5	8.6e-06
5	256	2 644 977	136 178 715	51.5	9.8e-07
6	64	1 709 857	31 829 977	18.6	9.5e-05
6	128	5 137 789	192 757 285	37.5	1.6e-05
6	256	14 977 209	1 400 567 254	93.5	2.3e-06
7	8	198 369	1 059 754	5.3	1.7e-02
7	16	716 985	7 798 320	10.9	3.9e-03
7	32	2 465 613	57 114 640	23.2	7.8e-04
8	8	768 609	6 027 975	7.8	2.0e-02
8	16	2 935 521	49 768 670	17.0	4.9e-03
8	32	10 665 297	359 896 131	33.7	1.0e-03
9	4	688 905	6 898 038	10.0	5.8e-02
9	8	2 910 897	34 112 281	11.7	2.3e-02
9	16	11 693 889	320 144 128	27.4	6.0e-03
10	2	452 709	4 315 343	9.5	2.1e-01
10	4	2 421 009	30 780 958	12.7	6.4e-02
10	8	10 819 089	194 144 634	17.9	2.6e-02

Table 5.1: Cardinalities $|I_N^{d,0}|$, rank-1 lattice sizes M, oversampling factors $\frac{M}{|I_N^{d,0}|}$ and approximation errors $\|f - \tilde{S}_{I_N^{d,0}} f|L^2(\mathbb{T}^d)\|$ of the function f from (5.1) for various values of d and N.

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