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„Extraktion quantifizierbarer Information aus komplexen Systemen"

On Embeddings of Weighted Tensor Product Hilbert Spaces

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ON EMBEDDINGS OF WEIGHTED TENSOR PRODUCT HILBERT SPACES

MARIO HEFTER AND KLAUS RITTER

Abstract. We study embeddings between tensor products of weighted reproducing kernel Hilbert spaces. The setting is based on a sequence of weights $\gamma_j > 0$ and sequences $1+\gamma_j k$ and $1+l_{\gamma_j}$ of reproducing kernels k such that $H(1+\gamma_j k) = H(1+l_{\gamma_j})$, in particular. We derive necessary and sufficient conditions for the norms on $\bigotimes_{j=1}^s H(1+\gamma_j k)$ and $\bigotimes_{j=1}^s H(1+l_{\gamma_j})$ to be equivalent uniformly in s. Furthermore, we study relaxed versions of uniform equivalence by modifying the sequence of weights, e.g., by constant factors, and by analyzing embeddings of the respective spaces. Likewise, we analyze the limiting case $s = \infty$.

1. Introduction

Embedding theorems deal with scales $(F_s^\alpha)_\alpha$ of function spaces on a common domain of dimension $s \in \mathbb{N}$, and one of the aims is to characterize those pairs of spaces F_s^α and F_s^β that permit a continuous embedding $i_s^{\alpha,\beta} : F_s^\alpha \hookrightarrow F_s^\beta$. A major application of embedding theorems in information-based complexity, approximation theory, and numerical mathematics is as follows: The existence of a continuous embedding $i_s^{\alpha,\beta}$ with norm $\|i_s^{\alpha,\beta}\|$ implies

$$e_n(F_s^\alpha) \leq \|i_s^{\alpha,\beta}\| \cdot e_n(F_s^\beta) \tag{1}$$

for many quantities e_n of interest, like n-th minimal errors or n-widths.

In the classical approach one studies the asymptotic behavior of $e_n(F_s^\alpha)$ as n tends to infinity with α and s being fixed, and the mere existence of continuous embeddings can already be exploited, since (1) yields $e_n(F_s^\alpha) = O\left(e_n(F_s^\beta)\right)$. In particular, if $F_s^\alpha = F_s^\beta$ as vector spaces with equivalent norms, then the sequences $(e_n(F_s^\alpha))_n$ and $(e_n(F_s^\beta))_n$ are weakly equivalent.

In contrast, tractability analysis studies the explicit dependence of $e_n(F_s^\alpha)$ on n and on the dimension s, which is crucial to fully understand the impact of a high dimension on the computational or approximation problem at hand. We refer to [7, 8, 9] for a comprehensive study and further references. Moreover, tractability analysis enables the study of the limiting case $s = \infty$, i.e., of computational or approximation problems for functions with infinitely many variables. Exploiting the existence of continuous embeddings $i_s^{\alpha,\beta}$ or the equivalence of norms on $F_s^\alpha = F_s^\beta$ for all $s \in \mathbb{N}$ in tractability analysis requires a tight control of the dependence of the norms of the respective embeddings on the dimension s.

In this paper we study scales of weighted tensor product Hilbert spaces, which are most often studied in tractability analysis. The starting point for the construction of these spaces is a reproducing kernel k on a domain $D \times D$ and a sequence $(\gamma_j)_{j\in\mathbb{N}}$ of positive weights. By assumption, the Hilbert space $H(1+k)$ with reproducing kernel $1+k$ is the orthogonal sum of the space $H(1)$ of constant functions and the space $H(k)$.

The corresponding norm of $f \in H(1+\gamma_j k)$ is therefore given by

$$\|f\|_{1+\gamma_j k}^2 = P(f)^2 + \frac{1}{\gamma_j} \cdot \|f - P(f)\|_k^2,$$

where P denotes the orthogonal projection onto $H(1)$. The second scale is derived from an equivalent norm

$$\|f\|_{1+l_{\gamma_j}}^2 = \langle f, f\rangle + \frac{1}{\gamma_j} \cdot \|f - P(f)\|_k^2$$

on the same vector space $H = H(1+k) = H(1+\gamma_j k)$. By assumption, $\langle \cdot, \cdot \rangle$ is a properly normalized symmetric bilinear form on H that is continuous on $H(k)$, and actually we get a new reproducing kernel Hilbert space $H(1+l_{\gamma_j})$ in this way. As it turns out,

$$\forall\, f \in H : P(f) = \langle f, 1\rangle \tag{2}$$

forms a particular instance, since (2) is equivalent to $H(1)$ and $H(k)$ being orthogonal in the spaces $H(1+l_{\gamma_j})$, too.

The first result of this paper, Theorem 1, deals with embeddings

$$\imath_s^{\boldsymbol{\eta},\boldsymbol{\gamma}} : H(K_s^{\boldsymbol{\eta}}) \hookrightarrow H(L_s^{\boldsymbol{\gamma}})$$

between the tensor product spaces

$$H(K_s^{\boldsymbol{\eta}}) = \bigotimes_{j=1}^{s} H(1+\eta_j k) \quad \text{and} \quad H(L_s^{\boldsymbol{\gamma}}) = \bigotimes_{j=1}^{s} H(1+l_{\gamma_j})$$

of functions on D^s, where $\boldsymbol{\eta} = (\eta_j)_{j\in\mathbb{N}}$ and $\boldsymbol{\gamma} = (\gamma_j)_{j\in\mathbb{N}}$ are arbitrary sequences of positive weights. Hence the reproducing kernels $K_s^{\boldsymbol{\eta}}$ and $L_s^{\boldsymbol{\gamma}}$ are given by

$$K_s^{\boldsymbol{\eta}}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{s}(1+\eta_j k(x_j,y_j)), \qquad \mathbf{x},\mathbf{y} \in D^s,$$

and

$$L_s^{\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{s}(1+l_{\gamma_j}(x_j,y_j)), \qquad \mathbf{x},\mathbf{y} \in D^s.$$

Here we present a particular consequence of Theorem 1 for summable weights, i.e.,

$$\sum_{j\in\mathbb{N}} \gamma_j < \infty. \tag{3}$$

This condition often arises in the context of tractability analysis and was first encountered in [10]. If (2) is satisfied, then (3) is equivalent to

$$\sup_{s\in\mathbb{N}} \max\left(\|\imath_s^{\boldsymbol{\gamma},\boldsymbol{\gamma}}\|, \|(\imath_s^{\boldsymbol{\gamma},\boldsymbol{\gamma}})^{-1}\|\right) < \infty,$$

i.e., we have a uniform equivalence of the norms on the spaces $H(K_s^{\boldsymbol{\gamma}})$ and $H(L_s^{\boldsymbol{\gamma}})$ for $s \in \mathbb{N}$. If (2) is not satisfied, then (3) is equivalent to the existence of $0 < c' < 1 < \tilde{c}$ such that

$$\sup_{s\in\mathbb{N}} \max\left(\|\imath_s^{c'\boldsymbol{\gamma},\boldsymbol{\gamma}}\|, \|(\imath_s^{\tilde{c}\boldsymbol{\gamma},\boldsymbol{\gamma}})^{-1}\|, \|\imath_s^{\boldsymbol{\gamma},\tilde{c}\boldsymbol{\gamma}}\|, \|(\imath_s^{\boldsymbol{\gamma},c'\boldsymbol{\gamma}})^{-1}\|\right) < \infty,$$

i.e., changing the weights γ_j by suitable constant factors leads to uniformly bounded norms of the embeddings corresponding to

$$H(K_s^{c'\boldsymbol{\gamma}}) \subseteq H(L_s^{\boldsymbol{\gamma}}) \subseteq H(K_s^{\tilde{c}\boldsymbol{\gamma}}) \quad \text{and} \quad H(L_s^{c'\boldsymbol{\gamma}}) \subseteq H(K_s^{\boldsymbol{\gamma}}) \subseteq H(L_s^{\tilde{c}\boldsymbol{\gamma}}).$$

For the second result of this paper, Theorem 2, we consider the limit $s \to \infty$ of the reproducing kernels $K_s^{\boldsymbol{\eta}}$ and $L_s^{\boldsymbol{\gamma}}$, namely,

$$K^{\boldsymbol{\eta}}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{\infty}(1+\eta_j k(x_j,y_j)), \qquad \mathbf{x},\mathbf{y} \in \mathfrak{X}^{\boldsymbol{\eta}},$$

where $\mathfrak{X}^{\boldsymbol{\eta}} = \{\mathbf{x} \in D^{\mathbb{N}} : \prod_{j=1}^{\infty}(1+\eta_j k(x_j,x_j)) < \infty\}$, and

$$L^{\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y}) = \prod_{j=1}^{\infty}(1+l_{\gamma_j}(x_j,y_j)), \qquad \mathbf{x},\mathbf{y} \in \mathfrak{Y}^{\boldsymbol{\gamma}},$$

where $\mathfrak{Y}^{\boldsymbol{\gamma}} = \{\mathbf{y} \in D^{\mathbb{N}} : \prod_{j=1}^{\infty}(1+l_{\gamma_j}(y_j,y_j)) < \infty\}$. Here we present a particular consequence of Theorem 2 and Remark 4, which again deals with summable weights. If (2) is satisfied, then (3) is equivalent to

$$H(K^{\boldsymbol{\gamma}}) = H(L^{\boldsymbol{\gamma}}).$$

If (2) is not satisfied, then (3) is equivalent to the existence of $0 < c' < 1 < \tilde{c}$ such that

$$H(K^{c'\boldsymbol{\gamma}}) \subseteq H(L^{\boldsymbol{\gamma}}) \subseteq H(K^{\tilde{c}\boldsymbol{\gamma}}) \quad \text{and} \quad H(L^{c'\boldsymbol{\gamma}}) \subseteq H(K^{\boldsymbol{\gamma}}) \subseteq H(L^{\tilde{c}\boldsymbol{\gamma}}).$$

Due to the closed graph theorem the respective embeddings are continuous, and in the case (2) we have equivalence of the norms on the spaces $H(K^{\boldsymbol{\gamma}})$ and $H(L^{\boldsymbol{\gamma}})$.

We refer to [3] for analytic properties of the spaces $H(K^{\boldsymbol{\eta}})$ and $H(L^{\boldsymbol{\gamma}})$ as well as for more general weighted superpositions of tensor products of reproducing kernel Hilbert spaces. Implications of Theorem 1 for tractability analysis and of Theorem 2 for computational or approximation problems for functions with infinitely many variables will be studied in the forthcoming paper [2].

Two prominent examples from tractability analysis are given by $D = [0,1]$ and

$$k^{(1)}(x,y) = \min(x,y)$$

as well as

$$k^{(2)}(x,y) = 1/2 + (x^2+y^2)/2 - \max(x,y)$$

for $x, y \in D$. We have $H(1+\gamma_j k^{(i)}) = W_2^1([0,1])$ for $i = 1,2$ and

$$\|f\|^2_{1+\gamma_j k^{(1)}} = f^2(0) + \frac{1}{\gamma_j} \cdot \int_0^1 (f')^2(x)\,dx$$

as well as

$$\|f\|^2_{1+\gamma_j k^{(2)}} = \left(\int_0^1 f(x)\,dx\right)^2 + \frac{1}{\gamma_j} \cdot \int_0^1 (f')^2(x)\,dx$$

for $f \in W_2^1([0,1])$, and these settings are called the anchored decomposition and the ANOVA decomposition of the space $W_2^1([0,1])$. An obvious choice of $\langle \cdot, \cdot \rangle$ leads to

$$\|f\|^2_{1+l_{\gamma_j}} = \int_0^1 f^2(x)\,dx + \frac{1}{\gamma_j} \cdot \int_0^1 (f')^2(x)\,dx,$$

which yields another decomposition of the space $W_2^1([0,1])$. See [7, Sec. A.2].

Theorem 1 applied to this example reveals that uniform equivalence of the norms on $\bigotimes_{j=1}^s H(1+\gamma_j k^{(1)})$ and $\bigotimes_{j=1}^s H(1+\gamma_j k^{(2)})$ holds iff $\sum_{j\in\mathbb{N}} \sqrt{\gamma_j} < \infty$, and the latter is also equivalent to uniform equivalence of the norms on $\bigotimes_{j=1}^s H(1+\gamma_j k^{(1)})$ and $\bigotimes_{j=1}^s H(1+l_{\gamma_j})$. However, uniform equivalence of the norms on $\bigotimes_{j=1}^s H(1+\gamma_j k^{(2)})$ and $\bigotimes_{j=1}^s H(1+l_{\gamma_j})$ is equivalent to $\sum_{j\in\mathbb{N}} \gamma_j < \infty$.

This paper is organized as follows. In Section 2 we present the essential assumptions, and in Section 3 we derive basic properties of the spaces $H(1+\gamma_j k)$ and $H(1+l_{\gamma_j})$ of functions of a single variable. Sections 4 and 5 contain the analysis for the spaces $H(K_s^{\boldsymbol{\eta}})$ and $H(L_s^{\boldsymbol{\gamma}})$ of functions of finitely many variables and for the spaces $H(K^{\boldsymbol{\eta}})$ and $H(L^{\boldsymbol{\gamma}})$ of functions of infinitely many variables, respectively.

2. Assumptions

We use basic results from [1] about reproducing kernels M and the corresponding Hilbert spaces $H(M)$ frequently without giving further references. We assume that

(A1) $k \neq 0$ is a reproducing kernel on $D \times D$ with $D \neq \emptyset$,

which satisfies

(A2) $H(1) \cap H(k) = \{0\}$.

Moreover, we assume that

(A3) $\boldsymbol{\gamma} = (\gamma_j)_{j\in\mathbb{N}}$ is a sequence of positive weights.

Finally, we assume that

(A4) $\|\cdot\|$ is a seminorm on $H(1+k)$, induced by a symmetric bilinear form $\langle\cdot,\cdot\rangle$,

which satisfies

(A5) $\|1\| = 1$

as well as

(A6) $\|f\| \leq c \cdot \|f\|_k$ for every $f \in H(k)$ with some constant $c > 0$.

The orthogonal projection from $H(1+k)$ onto $H(1)$ is denoted by P, and we do not distinguish between a function in $H(1)$ and its constant function value. Note that (A2) implies that $H(1)$ and $H(k)$ are orthogonal subspaces of $H(1+k)$. In the following we use H to denote the vector space $H(1+k)$, and for the moment, we merely consider a real number $\gamma > 0$ instead of the sequence $\boldsymbol{\gamma}$. Observe that $H(1+\gamma k) = H$ and

$$\|f\|_{1+\gamma k}^2 = P(f)^2 + \frac{1}{\gamma} \cdot \|f - P(f)\|_k^2$$

holds for all $f \in H$ and $\gamma > 0$.

Example 1. Let ξ denote a bounded linear functional on $H(1+k)$ such that $\xi(1) = 1$. Put

$$\|f\| = |\xi(f)|$$

for $f \in H$ to obtain (A4)–(A6) with c being the norm of ξ, restricted to $H(k)$. Important particular cases are

$$\xi(f) = f(b)$$

for some $b \in D$ and

$$\xi(f) = \int_D f \, d\nu,$$

where ν is a probability measure on (a σ-algebra in) D and $H(1+k) \subseteq L_1(\nu)$ is assumed to hold.

Example 2. Assume that $H(1+k) \subseteq L_2(\nu)$ with ν as before, and put

$$\|f\| = \left(\int_D f^2 \, d\nu \right)^{1/2}$$

for $f \in H(1+k)$. Then the assumptions (A4)–(A6) are satisfied with c being the norm of the embedding of $H(k)$ into $L_2(\nu)$.

3. Functions of a Single Variable

With a slight abuse of notation we define

$$\|f\|_\gamma^2 = \|f\|^2 + \frac{1}{\gamma} \cdot \|f - P(f)\|_k^2$$

for $f \in H$ and $\gamma > 0$. Due to (A4) and (A5), $\|\cdot\|_\gamma$ is a norm on H that is induced by an inner product, and $\|1\|_\gamma = \|1\|_{1+\gamma k} = 1$. Actually, we get a new reproducing kernel Hilbert space in this way, as shown in the following lemma.

Lemma 1. *For every $\gamma > 0$ there exists a uniquely determined reproducing kernel $l_\gamma \neq 0$ on $D \times D$ such that*

$$H(1 + l_\gamma) = H$$

and

$$\|f\|_{1+l_\gamma} = \|f\|_\gamma$$

for $f \in H$. Furthermore,

$$H(1) \cap H(l_\gamma) = \{0\}.$$

Finally, we have the following equivalences of norms. With c according to (A6),

$$(1 + c\sqrt{\gamma} + c^2\gamma)^{-1} \cdot \|f\|_{1+\gamma k}^2 \leq \|f\|_{1+l_\gamma}^2 \leq (1 + c\sqrt{\gamma} + c^2\gamma) \cdot \|f\|_{1+\gamma k}^2, \tag{4}$$

$$(1 + (1 + c^2) \cdot \gamma)^{-1} \cdot \|f\|_{1+k}^2 \leq \|f\|_{1+l_\gamma}^2 \leq (1 + c^2 + 1/\gamma) \cdot \|f\|_{1+k}^2, \tag{5}$$

and

$$(1 + (1 + c^2) \cdot \gamma)^{-1} \cdot \|f\|_{1+(1+c^2)\cdot\gamma k}^2 \leq \|f\|_{1+l_\gamma}^2 \leq (1 + \gamma) \cdot \|f\|_{1+\gamma/(1+c^2)k}^2. \tag{6}$$

Proof. First, we establish the equivalences of the norms. For $\eta_1 > 0$ and $u > 0$ we put

$$\eta_2 = \frac{1+u}{(1+1/u)\, c^2 + 1/\eta_1} > 0.$$

Then

$$\begin{aligned} \|f\|_{\eta_1}^2 &\leq (1+u) \cdot P(f)^2 + (1 + 1/u) \cdot \|f - P(f)\|^2 + \frac{1}{\eta_1} \cdot \|f - P(f)\|_k^2 \\ &\leq (1+u) \cdot \|f\|_{1+\eta_2 k}^2, \end{aligned}$$

and analogously

$$\begin{aligned} \|f\|_{1+\eta_1 k}^2 &\leq (1+u) \cdot \|f\|^2 + (1 + 1/u) \cdot \|f - P(f)\|^2 + \frac{1}{\eta_1} \cdot \|f - P(f)\|_k^2 \\ &\leq (1+u) \cdot \|f\|_{\eta_2}^2. \end{aligned}$$

To derive (4), let $\eta_1 = \gamma$ and

$$u = c^2\gamma/2 + \sqrt{c^2\gamma + c^4\gamma^2/4} \leq c\sqrt{\gamma} + c^2\gamma,$$

which yields $\eta_2 = \gamma$. To derive the right-hand side in (5), let $\eta_1 = \gamma$ and

$$u = \frac{1}{2\gamma} \cdot \left(1 - \gamma + c^2\gamma + \sqrt{4c^2\gamma^2 + (1 - \gamma + c^2\gamma)^2}\right) \leq c^2 + \frac{1}{\gamma},$$

which yields $\eta_2 = 1$. To derive the right-hand side in (6), let $\eta_1 = u = \gamma$, which yields

$$\eta_2 = \frac{\gamma \cdot (1+\gamma)}{(1+\gamma)c^2 + 1} \geq \frac{\gamma}{c^2+1}.$$

For $\eta_1 = 1$ and

$$u = \frac{1}{2} \cdot \left(-1 + \gamma + c^2\gamma + \sqrt{4c^2\gamma + (1 - \gamma - c^2\gamma)^2}\right) \leq (c^2+1)\gamma$$

we get $\eta_2 = \gamma$, and the left-hand side in (5) follows. For $\eta_1 = u = (c^2+1)\gamma$ we get

$$\eta_2 = \frac{(1 + (c^2+1) \cdot \gamma) \cdot (c^2+1) \cdot \gamma}{(1 + (c^2+1) \cdot \gamma) \cdot c^2 + 1} \geq \gamma,$$

and the left-hand side in (6) follows.

Let H be equipped with the norm $\|\cdot\|_\gamma$. Any of the above equivalences of norms implies in particular that this space is a reproducing kernel Hilbert space, too. For the orthogonal complement of $H(1)$ in this space we have

$$H(1)^\perp = \{f \in H : \langle f, 1 \rangle = 0\},$$

and $\langle f, 1 \rangle$ is the orthogonal projection of $f \in H$ onto $H(1)$. Furthermore, if l_γ denotes the reproducing kernel of $H(1)^\perp$, considered as a subspace of H, then $1 + l_\gamma$ is the reproducing kernel of H. □

Typically, we do not refer to the explicit form of the reproducing kernel l_γ according to Lemma 1, and later on, the equivalences (4)–(6) will be used for small values of γ.

Remark 1. Consider a reproducing kernel $l \neq 0$ on $D \times D$ such that $H(1) \cap H(l) = \{0\}$, and let Q denote the orthogonal projection from $H(1+l)$ onto $H(1)$. Then there exists a seminorm $\|\cdot\|$ that fulfills (A4)–(A6) and $1 + l_\gamma = 1 + \gamma l$ for all $\gamma > 0$ iff

$$H(1+l) = H \tag{7}$$

and

$$\forall f \in H : \|f - P(f)\|_k = \|f - Q(f)\|_l. \tag{8}$$

Given (7) and (8), we get

$$\forall f \in H : \|f\| = |Q(f)|. \tag{9}$$

In fact, if (A4)–(A6) are satisfied and $1 + l_\gamma = 1 + \gamma l$ for all $\gamma > 0$, then Lemma 1 implies (7) and

$$\|f\|^2 + \frac{1}{\gamma} \cdot \|f - P(f)\|_k^2 = Q(f)^2 + \frac{1}{\gamma} \cdot \|f - Q(f)\|_l^2$$

for all $f \in H$. This yields (8) and (9). Conversely, suppose that (7) and (8) are satisfied. Then Q defines a bounded linear functional on $H(1+k)$, which follows from the closed graph theorem and (7). Define $\|\cdot\|$ by (9) so that (A4)–(A6) are satisfied according to Example 1. Use (8) and Lemma 1 to verify $1 + l_\gamma = 1 + \gamma l$ for all $\gamma > 0$.

We refer to Lemma 5 and Example 7 for further discussion.

To simplify the presentation we exclude the trivial case $\gamma k = l_\gamma$ by assuming that

(A7) there exists a function $f \in H$ such that $|P(f)| \neq \|f\|$.

Subsequently we distinguish two cases, namely the case

$$\exists\, f \in H : P(f) \neq \langle f, 1 \rangle \tag{10}$$

and its complement

$$\forall\, f \in H : P(f) = \langle f, 1 \rangle. \tag{11}$$

The latter is equivalent to $H(1)$ and $H(k)$ being orthogonal in the space $H(1+l_\gamma)$, too, so that the only difference between $H(1+\gamma k)$ and $H(1+l_\gamma)$ is the definition of the norm on the vector space $H(k)$. It turns out that (11) is the stronger assumption for our purposes, see Theorems 1 and 2 below.

Example 3. In the situation of Example 1 we have (A7) iff the functional ξ is different from the projection P, and the latter is equivalent to (10).

In the situation of Example 2 we have (A7) iff there exists a function in $H(k)$ that does not vanish ν-a.s. Moreover, we have (10), unless the integral w.r.t. ν is equal to the projection P.

Example 4. We consider two classical Sobolev spaces of once differentiable functions, which are equipped with different norms in an analogous way. Either, let $D = [0, 1]$ as well as

$$k(x, y) = \min(x, y), \qquad x, y \in D,$$

or let $D = \mathbb{R}$ as well as

$$k(x, y) = k(-x, -y) = \int_0^{\min(x,y)} 1/\varphi(t)\, dt, \qquad x, y > 0,$$

and $k(x, y) = 0$ if $x \cdot y \leq 0$, where φ denotes the density of the standard normal distribution. In the first case, let ν denote the uniform distribution on D, while ν denotes the standard normal distribution in the second case. The study of the first case is motivated by the fact that a substantial number of papers on tractability of multivariate problems deal we the minimum kernel; we refer to [7, 8, 9] and in particular to [7, Sec. A.2] for results and further references. First of all the study of the second case is motivated by [5, 6], who analyze in particular the component by component algorithm on finite tensor products of this space. A further motivation comes from the fact that the Malliavin space $\mathbb{D}^{1,2}$ of once differentiable functionals is a generalization of this space to infinitely many variables in the context of stochastic processes. For a construction of $\mathbb{D}^{1,2}$ on the sequence space $\mathbb{R}^{\mathbb{N}}$, see [4].

In both cases, (A1) and (A2) are satisfied, and H is the Sobolev space of all locally absolutely continuous functions $f : D \to \mathbb{R}$ such that

$$\int_D (f')^2\, d\nu < \infty.$$

Furthermore,

$$\|f\|_{1+\gamma k}^2 = f^2(0) + \frac{1}{\gamma} \cdot \int_D (f')^2\, d\nu \tag{12}$$

and therefore

$$P(f) = f(0)$$

for $f \in H$. In the first case this is well-known and for the second case we refer to, e.g., [6, Sec. 2.1.1].

Let $f \in H(k)$. In the first case we have

$$f(y)^2 = \left(\int_0^y f'(x)\, dx \right)^2 \leq y \cdot \int_D (f'(x))^2\, dx \tag{13}$$

for all $y \in D$, and hence

$$\int_D f^2\, d\nu \leq \int_D y\, dy \cdot \int_D (f'(x))^2\, dx = 1/2 \cdot \int_D (f'(x))^2\, dx. \tag{14}$$

Analogously, in the second case,

$$\begin{aligned} \int_D f^2\, d\nu &= \int_D \left(\int_0^x f'(y)\, dy \right)^2 d\nu(x) \\ &\leq \int_D |x| \cdot \left| \int_0^x (f'(y))^2\, dy \right| d\nu(x) = \int_D (f')^2\, d\nu. \end{aligned} \tag{15}$$

In both cases we get $H(1+k) \subseteq L_2(\nu)$, which is well-known.

Consider the setting from Example 1. At first, assume that $\xi(f) = \int_D f\, d\nu$, i.e.,

$$\|f\| = \left| \int_D f\, d\nu \right| \tag{16}$$

for $f \in H$. Then the assumptions (A4)–(A7) are satisfied, and the minimal constant in (A6) is given by

$$c^2 = \int_D \int_D k(x,y)\, d\nu(x)\, d\nu(y).$$

We obtain

$$\|f\|^2_{1+l_\gamma} = \left(\int_D f\, d\nu \right)^2 + \frac{1}{\gamma} \cdot \int_D (f')^2\, d\nu \tag{17}$$

for $f \in H$. Secondly, fix $b \in D$ and put $\xi(f) = f(b)$, i.e.,

$$\|f\| = |f(b)| \tag{18}$$

for $f \in H$. Once more, the assumptions (A4)–(A6) are satisfied and (A7) is satisfied iff $b \neq 0$. The minimal constant in (A6) is given by

$$c^2 = k(b,b).$$

This leads to

$$\|f\|^2_{1+l_\gamma} = f^2(b) + \frac{1}{\gamma} \cdot \int_D (f')^2\, d\nu \tag{19}$$

for $f \in H$.

Alternatively, consider the setting from Example 2, where

$$\|f\| = \left(\int_D f^2\, d\nu \right)^{1/2} \tag{20}$$

for $f \in H$. The assumptions (A4)–(A7) are satisfied, with $c = 1$ being the minimal constant in (A6) in the second case, while $c = 1/\sqrt{2}$ is admissible in the first case. See

(14) and (15). For completeness, we add that $c = 2/\pi$ is the optimal constant in the first case. In both cases we obtain

$$\|f\|_{1+l_\gamma}^2 = \int_D f^2 \, d\nu + \frac{1}{\gamma} \cdot \int_D (f')^2 \, d\nu \tag{21}$$

for $f \in H$.

In all the above cases with $D = [0,1]$ an explicit formula for the kernel l_γ is presented in [7, Sec. A.2].

Lemma 1 provides a mutual control among the different norms on H given by (12), (17), (19), and (21). Furthermore, we see that different norms may lead to the same orthogonal complement of $H(1)$. In fact, for (17) as well as for (21), this orthogonal complement consists of all functions $f \in H$ with $\int_D f \, d\nu = 0$. Except for the trivial case (18) with $b = 0$ we always have (10), cf. Example 3.

Example 5. We illustrate that the case (11) arises in a natural way. Consider the Sobolev space H from Example 4 in any of the two variants for the domain D and the measure ν. But now, as a starting point, we choose the reproducing kernel k on the domain $D \times D$ such that

$$H(k) = \{f \in H : \int_D f \, d\nu = 0\}$$

with

$$\|f\|_k^2 = \int_D (f')^2 \, d\nu$$

for $f \in H(k)$. Assumptions (A1) and (A2) are satisfied, and $1 + \gamma k$ is the kernel that is characterized by the right-hand side from (17). On the space H we may choose the same seminorms $\|\cdot\|$ as in Example 4, and the non-trivial cases are (18) and (20). Then the assumptions (A4)–(A7) are satisfied, and of course the resulting kernels l_γ are as in Example 4, see (19) and (21). While (10) holds for (18), we have (11) for (20), cf. Example 3.

In the case (10) the upper bound in (4) turns out to be essentially sharp for small values of γ.

Lemma 2. *Suppose that* (10) *is satisfied. There exists a constant* $c_1 > 0$ *with the following property for all* $c_2 \geq 0$ *and* $\gamma > 0$. *If*

$$\|f\|_{1+l_\gamma}^2 \leq c_2 \cdot \|f\|_{1+\gamma k}^2$$

for all $f \in H$, *then*

$$c_2 \geq 1 + c_1 \cdot \sqrt{\gamma}.$$

Proof. Due to (10) there exists a function $f \in H(k)$ such that $\langle f, 1 \rangle > 0$ and $\|f\|_k = 1$. Define

$$g = 1 + \sqrt{\gamma} \cdot f. \tag{22}$$

Then we get

$$\|g\|_{1+l_\gamma}^2 = \|g\|^2 + 1 = 2 + 2\sqrt{\gamma} \cdot \langle f, 1 \rangle + \gamma \cdot \|f\|^2 \geq 2 \cdot (1 + \sqrt{\gamma} \cdot \langle f, 1 \rangle)$$

as well as $\|g\|_{1+\gamma k}^2 = 2$. Hereby the claim follows. $\square$

In the case (10) the upper bound in (6) turns out to be essentially sharp for small values of γ.

Lemma 3. *Suppose that* (10) *is satisfied. There exists a constant* $c_1 > 0$ *with the following property for all* $c_2 \geq 0$ *and* $\gamma > 0$*. If there exists* $\eta > 0$ *such that*

$$\|f\|^2_{1+l_\eta} \leq c_2 \cdot \|f\|^2_{1+\gamma k}$$

for all $f \in H$*, then*

$$c_2 \geq 1 + \frac{\gamma}{c_1 + \gamma}.$$

Proof. Take f as in the proof of Lemma 2. Define

$$c_1 = \frac{1}{\langle 1, f\rangle^2}$$

and

$$g = 1 + \gamma\langle 1, f\rangle \cdot f.$$

Then we get

$$\|g\|^2_{1+l_\eta} \geq \|1 + \gamma\langle 1, f\rangle \cdot f\|^2 = 1 + \gamma^2\langle 1, f\rangle^2\|f\|^2 + 2\gamma\langle 1, f\rangle^2 \geq 1 + 2\gamma\langle 1, f\rangle^2$$

as well as

$$\|g\|^2_{1+\gamma_k} = 1 + \gamma\langle 1, f\rangle^2.$$

Hereby the claim follows. □

Let c_0 be the minimal constant which fulfills (A6), i.e.,

$$c_0 = \inf\left\{c > 0 : \|f\| \leq c \cdot \|f\|_k \text{ for every } f \in H(k)\right\}. \tag{23}$$

Due to (A7)

$$c_0 > 0. \tag{24}$$

Lemma 4. *Suppose that* (11) *is satisfied. For all* $\gamma \geq \eta > 0$ *the norm of the embedding of* $H(1 + l_\eta)$ *into* $H(1 + \gamma k)$ *is one. Conversely, for all* $\gamma, \eta > 0$ *the norm of the embedding of* $H(1 + \eta k)$ *into* $H(1 + l_\gamma)$ *is* $\max\left(1, \sqrt{\eta\,(c_0^2 + 1/\gamma)}\right)$.

Proof. Let $f \in H$. Due to (11) we have

$$\|f\|^2 = P(f)^2 + \|f - P(f)\|^2,$$

so that $|P(f)| \leq \|f\|$. Hence $\|f\|_{1+\gamma k} \leq \|f\|_{1+l_\eta}$. On the other hand $\|1\|_{1+\gamma k} = 1 = \|1\|_{1+l_\eta}$.

Let $c_1 \geq 0$. We have

$$\|f\|^2_{1+l_\gamma} \leq c_1\|f\|^2_{1+\eta k}$$

for all $f \in H$, if and only if

$$\|f - P(f)\|^2 \leq (c_1 - 1)P(f)^2 + \left(\frac{c_1}{\eta} - \frac{1}{\gamma}\right)\|f - P(f)\|^2_k$$

for all $f \in H$, which is equivalent to $c_1 \geq 1$ and $c_1/\eta - 1/\gamma \geq c_0^2$. □

Now we study the sequence of spaces $H(1 + l_{\gamma_j})$ with $j \in \mathbb{N}$.

Example 6. Consider the setting from Example 1. Here we have

$$H(l_{\gamma_j}) = \{f \in H : \|f\| = 0\}$$

for $j \in \mathbb{N}$, and therefore

$$\|f\|^2_{l_{\gamma_j}} = \frac{1}{\gamma_j} \cdot \|f - P(f)\|^2_k$$

for every $f \in H(l_{\gamma_j})$. We conclude that there exists a uniquely determined kernel $l \neq 0$ on $D \times D$ such that

$$l_{\gamma_j} = \gamma_j l, \qquad j \in \mathbb{N}. \tag{25}$$

More generally, the following holds true.

Lemma 5. *There exists a reproducing kernel $l \neq 0$ on $D \times D$ and a sequence $(\eta_j)_{j\in\mathbb{N}}$ of positive weights such that*

$$l_{\gamma_j} = \eta_j l, \qquad j \in \mathbb{N}, \tag{26}$$

iff $\boldsymbol{\gamma}$ is a constant sequence or there exists a constant $c_1 \geq 0$ such that

$$\|f\|^2 = \langle f, 1\rangle^2 + c_1 \cdot \|f - P(f)\|_k^2 \tag{27}$$

for all $f \in H$.

Proof. Consider the non-trivial case of a non-constant sequence $\boldsymbol{\gamma}$.

First, assume that (26) is satisfied. Fix $i, j \in \mathbb{N}$ with $\gamma_i \neq \gamma_j$. For $f \in H$ with $\langle f, 1\rangle = 0$ we obtain

$$\|f\|^2 + \frac{1}{\gamma_j} \cdot \|f - P(f)\|_k^2 = \|f\|_{l_{\gamma_j}}^2 = \frac{\eta_i}{\eta_j} \cdot \|f\|_{l_{\gamma_i}}^2 = \frac{\eta_i}{\eta_j}\left(\|f\|^2 + \frac{1}{\gamma_i} \cdot \|f - P(f)\|_k^2\right).$$

Since $k \neq 0$, we get $\eta_i \neq \eta_j$, so that

$$\|f\|^2 = c_1 \cdot \|f - P(f)\|_k^2$$

with

$$c_1 = \frac{\eta_i/\eta_j \cdot 1/\gamma_i - 1/\gamma_j}{1 - \eta_i/\eta_j} \geq 0.$$

Hence, for all $f \in H$,

$$\|f\|^2 = \langle f, 1\rangle^2 + \|f - \langle f, 1\rangle\|^2 = \langle f, 1\rangle^2 + c_1 \cdot \|f - P(f)\|_k^2,$$

as claimed.

Conversely, assume that (27) is satisfied for all $f \in H$. Let $l = l_{\gamma_1}$, and put

$$\eta_j = \frac{c_1 + 1/\gamma_1}{c_1 + 1/\gamma_j}.$$

For $f \in H(l_{\gamma_j})$ we have

$$\|f\|_{l_{\gamma_j}}^2 = \left(c_1 + \frac{1}{\gamma_j}\right) \cdot \|f - P(f)\|_k^2 = \frac{1}{\eta_j} \cdot \|f\|_l^2,$$

which implies (26). □

Example 7. Consider the setting from Example 4. For (17) as well as for (19), any two kernels l_{γ_i} and l_{γ_j} only differ by a multiplicative factor, see Example 6. However, for (21) this property only holds true in the trivial case of a constant sequence $(\gamma_j)_{j\in\mathbb{N}}$. This follows from Lemma 5, since (27) is not fulfilled for $\|\cdot\|$ given by (20); for a proof take, e.g., functions $\sin(k\cdot)$ with $k \in \mathbb{N}$.

4. Functions of Finitely Many Variables

For $s \in \mathbb{N}$ we define the reproducing kernels

$$K_s^{\boldsymbol{\gamma}}(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^{s} (1 + \gamma_j k(x_j, y_j)), \qquad \mathbf{x}, \mathbf{y} \in D^s,$$

and

$$L_s^{\boldsymbol{\gamma}}(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^{s} (1 + l_{\gamma_j}(x_j, y_j)), \qquad \mathbf{x}, \mathbf{y} \in D^s,$$

on the domain $D^s \times D^s$. Note that these kernels depend on $\boldsymbol{\gamma}$ only through $\gamma_1, \ldots, \gamma_s$. Lemma 1 implies

$$H(L_s^{\boldsymbol{\gamma}}) = H(K_s^{\boldsymbol{\eta}})$$

for any two sequences $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ of positive weights, and we use $\imath_s^{\boldsymbol{\eta},\boldsymbol{\gamma}}$ to denote the embedding of $H(K_s^{\boldsymbol{\eta}})$ into $H(L_s^{\boldsymbol{\gamma}})$. Let H_s denote the vector space $H(K_s^{\boldsymbol{\eta}})$.

Obviously, we have equivalence of the norms $\|\cdot\|_{K_s^{\boldsymbol{\gamma}}}$ and $\|\cdot\|_{L_s^{\boldsymbol{\gamma}}}$, but the respective constants in this equivalence may depend on s. As the first result in the following theorem, we characterize the case when the equivalence holds uniformly in s, i.e., there exist constants $A, B > 0$ such that

$$A \cdot \|f\|_{K_s^{\boldsymbol{\gamma}}} \le \|f\|_{L_s^{\boldsymbol{\gamma}}} \le B \cdot \|f\|_{K_s^{\boldsymbol{\gamma}}}$$

for all $s \in \mathbb{N}$ and for all $f \in H_s$. Furthermore, we study two relaxed versions of uniform equivalence by modifying the sequence of weights. At first we permit multiplication of γ by constants $0 < c' < 1 < \tilde{c}$ to get a sequence $\tilde{\boldsymbol{\gamma}} = \tilde{c}\boldsymbol{\gamma}$ of larger weights and a sequence $\boldsymbol{\gamma}' = c'\boldsymbol{\gamma}$ of smaller weights, and secondly we consider any two sequences $\boldsymbol{\gamma}'$ and $\tilde{\boldsymbol{\gamma}}$ of positive weights such that $\boldsymbol{\gamma}' < \boldsymbol{\gamma} < \tilde{\boldsymbol{\gamma}}$. To retain the full symmetry we characterize the case when

$$A \cdot \|f\|_{K_s^{\tilde{\boldsymbol{\gamma}}}} \le \|f\|_{L_s^{\boldsymbol{\gamma}}} \le B \cdot \|f\|_{K_s^{\boldsymbol{\gamma}'}}$$

and

$$A \cdot \|f\|_{L_s^{\tilde{\boldsymbol{\gamma}}}} \le \|f\|_{K_s^{\boldsymbol{\gamma}}} \le B \cdot \|f\|_{L_s^{\boldsymbol{\gamma}'}}$$

for all $s \in \mathbb{N}$ and for all $f \in H_s$.

Theorem 1. *We have*

$$\sup_{s \in \mathbb{N}} \max\left(\|\imath_s^{\boldsymbol{\gamma},\boldsymbol{\gamma}}\|, \|(\imath_s^{\boldsymbol{\gamma},\boldsymbol{\gamma}})^{-1}\|\right) < \infty \tag{28}$$

iff

$$\sum_{j \in \mathbb{N}} \sqrt{\gamma_j} < \infty \ \vee \ \left(\sum_{j \in \mathbb{N}} \gamma_j < \infty \ \wedge \ (11)\right). \tag{29}$$

Furthermore, there exist $0 < c' < 1 < \tilde{c}$ *such that*

$$\sup_{s \in \mathbb{N}} \max\left(\|\imath_s^{c'\boldsymbol{\gamma},\boldsymbol{\gamma}}\|, \|(\imath_s^{\tilde{c}\boldsymbol{\gamma},\boldsymbol{\gamma}})^{-1}\|, \|\imath_s^{\boldsymbol{\gamma},\tilde{c}\boldsymbol{\gamma}}\|, \|(\imath_s^{\boldsymbol{\gamma},c'\boldsymbol{\gamma}})^{-1}\|\right) < \infty \tag{30}$$

iff

$$\sum_{j \in \mathbb{N}} \gamma_j < \infty \ \vee \ \left(\limsup_{j \to \infty} \gamma_j < 1/c_0^2 \ \wedge \ (11)\right), \tag{31}$$

where c_0 is given by (23)*. Finally, there exist sequences $\boldsymbol{\gamma}' < \boldsymbol{\gamma} < \tilde{\boldsymbol{\gamma}}$ of positive weights with*

$$\sup_{s\in\mathbb{N}} \max\left(\|\imath_s^{\boldsymbol{\gamma}',\boldsymbol{\gamma}}\|, \|(\imath_s^{\tilde{\boldsymbol{\gamma}},\boldsymbol{\gamma}})^{-1}\|, \|\imath_s^{\boldsymbol{\gamma},\tilde{\boldsymbol{\gamma}}}\|, \|(\imath_s^{\boldsymbol{\gamma},\boldsymbol{\gamma}'})^{-1}\| \right) < \infty \tag{32}$$

iff

$$\sum_{j\in\mathbb{N}} \gamma_j < \infty \ \vee \ \left(\sum_{j\in\mathbb{N}} \max\left(0, \gamma_j - 1/c_0^2\right) < \infty \ \wedge \ (11) \right). \tag{33}$$

Proof. At first we show that the assumptions (29), (31), and (33) imply the respective properties of the norms of the embedding. Let $f \in H_s$. From (4) and (6) in Lemma 1 we get

$$\prod_{j=1}^{s} \left(1 + c\sqrt{\gamma_j} + c^2\gamma_j\right)^{-1} \cdot \|f\|^2_{K_s^{\boldsymbol{\gamma}}} \le \|f\|^2_{L_s^{\boldsymbol{\gamma}}} \le \prod_{j=1}^{s} \left(1 + c\sqrt{\gamma_j} + c^2\gamma_j\right) \cdot \|f\|^2_{K_s^{\boldsymbol{\gamma}}}, \tag{34}$$

$$\prod_{j=1}^{s} \left(1 + (1+c^2)\gamma_j\right)^{-1} \cdot \|f\|^2_{K_s^{(1+c^2)\boldsymbol{\gamma}}} \le \|f\|^2_{L_s^{\boldsymbol{\gamma}}} \le \prod_{j=1}^{s} (1+\gamma_j) \cdot \|f\|^2_{K_s^{\boldsymbol{\gamma}/(1+c^2)}}, \tag{35}$$

and

$$\prod_{j=1}^{s} \left(1 + (1+c^2)\gamma_j\right)^{-1} \cdot \|f\|^2_{L_s^{(1+c^2)\boldsymbol{\gamma}}} \le \|f\|^2_{K_s^{\boldsymbol{\gamma}}} \le \prod_{j=1}^{s} (1+\gamma_j) \cdot \|f\|^2_{L_s^{\boldsymbol{\gamma}/(1+c^2)}}. \tag{36}$$

In the case (11) we apply Lemma 4 to obtain

$$\|f\|^2_{K_s^{\boldsymbol{\gamma}}} \le \|f\|^2_{L_s^{\boldsymbol{\gamma}}} \le \prod_{j=1}^{s} (1 + c^2\gamma_j) \cdot \|f\|^2_{K_s^{\boldsymbol{\gamma}}}, \tag{37}$$

$$\|f\|^2_{K_s^{\boldsymbol{\gamma}}} \le \|f\|^2_{L_s^{\boldsymbol{\gamma}}} \le \|f\|^2_{K_s^{\boldsymbol{\gamma}'}} \tag{38}$$

for

$$\gamma_j' = \frac{\gamma_j}{1 + c^2\gamma_j}, \tag{39}$$

and

$$\prod_{j=1}^{s} \max\left(1, \gamma_j\left(c_0^2 + 1/\tilde{\gamma}_j\right)\right)^{-1} \cdot \|f\|^2_{L_s^{\tilde{\boldsymbol{\gamma}}}} \le \|f\|^2_{K_s^{\boldsymbol{\gamma}}} \le \|f\|^2_{L_s^{\boldsymbol{\gamma}}} \tag{40}$$

for every sequence $\tilde{\boldsymbol{\gamma}}$ of positive weights.

The constants in the upper bounds in (34)–(37) are uniformly bounded if $\sum_{j\in\mathbb{N}} \sqrt{\gamma_j} < \infty$ or $\sum_{j\in\mathbb{N}} \gamma_j < \infty$, respectively. Under the same conditions the constants in the lower bounds in (34)–(36) are uniformly bounded away from zero. We conclude that (29) implies (28). Moreover, $\sum_{j\in\mathbb{N}} \gamma_j < \infty$ implies (30) with

$$\tilde{c} = 1 + c_0^2$$

and

$$c' = \frac{1}{1 + c_0^2},$$

which obviously yields (32) for $\tilde{\boldsymbol{\gamma}} = \tilde{c}\,\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}' = c'\,\boldsymbol{\gamma}$.

Suppose that $\limsup_{j\to\infty} \gamma_j < 1/c_0^2$, take

$$\tilde{c} > \frac{1}{1 - c_0^2 \cdot \limsup_{j\to\infty} \gamma_j},$$

and define $\tilde{\boldsymbol{\gamma}} = \tilde{c}\,\boldsymbol{\gamma}$ to obtain $\gamma_j\,(c_0^2 + 1/\tilde{\gamma}_j) \le 1$ for j sufficiently large. With this choice the constant in the lower bound in (40) is therefore uniformly bounded away from zero. Furthermore, take

$$c' = \frac{1}{1 + c^2 \sup_{j\in\mathbb{N}} \gamma_j}$$

to obtain $c'\boldsymbol{\gamma} \le \boldsymbol{\gamma}'$ for γ_j' according to (39). Use (38) and (40) to derive (30) from $\limsup_{j\to\infty} \gamma_j < 1/c_0^2$ and (11).

Finally, suppose that $\sum_{j\in\mathbb{N}} \max\left(0, \gamma_j - 1/c_0^2\right) < \infty$ and that (11) is satisfied. Consider $\boldsymbol{\gamma}'$ given by (39) as well as

$$\tilde{\gamma}_j = 2^j \gamma_j.$$

Use (38) and (40) to derive (32).

Next we show that (29), (31), and (33) are necessary conditions for the respective properties of the norms of the embedding to hold. Lemma 2 yields the existence of a constant $c_1 > 0$ such that

$$\|\imath_s^{\boldsymbol{\gamma},\boldsymbol{\gamma}}\|^2 \ge \prod_{j=1}^{s} (1 + c_1 \cdot \sqrt{\gamma_j})$$

in the case (10), and in the case (11) we have

$$\|\imath_s^{\boldsymbol{\gamma},\boldsymbol{\gamma}}\|^2 \ge \prod_{j=1}^{s} (1 + c_0^2 \cdot \gamma_j)$$

due to Lemma 4. We conclude that (28) implies (29). Now we turn to (33). Let $\tilde{\boldsymbol{\gamma}} > 0$. Lemma 3 yields the existence of a constant $c_1 > 0$ such that

$$\|\imath_s^{\boldsymbol{\gamma},\tilde{\boldsymbol{\gamma}}}\|^2 \ge \prod_{j=1}^{s} \left(1 + \frac{\gamma_j}{c_1 + \gamma_j}\right)$$

in the case (10), and in the case (11) we have

$$\|\imath_s^{\boldsymbol{\gamma},\tilde{\boldsymbol{\gamma}}}\|^2 = \prod_{j=1}^{s} \max\left(1, \gamma_j c_0^2 + \gamma_j/\tilde{\gamma}_j\right) \ge \prod_{j=1}^{s} \left(1 + \max\left(0, \gamma_j c_0^2 - 1\right)\right)$$

due to Lemma 4. We conclude that $\sup_{s\in\mathbb{N}} \|\imath_s^{\boldsymbol{\gamma},\tilde{\boldsymbol{\gamma}}}\| < \infty$ implies $\sum_{j\in\mathbb{N}} \gamma_j/(c_1 + \gamma_j) < \infty$, i.e., $\sum_{j\in\mathbb{N}} \gamma_j < \infty$, in the case (10) and $\sum_{j\in\mathbb{N}} \max\left(0, \gamma_j - 1/c_0^2\right) < \infty$ in the case (11). Consequently, (32) implies (33). It remains to show that (30) and (11) implies $\limsup_{j\to\infty} \gamma_j < 1/c_0^2$. In fact, Lemma 4 yields

$$\|\imath_s^{\boldsymbol{\gamma},\tilde{c}\boldsymbol{\gamma}}\|^2 = \prod_{j=1}^{s} \max\left(1, \gamma_j c_0^2 + 1/\tilde{c}\right)$$

for all $\tilde{c} > 0$ in the case (11), and hereby the statement follows. □

Remark 2. According to the proof of Theorem 1, $\sup_{s\in\mathbb{N}} \|\imath_s^{\gamma,\gamma}\| < \infty$ already implies (29), $\sup_{s\in\mathbb{N}} \|\imath_s^{\gamma,\tilde{c}\gamma}\| < \infty$ already implies (31), and $\sup_{s\in\mathbb{N}} \|\imath_s^{\gamma,\tilde{\gamma}}\| < \infty$ already implies (33). A suitable modification of Lemma 2, with a reversed role of $1 + l_\gamma$ and $1 + \gamma k$, yields that $\sup_{s\in\mathbb{N}} \|(\imath_s^{\gamma,\gamma})^{-1}\| < \infty$ and (10) implies (29), too. In the case (11), however, Lemma 4 implies $\|(\imath_s^{\gamma,\gamma'})^{-1}\| = 1$ for all $s \in \mathbb{N}$ as soon as $\boldsymbol{\gamma}' \leq \boldsymbol{\gamma}$.

Example 8. Consider the situation of Example 4 and denote by $L_s^{\gamma,1}$ the reproducing kernel corresponding to (17) and by $L_s^{\gamma,2}$ the reproducing kernel corresponding to (21). Recall that (10) holds in both cases. We derive conditions for $\boldsymbol{\gamma}$ that guarantee the uniform equivalence of the norms on the spaces $H(L_s^{\gamma,1})$ and $H(L_s^{\gamma,2})$.

We may apply Theorem 1 twice, with K_s^γ corresponding to (12) as the intermediate kernel, and since (10) is satisfied for $L_s^{\gamma,1}$ and for $L_s^{\gamma,2}$, we get $\sum_{j\in\mathbb{N}} \sqrt{\gamma_j} < \infty$ as a sufficient condition for the uniform equivalence. However, using $L_s^{\gamma,1}$ as the starting point, condition (11) is satisfied for the kernel $L_s^{\gamma,2}$, see Example 5. According to Theorem 1 we therefore get $\sum_{j\in\mathbb{N}} \gamma_j < \infty$ as a necessary and sufficient condition for uniform equivalence.

We benefit from condition (11) in the same way with respect to the other two statements from Theorem 1.

5. Functions of Infinitely Many Variables

We define

$$\mathfrak{X}^\gamma = \{\mathbf{x} \in D^{\mathbb{N}} : \prod_{j=1}^{\infty} (1 + \gamma_j k(x_j, x_j)) < \infty\}$$

as well as

$$\mathfrak{Y}^\gamma = \{\mathbf{y} \in D^{\mathbb{N}} : \prod_{j=1}^{\infty} (1 + l_{\gamma_j}(y_j, y_j)) < \infty\}.$$

Put

$$m = \inf_{x\in D} \sup\{f^2(x) : f \in H,\ P(f) = 0,\ \|f\|_{1+k} \leq 1\}$$

and

$$\hat{m} = \inf_{x\in D} \sup\{f^2(x) : f \in H,\ \langle f, 1\rangle = 0,\ \|f\|_{1+k} \leq 1\}.$$

Clearly $m = \inf_{x\in D} k(x, x)$, and (11) implies $m = \hat{m}$.

The following fact is the particular instance of [3, Lemma 1] for product weights.

Lemma 6. *We have $\mathfrak{X}^\gamma \neq \emptyset$ iff*

$$\sum_{j\in\mathbb{N}} \gamma_j < \infty \ \vee\ m = 0.$$

The counterpart for the domain $\mathfrak{Y}^\gamma$ is as follows.

Lemma 7. *We have $\mathfrak{Y}^\gamma \neq \emptyset$ iff*

$$\sum_{j\in\mathbb{N}} \gamma_j < \infty \ \vee\ \hat{m} = 0.$$

Proof. Let $\hat{k}$ denote the reproducing kernel on $D \times D$ such that

$$H(\hat{k}) = \{f \in H : \langle f, 1\rangle = 0\}$$

and $\|f\|_{\hat{k}} = \|f\|_{1+k}$ for $f \in H(\hat{k})$. Note that

$$\hat{k}(x, x) = \sup\{f^2(x) : f \in H,\ \langle f, 1\rangle = 0,\ \|f\|_{1+k} \leq 1\}$$

for all $x \in D$. Hence $\hat{m} = \inf_{x\in D} \hat{k}(x,x)$. Let $\mathbf{x} \in D^{\mathbb{N}}$. Lemma 1, and here in particular (5) and the lower bound in (6), imply

$$\hat{k}(x_j, x_j) \leq (1 + c^2 + 1/\gamma_j) \cdot l_{\gamma_j}(x_j, x_j),$$
$$l_{\gamma_j}(x_j, x_j) \leq \big(1 + (1 + c^2) \cdot \gamma_j\big) \cdot \hat{k}(x_j, x_j),$$

and

$$1 + l_{\gamma_j}(x_j, x_j) \leq \big(1 + (1 + c^2)\gamma_j\big) \cdot \big(1 + (1 + c^2) \cdot \gamma_j k(x_j, x_j)\big)$$

for all $j \in \mathbb{N}$. Hereby the claim follows. □

Example 9. Consider the situation of Example 4. Here we have $m = 0$, so that $\mathfrak{X}^{\boldsymbol{\gamma}} \neq \emptyset$ due to Lemma 6. Analogously, $\mathfrak{Y}^{\gamma} \neq \emptyset$ if $\|\cdot\|$ is given by (18), which corresponds to (19). For the other two choices of $\|\cdot\|$, namely (16) and (20), which correspond to (17) and (21), one can easily show that $\hat{m} > 0$, and due to Lemma 7 we then have $\mathfrak{Y}^{\gamma} \neq \emptyset$ iff $\sum_{j\in\mathbb{N}} \gamma_j < \infty$.

In the sequel, u varies over all finite subsets of $\mathbb{N}$. If $\mathfrak{X}^{\boldsymbol{\gamma}} \neq \emptyset$, we consider the reproducing kernels

$$K^{\boldsymbol{\gamma}}(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^{\infty} (1 + \gamma_j k(x_j, y_j)), \qquad \mathbf{x}, \mathbf{y} \in \mathfrak{X}^{\boldsymbol{\gamma}},$$

as well as

$$k_u^{\boldsymbol{\gamma}}(\mathbf{x}, \mathbf{y}) = \prod_{j\in u} \gamma_j k(x_j, y_j), \qquad \mathbf{x}, \mathbf{y} \in \mathfrak{X}^{\boldsymbol{\gamma}},$$

on the domain $\mathfrak{X}^{\boldsymbol{\gamma}} \times \mathfrak{X}^{\boldsymbol{\gamma}}$. If $\mathfrak{Y}^{\boldsymbol{\gamma}} \neq \emptyset$, we consider the reproducing kernels

$$L^{\boldsymbol{\gamma}}(\mathbf{x}, \mathbf{y}) = \prod_{j=1}^{\infty} (1 + l_{\gamma_j}(x_j, y_j)), \qquad \mathbf{x}, \mathbf{y} \in \mathfrak{Y}^{\boldsymbol{\gamma}},$$

as well as

$$l_u^{\boldsymbol{\gamma}}(\mathbf{x}, \mathbf{y}) = \prod_{j\in u} l_{\gamma_j}(x_j, y_j), \qquad \mathbf{x}, \mathbf{y} \in \mathfrak{Y}^{\boldsymbol{\gamma}},$$

on the domain $\mathfrak{Y}^{\boldsymbol{\gamma}} \times \mathfrak{Y}^{\boldsymbol{\gamma}}$.

In the sequel, we either choose

$$\mathfrak{E} = \mathfrak{X}^{\gamma}, \qquad N = K^{\gamma}, \qquad n_u = k_u^{\gamma}, \qquad N_s = K_s^{\gamma}, \tag{41}$$

provided that $\mathfrak{X}^{\gamma} \neq \emptyset$, or

$$\mathfrak{E} = \mathfrak{Y}^{\gamma}, \qquad N = L^{\gamma}, \qquad n_u = l_u^{\gamma}, \qquad N_s = L_s^{\gamma}, \tag{42}$$

provided that $\mathfrak{Y}^{\gamma} \neq \emptyset$. By definition, the direct sum of the spaces $H(n_u)$ consists of all sequences $(f_u)_u$ with $f_u \in H(n_u)$ such that $\sum_u \|f_u\|_{n_u}^2 < \infty$.

Lemma 8. *The direct sum of the spaces $H(n_u)$ is isometrically isomorphic to $H(N)$ via $(f_u)_u \mapsto \sum_u f_u$.*

Proof. In the case (41) the result is a particular instance of [3, Proposition 2] for product weights. The same proof is applicable in the case (42), too. □

For $s \in \mathbb{N}$ and $f : D^s \to \mathbb{R}$ we define $\psi_s^{\mathfrak{E}} f : \mathfrak{E} \to \mathbb{R}$ by

$$\psi_s^{\mathfrak{E}} f(\mathbf{x}) = f(x_1, \ldots, x_s), \qquad \mathbf{x} \in \mathfrak{E}.$$

Recall that $H_s = H(N_s)$.

Lemma 9. *The mapping $\psi_s^{\mathfrak{E}}$ is a linear isometry from $H(N_s)$ to $H(N)$, and $\bigcup_{s\in\mathbb{N}} \psi_s^{\mathfrak{E}}(H_s)$ is a dense subspace of $H(N)$.*

Proof. Let $u \subseteq \{1,\ldots,s\}$. Define the reproducing kernel $n_{u,s}$ on the domain $D^s \times D^s$ by

$$n_{u,s}(\mathbf{x},\mathbf{y}) = \prod_{j\in u} \gamma_j k(x_j,y_j), \qquad \mathbf{x},\mathbf{y} \in D^s,$$

in the case (41), and by

$$n_{u,s}(\mathbf{x},\mathbf{y}) = \prod_{j\in u} l_{\gamma_j}(x_j,y_j), \qquad \mathbf{x},\mathbf{y} \in D^s,$$

in the case (42). It is well-known that the direct sum of the spaces $H(n_{u,s})$ is isometrically isomorphic to $H(N_s)$ via $(f_u)_u \mapsto \sum_u f_u$. Since $n_u(\mathbf{x}) = n_{u,s}(x_1,\ldots,x_s)$ for $\mathbf{x} \in \mathfrak{E}$, the space $H(n_{u,s})$ is isometrically isomorphic to $H(n_u)$ via $\psi_s^{\mathfrak{E}}$, see [1, p. 357 ff]. The claim now follows from Lemma 8. □

We extend Theorem 1 to the case of functions of infinitely many variables. We write $H(K) \sqsubseteq H(L)$, if K and L are reproducing kernels on $\mathfrak{X}\times\mathfrak{X}$ and $\mathfrak{Y}\times\mathfrak{Y}$, respectively, such that $\mathfrak{Y} \subseteq \mathfrak{X}$ and $f|_{\mathfrak{Y}} \in H(L)$ for every $f \in H(K)$. To exclude a trivial case, we henceforth assume that

(A8) $\mathfrak{X}^{\boldsymbol{\gamma}} \neq \emptyset$ or $\mathfrak{Y}^{\boldsymbol{\gamma}} \neq \emptyset$.

See also Remark 3.

Theorem 2. *We have*

$$H(K^{\boldsymbol{\gamma}}) = H(L^{\boldsymbol{\gamma}}) \tag{43}$$

iff (29) *holds true. Furthermore, there exist* $0 < c' < 1 < \tilde{c}$ *such that*

$$H(K^{c'\boldsymbol{\gamma}}) \sqsubseteq H(L^{\boldsymbol{\gamma}}) \sqsubseteq H(K^{\tilde{c}\boldsymbol{\gamma}}) \tag{44}$$

and

$$H(L^{c'\boldsymbol{\gamma}}) \sqsubseteq H(K^{\boldsymbol{\gamma}}) \sqsubseteq H(L^{\tilde{c}\boldsymbol{\gamma}}) \tag{45}$$

iff (31) *holds true. Finally, there exist sequences* $\boldsymbol{\gamma}' < \boldsymbol{\gamma} < \tilde{\boldsymbol{\gamma}}$ *of positive weights with*

$$H(K^{\boldsymbol{\gamma}'}) \sqsubseteq H(L^{\boldsymbol{\gamma}}) \sqsubseteq H(K^{\tilde{\boldsymbol{\gamma}}}) \tag{46}$$

and

$$H(L^{\boldsymbol{\gamma}'}) \sqsubseteq H(K^{\boldsymbol{\gamma}}) \sqsubseteq H(L^{\tilde{\boldsymbol{\gamma}}}) \tag{47}$$

iff (33) *holds true.*

Proof. First, we establish some auxiliary facts involving arbitrary sequences $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ of positive weights.

We have

$$\sup_{s\in\mathbb{N}} \|i_s^{\boldsymbol{\eta},\boldsymbol{\gamma}}\| < \infty \quad \Rightarrow \quad \mathfrak{Y}^{\boldsymbol{\gamma}} \subseteq \mathfrak{X}^{\boldsymbol{\eta}}. \tag{48}$$

For the proof, put $c = \sup_{s\in\mathbb{N}} \|i_s^{\boldsymbol{\eta},\boldsymbol{\gamma}}\|$ to obtain

$$K_s^{\boldsymbol{\eta}}(\mathbf{x},\mathbf{x}) \leq c^2 \cdot L_s^{\boldsymbol{\gamma}}(\mathbf{x},\mathbf{x})$$

for all $s \in \mathbb{N}$ and $\mathbf{x} \in D^s$. Hereby the claim follows. Analogously, we obtain

$$\sup_{s\in\mathbb{N}} \|\,(i_s^{\boldsymbol{\eta},\boldsymbol{\gamma}})^{-1}\,\| < \infty \quad \Rightarrow \quad \mathfrak{X}^{\boldsymbol{\eta}} \subseteq \mathfrak{Y}^{\boldsymbol{\gamma}}. \tag{49}$$

Suppose that $\mathfrak{Y}^{\gamma} \neq \emptyset$. Then we have

$$\sup_{s\in\mathbb{N}} \|i_s^{\eta,\gamma}\| < \infty \quad \Leftrightarrow \quad H(K^{\eta}) \sqsubseteq H(L^{\gamma}). \tag{50}$$

For the proof, we first assume that $c < \infty$ with c as before. Then we have $\emptyset \neq \mathfrak{Y}^{\gamma} \subseteq \mathfrak{X}^{\eta}$, see (48). In this case

$$\psi_s^{\mathfrak{Y}^{\gamma}} f = \left(\psi_s^{\mathfrak{X}^{\eta}} f\right)|_{\mathfrak{Y}^{\gamma}} \tag{51}$$

for $s \in \mathbb{N}$ and $f \in H_s$, and Lemma 9 implies

$$\| \left(\psi_s^{\mathfrak{X}^{\eta}} f\right)|_{\mathfrak{Y}^{\gamma}}\|_{L^{\gamma}} = \|f\|_{L_s^{\gamma}} \leq c \cdot \|f\|_{K_s^{\eta}} = c \cdot \|\psi_s^{\mathfrak{X}^{\eta}} f\|_{K^{\eta}}.$$

Using a standard continuation argument and Lemma 9 we get $H(K^{\eta}) \sqsubseteq H(L^{\gamma})$.

Conversely, assume that $H(K^{\eta}) \sqsubseteq H(L^{\gamma})$. In particular, $\emptyset \neq \mathfrak{Y}^{\gamma} \subseteq \mathfrak{X}^{\eta}$. Due to the closed graph theorem there exists a constant $c \geq 0$ such that

$$\|f|_{\mathfrak{Y}^{\gamma}}\|_{L^{\gamma}} \leq c \cdot \|f\|_{K^{\eta}}$$

for all $f \in H(K^{\eta})$. For $s \in \mathbb{N}$ and $f \in H_s$ we employ Lemma 9 and (51) to obtain

$$\|f\|_{L_s^{\gamma}} = \| \left(\psi_s^{\mathfrak{X}^{\eta}} f\right)|_{\mathfrak{Y}^{\gamma}}\|_{L^{\gamma}} \leq c \cdot \|\psi_s^{\mathfrak{X}^{\eta}} f\|_{K^{\eta}} = c \cdot \|f\|_{K_s^{\eta}},$$

so that $\sup_{s\in\mathbb{N}} \|i_s^{\eta,\gamma}\| \leq c$. Analogously to (50) we obtain

$$\sup_{s\in\mathbb{N}} \| (i_s^{\eta,\gamma})^{-1} \| < \infty \quad \Leftrightarrow \quad H(L^{\gamma}) \sqsubseteq H(K^{\eta}), \tag{52}$$

if $\mathfrak{X}^{\eta} \neq \emptyset$.

Suppose that (29) holds true. Theorem 1 implies (28). From (48), (49), and (A8) we get $\mathfrak{X}^{\gamma} = \mathfrak{Y}^{\gamma} \neq \emptyset$. Use (50) and (52) to establish (43).

Conversely, suppose that (43) holds true. Together with (A8) this implies $\mathfrak{X}^{\gamma} = \mathfrak{Y}^{\gamma} \neq \emptyset$. Apply Theorem 1 and (50) as well as (52) to obtain (29).

Suppose that (31) holds true. Theorem 1 implies (30) with constants $0 < c' < 1 < \tilde{c}$. Use (48) and (49) to conclude that $\mathfrak{X}^{\tilde{c}\gamma} \subseteq \mathfrak{Y}^{\gamma} \subseteq \mathfrak{X}^{c'\gamma}$ and $\mathfrak{Y}^{\tilde{c}\gamma} \subseteq \mathfrak{X}^{\gamma} \subseteq \mathfrak{Y}^{c'\gamma}$. Due to Lemma 6, Lemma 7, and (A8) all these domains are non-empty. Use (50) and (52) to establish (44) and (45).

Conversely, suppose that (44) and (45) holds true with $0 < c' < 1 < \tilde{c}$. Hence, in particular, $\mathfrak{X}^{\tilde{c}\gamma} \subseteq \mathfrak{Y}^{\gamma} \subseteq \mathfrak{X}^{c'\gamma}$ and $\mathfrak{Y}^{\tilde{c}\gamma} \subseteq \mathfrak{X}^{\gamma} \subseteq \mathfrak{Y}^{c'\gamma}$, and as before it follows that all the domains are non-empty. Apply Theorem 1 and (50) as well as (52) to obtain (31).

Suppose (33) holds true. If $\sum_{j\in\mathbb{N}} \gamma_j < \infty$, then Theorem 1 yields (30), so that (46) and (47) hold with $\tilde{\gamma} = \tilde{c}\gamma$ and $\gamma' = c'\gamma$. Assume $\sum_{j\in\mathbb{N}} \gamma_j = \infty$. Then we have (11), so that $m = \hat{m} = 0$ due to (A8), Lemma 6, and Lemma 7. Theorem 1 implies (32) for sequences $\gamma' < \gamma < \tilde{\gamma}$ of positive weights. As before, we obtain $\mathfrak{X}^{\tilde{\gamma}} \subseteq \mathfrak{Y}^{\gamma} \subseteq \mathfrak{X}^{\gamma'}$ as well as $\mathfrak{Y}^{\tilde{\gamma}} \subseteq \mathfrak{X}^{\gamma} \subseteq \mathfrak{Y}^{\gamma'}$, and all these domains are non-empty. Use (50) and (52) to establish (46) and (47).

Conversely, suppose that (46) and (47) holds true with sequences $\gamma' < \gamma < \tilde{\gamma}$ of positive weights. We claim that $m > 0$ or $\hat{m} > 0$ implies $\sum_{j\in\mathbb{N}} \gamma_j < \infty$, so that (33) is satisfied in this case. If $m > 0$ and $\mathfrak{X}^{\gamma} \neq \emptyset$, then $\sum_{j\in\mathbb{N}} \gamma_j < \infty$ due to Lemma 6. Assume that $m > 0$ and $\mathfrak{X}^{\gamma} = \emptyset$. Then we have $\mathfrak{Y}^{\gamma} \neq \emptyset$ due to (A8), and Lemma 7 implies $\sum_{j\in\mathbb{N}} \gamma_j < \infty$ or $\hat{m} = 0$. The former contradicts $\mathfrak{X}^{\gamma} = \emptyset$, see Lemma 6. The latter implies $\mathfrak{Y}^{\tilde{\gamma}} \neq \emptyset$, and we have $\mathfrak{Y}^{\tilde{\gamma}} \subseteq \mathfrak{X}^{\gamma}$ as a consequence of (47), which leads to a contradiction, too. The case $\hat{m} > 0$ is handled in the same way. It remains to discuss the case $m = \hat{m} = 0$. From (46) and (47) we get $\mathfrak{X}^{\tilde{\gamma}} \subseteq \mathfrak{Y}^{\gamma} \subseteq \mathfrak{X}^{\gamma'}$ and $\mathfrak{Y}^{\tilde{\gamma}} \subseteq \mathfrak{X}^{\gamma} \subseteq \mathfrak{Y}^{\gamma'}$, and due to Lemma 6 and Lemma 7 all theses domains are non-empty. Apply Theorem 1 and (50) as well as (52) to obtain (33). □

Remark 3. The proof of Theorem 2 reveals that the conditions (29), (31), and (33) actually ensure that the domains for the Hilbert spaces in (43), in (44) and (45), and in (46) and (47), respectively, are non-empty.

Remark 4. Suppose that

$$\sum_{j\in\mathbb{N}} \gamma_j < \infty \ \vee \ \left(\lim_{j\to\infty} \gamma_j = 0 \ \wedge \ (11) \right), \tag{53}$$

which is stronger than (31), and observe that $\mathfrak{X}^{\boldsymbol{\gamma}} = \mathfrak{X}^{c\boldsymbol{\gamma}}$ for every $c > 0$ in any case. Applying the second statement from Theorem 2 three times we get

$$H(K^{c'\boldsymbol{\gamma}}) \subseteq H(L^{\boldsymbol{\gamma}}) \subseteq H(K^{\tilde{c}\boldsymbol{\gamma}}) \tag{54}$$

and

$$H(L^{c'\boldsymbol{\gamma}}) \subseteq H(K^{\boldsymbol{\gamma}}) \subseteq H(L^{\tilde{c}\boldsymbol{\gamma}}) \tag{55}$$

with constants $0 < c' < 1 < \tilde{c}$, and the domains for these Hilbert spaces coincide. We add that (54) already follows from (31).

Remark 5. According to the closed graph theorem, all the embeddings that are stated in Theorem 2 are continuous. Explicit estimates of the respective norms in terms of the constant c from assumption (A6) and $\boldsymbol{\gamma}$ are easily derived from the proofs of Theorem 1 and Theorem 2 and from Remark 2.

Example 10. Consider the situation of Example 4 and denote by $L^{\boldsymbol{\gamma},1}$ the reproducing kernel corresponding to (17) and by $L^{\boldsymbol{\gamma},2}$ the reproducing kernel corresponding to (21). For both spaces $H(L^{\boldsymbol{\gamma},1})$ and $H(L^{\boldsymbol{\gamma},2})$ we have a non-empty domain iff $\sum_{j\in\mathbb{N}} \gamma_j < \infty$, see Example 9. Applying Theorem 2 with $L^{\boldsymbol{\gamma},1}$ as the starting point, we obtain $H(L^{\boldsymbol{\gamma},1}) = H(L^{\boldsymbol{\gamma},2})$ if $\sum_{j\in\mathbb{N}} \gamma_j < \infty$, cf. Example 8.

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