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On Weighted Hilbert Spaces and Integration of Functions of Infinitely Many Variables

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ON WEIGHTED HILBERT SPACES AND INTEGRATION OF FUNCTIONS OF INFINITELY MANY VARIABLES

MICHAEL GNEWUCH, SEBASTIAN MAYER, AND KLAUS RITTER

ABSTRACT. We study aspects of the analytic foundations of integration and closely related problems for functions of infinite many variables $x_1, x_2, \ldots \in D$. The setting is based on a reproducing kernel k for functions on D, a family of non-negative weights γ_u, where u varies over all finite subsets of $\mathbb{N}$, and a probability measure ρ on D. We consider the weighted superposition $K = \sum_u \gamma_u k_u$ of finite tensor products k_u of k. Under mild assumptions we show that K is a reproducing kernel on a properly chosen domain in the sequence space $D^{\mathbb{N}}$, and that the reproducing kernel Hilbert space $H(K)$ is the orthogonal sum of the spaces $H(\gamma_u k_u)$. Integration on $H(K)$ can be defined in two ways, via a canonical representer or with respect to the product measure $\rho^{\mathbb{N}}$ on $D^{\mathbb{N}}$. We relate both approaches and provide sufficient conditions for the two approaches to coincide.

Dedicated to J. F. Traub and G. W. Wasilkowski
on the occasion of their 80th and 60th birthdays

1. Introduction

For functions of infinitely many variables $x_1, x_2, \ldots \in D$ with D denoting a non-empty set, the study of quadrature problems and their complexity was initiated in [10], and it has intensively been studied recently, see [2, 7, 8, 9, 15, 16, 17, 20] and the preprints [3, 4, 6]. In the same setting function approximation is studied in [24, 25, 26], linear tensor product problems are studied in [22], and a non-linear problem associated with elliptic PDEs with random coefficients is studied in [14, 13]. See [23] for a survey.

The present paper is devoted to some aspects of the analytic foundations of computational problems of this kind.

Let us outline the setting in the references mentioned above together with a discussion of our results. At first we consider the underlying function spaces, and then we turn to the integration functional, which is to be approximated.

The construction of spaces of functions with an infinite number of variables is based on a reproducing kernel k for functions of a single variable $x \in D$ and on a family of weights $\gamma_u \geq 0$, which indicate the importance of the group $(x_j)_{j \in u}$ of variables for finite sets $u \subseteq \mathbb{N}$. Formally, this leads to

$$K = \sum_u \gamma_u k_u,$$

where u varies over all finite subsets of $\mathbb{N}$ and where k_u is the $|u|$-fold tensor product of k such that the functions in the associated reproducing kernel Hilbert space $H(k_u)$ only depend on $(x_j)_{j \in u}$.

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In this paper we study a domain $\mathfrak{X} \subseteq D^{\mathbb{N}}$ such that K is actually a reproducing kernel on $\mathfrak{X} \times \mathfrak{X}$ and that the spaces $H(\gamma_u k_u)$ form an orthogonal decomposition of $H(K)$ under mild assumptions. The latter fact has been used in [9, 10], e.g., without providing a rigorous proof. Moreover, we show that the space $H(K)$ is isometrically isomorphic in a natural way to the quasi-reproducing kernel Hilbert space introduced and studied in [15, 20, 25, 26] for integration and function approximation.

Two different ways are used to define the integration functional for $f \in H(K)$. Both of these constructions are based on a probability measure ρ on D such that $H(k) \subseteq L_1(\rho)$, which implies

$$\int_D g \, d\rho = \langle g, h \rangle_k$$

for every $g \in H(k)$ with a representer $h \in H(k)$.

Either, one studies the Lebesgue integral with respect to the product measure $\mu = \rho^{\mathbb{N}}$ on $D^{\mathbb{N}}$. Taking into account that $\mu(\mathfrak{X}) \in \{0, 1\}$, the functions $f \in H(K)$ have to be properly extended from $\mathfrak{X}$ to $D^{\mathbb{N}}$, in particular if $\mu(\mathfrak{X}) = 0$. At this point we are free to think of the kernels k_u as being defined on $\mathfrak{X} \times \mathfrak{X}$ or $D^u \times D^u$. This distinction is indeed only of a technical nature, which will become clear when we introduce the kernels rigorously. The extension Tf is given as the L_1-limit of the orthogonal projections of f onto the spaces $H(\sum_{u \subseteq \{1,\dots,s\}} \gamma_u k_u)$, and it leads to the integral

$$I_1(f) = \int_{D^{\mathbb{N}}} Tf \, d\mu.$$

Clearly

$$I_1(f) = \int_{\mathfrak{X}} f \, d\mu$$

if $\mu(\mathfrak{X}) = 1$. Cf. [2, 3, 4, 7, 8, 6, 9, 10, 17].

Alternatively, one studies the bounded linear functional

$$I_2(f) = \langle f, \sum_u \gamma_u h_u \rangle_K$$

on $H(K)$, where

$$h_u(\mathbf{x}) = \prod_{j \in u} h(x_j), \qquad \mathbf{x} \in \mathfrak{X}.$$

Here, as previously, u varies over all finite subsets of $\mathbb{N}$. We are free to think of h_u as being defined on $\mathfrak{X}$ or D^u, so that this function is the representer of integration with respect to the product measure ρ^u on D^u. Cf. [15, 20].

We provide necessary and sufficient conditions for I_1 and I_2 to be well-defined and for $I_1 = I_2$ to hold true. In particular, we show that

$$I_1 = I_2$$

if

$$\sum_u \gamma_u \|J_u\|^2 < \infty,$$

where J_u denotes the embedding from $H(k_u)$ to $L_1(\rho^u)$.

The paper is organized as follows. In Section 2 we present the basic assumptions on the kernel k and the weights γ_u, and we introduce the domain $\mathfrak{X}$ for functions of infinitely many variables. The associated spaces of functions of finitely many and infinitely many variables are studied in Section 3, with Proposition 2 being the main result. In Section 4 we present further assumptions on k and ρ, which are then used to study the associated finite and infinite dimensional integration problems. Here the main result is Proposition 3. An appendix contains some basic results on reproducing kernel Hilbert spaces.

2. Preliminaries

Throughout this paper we use the following notation. For $\mathbf{x} = (x_j)_{j\in\mathbb{N}} \in D^{\mathbb{N}}$ and $\emptyset \neq u \subseteq \mathbb{N}$ we put $\mathbf{x}_u = (x_j)_{j\in u} \in D^u$, and u^c denotes the complement of any set $u \subseteq \mathbb{N}$. Unless stated otherwise we use u, v, and w to denote finite subsets of $\mathbb{N}$ in the sequel. For $s \in \mathbb{N}$ we let $1:s$ denote the set $\{1,\dots,s\}$.

We use basic results from [1] about reproducing kernels and the corresponding Hilbert spaces frequently without giving further references.

2.1. **Assumptions.** We assume that

(A1) $k \neq 0$ is a reproducing kernel on $D \times D$ with $D \neq \emptyset$,

which satisfies

(A2) $H(k) \cap H(1) = \{0\}$.

With

$$m = \inf_{x\in D} k(x,x)$$

we furthermore assume that

(A3) $(\gamma_u)_u$ is a family of non-negative weights such that

$$\sum_u \gamma_u\, m^{|u|} < \infty.$$

Put

$$D_0 = \{x \in D : k(x,x) = 0\}.$$

If $D_0 \neq \emptyset$ then any point in D_0 is called an anchor for the kernel k. Accordingly, we often distinguish between the anchored case $D_0 \neq \emptyset$ and the unanchored case $D_0 = \emptyset$.

Remark 1. Assumption (A3) trivially holds in the anchored case and, more generally, if $m = 0$. In the sequel, let $m > 0$. For product weights

$$\gamma_u = \prod_{j\in u} \gamma_j$$

with any sequence of real numbers $\gamma_j \geq 0$, which were introduced in [21], assumption (A3) is equivalent to

$$\sum_{j=1}^{\infty} \gamma_j < \infty. \tag{1}$$

For product and order dependent (POD) weights of the form

$$\gamma_u = |u|! \cdot \prod_{j\in u} \gamma_j,$$

see [14], condition (1) is a necessary condition for (A3), while sufficient conditions are, e.g.,

$$\sum_{j=1}^{\infty} \gamma_j < 1/m,$$

see [14, Lemma 6.2], or

$$\sum_{j=1}^{\infty} \gamma_j^{1-\varepsilon} < \infty$$

for some $\varepsilon > 0$, see [4, Cor. 1]. For finite order weights, i.e., $\gamma_u = 0$ if $|u| > r$ for some constant r, see [5], we have (A3) if and only if

$$\sum_u \gamma_u < \infty.$$

2.2. **The Domain.** In the present setting

$$\mathfrak{X} = \{\mathbf{x} \in D^{\mathbb{N}} : \sum_u \gamma_u \prod_{j\in u} k(x_j, x_j) < \infty\}$$

turns out to be the natural domain for functions of infinitely many variables.

Lemma 1. *Given* (A1)*, the assumption* (A3) *is equivalent to* $\mathfrak{X} \neq \emptyset$*. Moreover, if there exists a minimizing sequence* $(x_j)_{j\in\mathbb{N}}$ *for* k *such that* $k(x_j, x_j) > m$*, then* $\mathfrak{X}$ *contains elements with pairwise different components.*

Proof. Obviously $\mathfrak{X} \neq \emptyset$ implies (A3). To prove the reverse implication choose $\varepsilon_j > 0$ such that $\sum_{j=1}^{\infty} \varepsilon_j < \infty$. In the case $m > 0$ we may take $x_j \in D$ such that $k(x_j, x_j) \leq (1+\varepsilon_j)\, m$ to obtain

$$\sum_u \gamma_u \prod_{j\in u} k(x_j, x_j) \leq \sum_u \gamma_u m^{|u|} \cdot \prod_{j=1}^{\infty} (1+\varepsilon_j) < \infty$$

from (A3). In the case $m = 0$ we choose $x_j \in D$ such that $k(x_j, x_j) \leq 1$ and

$$2^{j-1} \cdot \max_{u\subseteq 1:j} \gamma_u \cdot k(x_j, x_j) \leq \varepsilon_j.$$

This yields $\prod_{\ell\in u} k(x_\ell, x_\ell) \leq k(x_j, x_j)$ with any $j \in u$ as well as

$$\sum_{u\neq\emptyset} \gamma_u \prod_{j\in u} k(x_j, x_j) \leq \sum_{j=1}^{\infty} 2^{j-1} \cdot \max_{u\subseteq 1:j} \gamma_u \cdot k(x_j, x_j) < \infty.$$

The second statement of the lemma is obvious now. □

For $s \in \mathbb{N}$ put

$$\mathfrak{R}_s = \bigcap_{u\subseteq 1:s} \{\mathbf{x} \in D^{(1:s)^c} : \sum_{w\subseteq (1:s)^c} \gamma_{u\cup w} \prod_{j\in w} k(x_j, x_j) < \infty\}$$

and

$$\mathfrak{X}_s = D^{1:s} \times \mathfrak{R}_s$$

as well as

$$\mathfrak{N}_s = \bigcup_{j\in 1:s} \{\mathbf{x} \in D^{\mathbb{N}} : x_j \in D_0\}.$$

Lemma 2. *For every* $s \in \mathbb{N}$ *we have*

$$\mathfrak{X} = \mathfrak{X}_s$$

in the unanchored case, and

$$D^{1:s} \times D_0^{(1:s)^c} \subseteq \mathfrak{X}_s \subseteq \mathfrak{X} \subseteq \mathfrak{X}_s \cup \mathfrak{N}_s$$

in the anchored case. In particular, $\mathfrak{X}_s \neq \emptyset$ *in both cases.*

Proof. The lemma follows easily from

$$\sum_u \gamma_u \prod_{j\in u} k(x_j,x_j) = \sum_{u\subseteq 1:s} \Big(\prod_{j\in u} k(x_j,x_j) \cdot \sum_{w\subseteq (1:s)^c} \gamma_{u\cup w} \prod_{j\in w} k(x_j,x_j) \Big)$$

and the fact that $\mathfrak{X} \neq \emptyset$, see Lemma 1. $\square$

Example 1. Let $D = \{0,1\}$, and let $k(1,1) = 1$, while $k(x,y) = 0$ otherwise. Moreover, let $\gamma_u = 1$ if $1 \in u$ or $u = \emptyset$, while $\gamma_u = 0$ otherwise. Then $\mathbf{x} \in D^{\mathbb{N}}$ belongs to $\mathfrak{X}$ if and only if $x_1 = 0$ or $x_j = 1$ for at most finitely many $j \in \mathbb{N}$. In particular, $\mathfrak{X}$ is not a Cartesian product $E_1 \times E_2$ with any sets $E_1 \subseteq D^{1:s}$ and $E_2 \subseteq D^{(1:s)^c}$ for any $s \in \mathbb{N}$. See, however, Lemma 8 below.

3. The Function Space

In this section we study the superposition of weighted tensor products of the kernel k and the associated reproducing kernel Hilbert space.

3.1. **Functions of Finitely Many Variables.**

At first we consider the reproducing kernels

$$k_u(\mathbf{x},\mathbf{y}) = \prod_{j\in u} k(x_j,y_j), \qquad \mathbf{x},\mathbf{y} \in \mathfrak{X},$$

as well as the associated reproducing kernel Hilbert spaces $H(k_u)$. By definition, $k_\emptyset = 1$ so that $H(k_\emptyset)$ consists of all constant functions on $\mathfrak{X}$.

The following fact is an immediate consequence of the reproducing property, and it shows in particular that the functions in $H(k_u)$ only depend on finitely many variables.

Lemma 3. *For $\mathbf{x},\mathbf{y} \in \mathfrak{X}$ and $f \in H(k_u)$ we have*

$$\mathbf{x}_u = \mathbf{y}_u \quad \Rightarrow \quad f(\mathbf{x}) = f(\mathbf{y}).$$

In a second step we define for any v the weighted sums

$$K_v(\mathbf{x},\mathbf{y}) = \sum_{u\subseteq v} \gamma_u k_u(\mathbf{x},\mathbf{y}), \qquad \mathbf{x},\mathbf{y} \in \mathfrak{X},$$

of reproducing kernels k_u. Clearly K_v is a reproducing kernel, too, and

$$H(K_v) = \Big\{ \sum_{u\subseteq v} f_u : f_u \in H(\gamma_u k_u) \Big\}.$$

If $\gamma_u > 0$ then $H(\gamma_u k_u) = H(k_u)$ and

$$\|f_u\|_{\gamma_u k_u} = \gamma_u^{-1/2} \, \|f_u\|_{k_u}, \qquad f \in H(k_u).$$

The tensor product form of the kernels k_u allows to deduce that $H(k_u) \cap H(k_v) = \{0\}$ if $u \neq v$. Inductively, it follows that $\sum_{u\subseteq 1:s} f_u = 0$ with $f_u \in H(k_u)$ implies $f_u = 0$ for all $u \subseteq 1:s$. See, e.g., [9, p. 233] for a proof in the case $D \subseteq \mathbb{R}$, which literally carries over to the case of arbitrary sets D. We refer to Lemma 11 for further equivalent formulations. In particular, this yields the following well-known fact.

Proposition 1. *The spaces $H(\gamma_u k_u)$ with $u \subseteq 1:s$ are pairwise orthogonal in $H(K_{1:s})$.*

Remark 2. In general, Proposition 1 does not hold for arbitrary kernels k_u such that $H(k_u) \cap H(k_v) = \{0\}$ for $u \neq v$ and that $f_u \in H(k_u)$ only depends on the variables x_j with $j \in u$, cf. Lemma 3. As a counterexample, let $\gamma_u = 1$ for every u, and consider $D = \mathbb{R}$ as well as

$$\begin{aligned} k_\emptyset(\mathbf{x}, \mathbf{y}) &= 1, \\ k_{\{1\}}(\mathbf{x}, \mathbf{y}) &= x_1 \cdot y_1, \\ k_{\{2\}}(\mathbf{x}, \mathbf{y}) &= x_2 \cdot y_2, \\ k_{\{1,2\}}(\mathbf{x}, \mathbf{y}) &= (x_1 + x_2) \cdot (y_1 + y_2). \end{aligned}$$

Then $H(k_\emptyset) = \operatorname{span}\{1\}$, $H(k_{\{1\}}) = \operatorname{span}\{x_1\}$, $H(k_{\{2\}}) = \operatorname{span}\{x_2\}$, and $H(k_{\{1,2\}}) = \operatorname{span}\{x_1 + x_2\}$.

For the sake of completeness we show that every function $f \in H(K_{1:s})$ may indeed be identified with a function on $D^{1:s}$ or $D^{\mathbb{N}}$, and $K_{1:s}$ may be identified with a kernel on $D^{1:s} \times D^{1:s}$ or $D^{\mathbb{N}} \times D^{\mathbb{N}}$, as well. It will be an immediate consequence of Proposition 1, Lemma 3, and Lemma 4 below. Put

$$l_u(\mathbf{x}, \mathbf{y}) = \prod_{j \in u} k(x_j, y_j), \qquad \mathbf{x}, \mathbf{y} \in D^u,$$

which formally differs from k_u, since the underlying domain is D^u instead of $\mathfrak{X}$. For $f \in H(l_u)$ we define the mapping $\psi_u f : \mathfrak{X} \to \mathbb{R}$ by

$$\psi_u f(\mathbf{x}) = f(\mathbf{x}_u), \qquad \mathbf{x} \in \mathfrak{X}.$$

Lemma 4. *The mapping ψ_u defines an isometric isomorphism between $H(l_u)$ and $H(k_u)$.*

Proof. Let $a^{(i)} \in \mathbb{R}$ and $\mathbf{y}^{(i)} \in D^u$ for $i = 1, \ldots, n$. Due to Lemma 2 there exist $\mathbf{x}^{(i)} \in \mathfrak{X}$ such that $\mathbf{x}_u^{(i)} = \mathbf{y}^{(i)}$. Put

$$f = \sum_{i=1}^{n} a^{(i)}\, l_u(\cdot, \mathbf{y}^{(i)})$$

to obtain

$$\psi_u f = \sum_{i=1}^{n} a^{(i)}\, k_u(\cdot, \mathbf{x}^{(i)}),$$

which in particular yields $\|f\|_{l_u} = \|\psi_u f\|_{k_u}$. We conclude that ψ_u defines a linear isometry between $\operatorname{span}\{l_u(\cdot, \mathbf{x}) : \mathbf{x} \in D^u\}$ and $H(k_u)$ with a dense range. Use the reproducing property to complete the proof. □

Clearly, we have $\psi_u^{-1} f(\mathbf{y}) = f(\mathbf{x})$ for every $f \in H(k_u)$ and all $\mathbf{y} \in D^u$ and $\mathbf{x} \in \mathfrak{X}$ with $\mathbf{x}_u = \mathbf{y}$.

3.2. **Functions of Infinitely Many Variables.** Finally, we consider the limit

$$K(\mathbf{x}, \mathbf{y}) = \sum_u \gamma_u k_u(\mathbf{x}, \mathbf{y}), \qquad \mathbf{x}, \mathbf{y} \in \mathfrak{X},$$

of the sequence of kernels $K_{1:s}$. Note that $\mathfrak{X}$ is the set of points $\mathbf{x} \in D^{\mathbb{N}}$ such that $K_{1:s}(\mathbf{x}, \mathbf{x})$ converges as s tends to infinity. Hence $\sum_u \gamma_u |k_u(\mathbf{x}, \mathbf{y})| < \infty$ for all $\mathbf{x}, \mathbf{y} \in \mathfrak{X}$, and K is a reproducing kernel on $\mathfrak{X} \times \mathfrak{X}$. Since $K - K_v$ is a reproducing kernel, too, we have $H(K_v) \subseteq H(K)$ and $\|f\|_{K_v} \geq \|f\|_K$ for all $f \in H(K_v)$. As a consequence of Proposition 1 and Proposition 2 below we actually have equality of the norms.

Consider the direct sum

$$H = \{(f_u)_u : f_u \in H(\gamma_u k_u),\ \sum_u \|f_u\|^2_{\gamma_u k_u} < \infty\}$$

of the spaces $H(\gamma_u k_u)$, equipped with the scalar product

$$\langle (f_u)_u, (g_u)_u \rangle = \sum_u \langle f_u, g_u \rangle_{\gamma_u k_u}.$$

For $(f_u)_u \in H$ and $\mathbf{x} \in \mathfrak{X}$ we have

$$\sum_u |f_u(\mathbf{x})| \le \sum_u \|f_u\|_{\gamma_u k_u} \cdot (\gamma_u k_u)^{1/2}(\mathbf{x}, \mathbf{x}) \le \|(f_u)_u\| \cdot K^{1/2}(\mathbf{x}, \mathbf{x}) < \infty.$$

Hence the mapping

$$\phi : H \to \mathbb{R}^{\mathfrak{X}}, \quad (f_u)_u \mapsto \sum_u f_u,$$

where we have absolute convergence of the series at every point in $\mathfrak{X}$, is well-defined.

Proposition 2. *The direct sum H is isometrically isomorphic to $H(K)$ via ϕ. This means that the spaces $H(\gamma_u k_u)$ with finite sets $u \subset \mathbb{N}$ are pairwise orthogonal in $H(K)$.*

Note that Proposition 1 is a particular case of Proposition 2, namely, if $\gamma_u = 0$ for $u \not\subseteq 1:s$, see Lemma 11. Proposition 2 is used in [9, 10] in the anchored and unanchored case without providing a rigorous proof.

To cover the unanchored case in the proof of Proposition 2, we need some additional lemmas. Hence we restrict considerations temporarily to the unanchored case $D_0 = \emptyset$. For $s \in \mathbb{N}$ and $u \subseteq 1:s$ we consider the reproducing kernel

$$J_u^{(s)}(\mathbf{x}, \mathbf{y}) = k_u(\mathbf{x}, \mathbf{y}) \cdot \sum_{w \subseteq (1:s)^c} \gamma_{u \cup w}\, k_w(\mathbf{x}, \mathbf{y}) = \sum_{w \subseteq (1:s)^c} \gamma_{u \cup w}\, k_{u \cup w}(\mathbf{x}, \mathbf{y})$$

on $\mathfrak{X} \times \mathfrak{X}$, which is well-defined due to Lemma 2. Clearly

$$K = \sum_{u \subseteq 1:s} J_u^{(s)} \tag{2}$$

and $H(\gamma_u k_u) \subseteq H(J_u^{(s)})$.

Lemma 5. *In the unanchored case the spaces $H(J_u^{(s)})$ with $u \subseteq 1:s$ are pairwise orthogonal in $H(K)$ for every $s \in \mathbb{N}$.*

Proof. Apply Proposition 1 with equal weights $\gamma_u = 1$ and for the domain $D^{1:s}$ to conclude that the spaces $H(k_u)$ with $u \subseteq 1:s$ are pairwise orthogonal in the Hilbert space with reproducing kernel $\sum_{u \subseteq 1:s} k_u$. By Lemma 13 and (2) the same property holds for the spaces $H(J_u^{(s)})$ in $H(K)$. □

Lemma 6. *Consider the unanchored case. For every u there exists a constant $c_u \ge 1$ such that*

$$\|f\|_{\gamma_u k_u} = c_u \cdot \|f\|_K$$

for every $f \in H(\gamma_u k_u)$.

Proof. If $\gamma_u = 0$ then $H(\gamma_u k_u) = \{0\}$, and we choose $c_u = 1$. In the sequel we assume that $\gamma_u > 0$. Put $s = \max(u \cup \{1\})$, which yields $u \subseteq 1:s$. Moreover, put $E_1 = D^{1:s}$ and $E_2 = \mathfrak{R}_s$. We have $\mathfrak{X} = E_1 \times E_2$ according to Lemma 2. Note that $J_u^{(s)} = L \otimes M$ with reproducing kernels

$$L(\mathbf{x}, \mathbf{y}) = l_u(\mathbf{x}_u, \mathbf{y}_u), \qquad \mathbf{x}, \mathbf{y} \in E_1,$$

and

$$M(\mathbf{x}, \mathbf{y}) = \sum_{w \subseteq (1:s)^c} \gamma_{u \cup w} \, l_w(\mathbf{x}_w, \mathbf{y}_w), \qquad \mathbf{x}, \mathbf{y} \in E_2.$$

Let $f \in H(k_u)$. For the section

$$g(\mathbf{x}) = f(\mathbf{x}, x_{s+1}, \dots), \qquad \mathbf{x} \in E_1,$$

with any $(x_{s+1}, \dots) \in E_2$ we have $g \in H(L)$ and $\|g\|_L = \|f\|_{k_u}$, see, e.g., [9, Lemma 16] and its proof. Note that $1 \in H(M)$, since $\gamma_u > 0$. Put $e_0 = 1/\|1\|_M$ and extend this element to an orthonormal base $(e_m)_m$ of $H(M)$. Furthermore, let $(d_\ell)_\ell$ be an orthonormal base of $H(L)$. For

$$h = \sum_\ell \langle g, d_\ell \rangle_L \cdot d_\ell \otimes e_0$$

we have $h \in H(J_u^{(s)})$ and

$$h(\mathbf{x}) = g(x_1, \dots, x_s) \cdot e_0 = f(\mathbf{x}) \cdot e_0, \qquad \mathbf{x} \in \mathfrak{X},$$

due to Lemma 3. Furthermore, we have $\|g\|_L = \|h\|_{J_u^{(s)}} = \|h\|_K$, where the last identity is due to Lemma 5. Consequently,

$$\|f\|^2_{\gamma_u k_u} = \gamma_u^{-1} \cdot \|f\|^2_{k_u} = \gamma_u^{-1} \cdot \|g\|^2_L = \gamma_u^{-1} \cdot \|h\|^2_K = c_u^2 \cdot \|f\|^2_K$$

with

$$c_u^2 = e_0^2/\gamma_u = 1/\left(\|1\|^2_M \cdot \gamma_u\right).$$

Since $M - \gamma_u$ is a reproducing kernel, we get $\|1\|_M \leq \|1\|_{\gamma_u} = \gamma_u^{-1/2}$, which implies $c_u \geq 1$. □

Proof of Proposition 2. We commence by showing that ϕ is injective.

First we consider the unanchored case. Let H_0 be the subspace of H that consists of all sequences $(f_u)_u$ such that $f_u = 0$ for all but finitely many u. Then the mapping

$$\chi : H_0 \to H(K), \quad (f_u)_u \mapsto \sum_u c_u f_u,$$

where the constants c_u are as in Lemma 6, is well-defined. Moreover, if $s \in \mathbb{N}$ and $f_u = 0$ for all $u \not\subseteq 1:s$, then

$$\|\chi(f_u)_u\|^2_K = \sum_{u \subseteq 1:s} \|c_u f_u\|^2_K = \sum_{u \subseteq 1:s} \|f_u\|^2_{\gamma_u k_u} = \|(f_u)_u\|^2,$$

see Lemma 5 and 6. Thus χ can be uniquely extended to a linear isometry $\chi : H \to H(K)$. Notice that for $(f_u)_u \in H$ we have necessarily $\chi(f_u)_u = \lim_{s \to \infty} \sum_{u \subseteq 1:s} f_u$, where the sequence converges in $H(K)$ and therefore also pointwise. Fix $\mathbf{x} \in \mathfrak{X}$ and consider the special choice $f_u = \gamma_u k_u(\cdot, \mathbf{x})$ for all u. Then

$$\sum_u c_u \gamma_u k_u(\mathbf{x}, \mathbf{x}) = |\chi(f_u)_u(\mathbf{x})| \le \|\chi(f_u)_u\|_K \cdot K^{1/2}(\mathbf{x}, \mathbf{x})$$
$$= \|(f_u)_u\| \cdot K^{1/2}(\mathbf{x}, \mathbf{x}) = \sum_u \gamma_u k_u(\mathbf{x}, \mathbf{x}).$$

Since $k_u(\mathbf{x}, \mathbf{x}) > 0$ and $c_u \ge 1$, it follows that $c_u = 1$ for each u with $\gamma_u > 0$. This means $\chi = \phi$ so that, in particular, ϕ is injective.

Let us now consider the anchored case. Assume that $\phi(f_u)_u = 0$ for some $(f_u)_u \in H$. For $\mathbf{x} \in \mathfrak{X}$ and a given u we define $\mathbf{y} \in D^{\mathbb{N}}$ by $y_j = x_j$ if $j \in u$ and $y_j = a$ for $j \notin u$, where $a \in D_0$. Note that $\mathbf{y} \in \mathfrak{X}$ due to Lemma 2, and $f_v(\mathbf{y}) = \langle f_v, k_v(\cdot, \mathbf{y})\rangle_{k_v} = 0$ if $v \not\subseteq u$. Thus

$$0 = \sum_v f_v(\mathbf{y}) = \sum_{v \subseteq u} f_v(\mathbf{y}) = \sum_{v \subseteq u} f_v(\mathbf{x}),$$

see Lemma 3. Via induction over the cardinality of u we obtain $(f_u)_u = 0$, so that ϕ is injective.

In both cases we consider the Hilbert space $\phi(H)$, endowed with the scalar product

$$\langle f, g\rangle_\phi = \langle \phi^{-1}(f), \phi^{-1}(g)\rangle, \qquad f, g \in \phi(H).$$

Choosing $f_u = \gamma_u k_u(\cdot, \mathbf{x})$ for all u, we see that $\phi(f_u)_u = K(\cdot, \mathbf{x}) \in \phi(H)$ for every $\mathbf{x} \in \mathfrak{X}$. For $(g_u)_u \in H$ we get

$$\langle \phi((g_u)_u), K(\cdot, \mathbf{x})\rangle_\phi = \sum_u \langle g_u, \gamma_u k_u(\cdot, \mathbf{x})\rangle_{\gamma_u k_u} = \sum_u g_u(\mathbf{x}) = \phi(g_u)_u(\mathbf{x}).$$

Hence $\phi(H) = H(K)$ and ϕ is an isometric isomorphism between H and $H(K)$. $\square$

Remark 3. The direct sum H, which is a completion of $\widetilde{H} = \operatorname{span} \bigcup_u H(\gamma_u k_u)$, is studied in [15, 20, 24, 25, 26], and it is called a quasi-reproducing kernel Hilbert space. In the sense of Proposition 2, H actually is a reproducing kernel Hilbert space. However, while the elements in $\widetilde{H}$ may be considered as functions on $D^{\mathbb{N}}$ this is no longer true, in general, for the elements in H in the sense that

$$\forall\, (f_u)_u \in H: \ \sum_u |\psi_u^{-1} f_u(\mathbf{y}_u)| < \infty$$

does not necessarily hold for every $\mathbf{y} \in D^{\mathbb{N}}$, see, e.g., Example 2 below. This is avoided, if $\mathfrak{X}$ is considered as the underlying domain instead of $D^{\mathbb{N}}$.

Remark 4. Let us impose (A1) and (A2) only. Furthermore, we denote by G the direct sum of the spaces $H(\gamma_u l_u)$. Now condition (A3) is equivalent to the following: There exists a point $\mathbf{y} \in D^{\mathbb{N}}$ such that the series $\sum_u |g_u(\mathbf{y}_u)|$ converges for every $(g_u)_u \in G$ and $(g_u)_u \mapsto \sum_u g_u(\mathbf{y}_u)$ yields a bounded linear functional on G. For the proof observe that for the latter functional the representer is then given by $(\gamma_u l_u(\cdot, \mathbf{y}_u))_u$, and therefore

$$\sum_u \gamma_u l_u(\mathbf{y}_u, \mathbf{y}_u) < \infty.$$

Consequently, we have $\mathfrak{X} \neq \emptyset$, and Lemma 1 yields (A3). See Proposition 2 for the reverse implication.

4. The Integration Problem

In this section we study the integration problem for the functions from $H(K)$, when a probability measure ρ on D with $H(k) \subseteq L_1(\rho)$ is given. We analyze and compare two different approaches, namely integration with respect to the product measure $\rho^{\mathbb{N}}$ on $D^{\mathbb{N}}$ and the definition of an integration functional by means of a representer $h^* \in H(K)$.

4.1. **Assumptions.** Let D be equipped with a σ-algebra, and let Cartesian products of D be equipped with the respective product σ-algebras.

In addition to (A1)–(A3) we assume that

(A4) ρ is a probability measure on D,
(A5) k is measurable and $\rho(D_0) = 0$,

and

(A6) $H(k) \subseteq L_1(\rho)$.

Notice that $\rho(D_0) = 0$ holds if and only if there does not exist a measurable subset of D with positive measure such that all functions in $H(k)$ vanish on this subset. Furthermore, by (A5), the sets $\mathfrak{X}$, $\mathfrak{X}_s$, $\mathfrak{Y}_s$, and $\mathfrak{R}_s$, which are introduced in Section 2.2, are measurable subsets of $D^{\mathbb{N}}$ or $D^{(1:s)^c}$, respectively.

A sufficient condition to ensure (A6) is

$$\int_D k^{1/2}(x,x)\, d\rho(x) < \infty, \tag{3}$$

since

$$\int_D |g(x)|\, d\rho(x) \le \|g\|_k \int_D k^{1/2}(x,x)\, d\rho(x), \qquad g \in H(k). \tag{4}$$

The closed graph theorem implies that if a reproducing kernel Hilbert space is contained in an L_1-space, then the respective embedding is continuous, see, e.g., [12, p. 126]. Thus, (A6) already yields the continuity of the embedding

$$J : H(k) \to L_1(\rho),$$

so that there exists a function $h \in H(k)$ with

$$\int_D g\, d\rho = \langle g, h\rangle_k, \qquad g \in H(k).$$

Clearly $h(x) = \langle h, k(\cdot,x)\rangle_k = \int_D k(y,x)\, d\rho(y)$ for every $x \in D$, and therefore

$$\|h\|_k^2 = \int_D \int_D k(x,y)\, d\rho(x)\, d\rho(y).$$

We stress that the right-hand side is well defined as an iterated integral, while we do not claim that k is integrable with respect to $\rho \otimes \rho$, in general. We refer to [18, NR 23.4.2] for further discussion of assumption (A6).

4.2. **Finite-Dimensional Integration.** For every set $\emptyset \neq u \subseteq \mathbb{N}$, not necessarily finite, we let ρ^u denote the corresponding product of ρ on D^u. Put

$$h_u(\mathbf{x}) = \prod_{j \in u} h(x_j), \qquad \mathbf{x} \in D^u.$$

Clearly $h_u \in H(l_u)$ and

$$\|h_u\|_{l_u} = \|h\|_k^{|u|}. \tag{5}$$

On the other hand, $l_u(\cdot, \mathbf{x}) \in L_1(\rho^u)$ and

$$\int_{D^u} l_u(\cdot, \mathbf{x})\, d\rho^u = h_u(\mathbf{x}).$$

We conclude that $f \in L_1(\rho^u)$ and

$$\int_{D^u} f\, d\rho^u = \langle f, h_u \rangle_{l_u} \tag{6}$$

for every $f \in \operatorname{span}\{l_u(\cdot, \mathbf{x}) : \mathbf{x} \in D^u\}$. We are not aware of an elementary proof of the following result, which only employs the assumptions (A1)–(A6).

Lemma 7. *Suppose that $u \neq \emptyset$. Then $H(l_u) \subseteq L_1(\rho^u)$, and the norm of the respective embedding J_u satisfies*

$$\|h\|_k^{|u|} \leq \|J_u\| \leq (\sqrt{\pi/2} \cdot \|J\|)^{|u|}.$$

Furthermore,

$$\int_{D^u} f\, d\rho^u = \langle f, h_u \rangle_{l_u}, \qquad f \in H(l_u).$$

Proof. We use some argument from [12]. Due to the Little Grothendieck Theorem, see, e.g., [19, 22.4.2], the dual operator $J' : L_\infty(\rho) \to H(k)$ of J is 2-summable with 2-summing norm

$$\pi_2(J') \leq \sqrt{\pi/2} \cdot \|J\|.$$

This implies that there exists a probability density $\varphi : D \to]0, \infty[$ with respect to ρ such that $H(k) \subseteq L_2(1/\varphi\, d\rho)$ and the norm of the embedding

$$A : H(k) \to L_2(1/\varphi\, d\rho)$$

satisfies

$$\|A\| \leq \pi_2(J').$$

See [12, p. 129]. Take tensor products to obtain

$$\|A_u\| \leq (\pi_2(J'))^{|u|}$$

for the embedding

$$A_u : H(l_u) \to L_2(\otimes_{j \in u}(1/\varphi\, d\rho)).$$

Due to the Cauchy Schwarz inequality $L_2(\otimes_{j \in u}(1/\varphi\, d\rho))$ is embedded into $L_1(\rho^u)$ with norm one. Consequently, $H(l_u) \subseteq L_1(\rho^u)$ with an embedding of norm at most $(\pi_2(J'))^{|u|}$.

This shows in particular that integration with respect to ρ^u defines a bounded linear functional on $H(l_u)$, and by (6) the function h_u necessarily is the representer of this functional. Together with (5) this yields the lower bound for $\|J_u\|$ as claimed. □

Remark 5. Let us discuss the estimate for the norm of J_u from Lemma 7 for different types of kernels k.

For ANOVA-type kernels k, i.e., if $h = 0$, the lower bound is sharp only in the trivial case that all functions from $H(k)$ vanish ρ-a.e.

If k is non-negative then

$$\|J_u\| = \|J\|^{|u|} = \|h\|_k^{|u|}.$$

We present a short proof of this fact, see [11]. For the dual operator $J_u' : L_\infty(\rho^u) \to H(l_u)$ of J_u and for $g \in L_\infty(\rho^u)$ and $\mathbf{x} \in D^u$ we have

$$J_u' g(\mathbf{x}) = \langle J_u' g, l_u(\cdot, \mathbf{x}) \rangle_{l_u} = \int_{D^u} l_u(\mathbf{x}, \mathbf{y}) g(\mathbf{y}) \, d\rho^u(\mathbf{y}).$$

Since $l_u(\mathbf{x}, \mathbf{y}) \geq 0$ for all $x, y \in D^u$, we obtain

$$\begin{aligned} \|J_u\|^2 &= \sup_{\|g\|_\infty = 1} \|J_u' g\|_{l_u}^2 = \sup_{\|g\|_\infty = 1} \int_{D^u} \int_{D^u} l_u(\mathbf{x}, \mathbf{y}) g(\mathbf{x}) g(\mathbf{y}) \, d\rho^u(\mathbf{x}) \, d\rho^u(\mathbf{y}) \\ &= \int_{D^u} \int_{D^u} l_u(\mathbf{x}, \mathbf{y}) \, d\rho^u(\mathbf{x}) \, d\rho^u(\mathbf{y}) = \left(\int_D \int_D k(x, y) \, d\rho(x) \, d\rho(y) \right)^{|u|} = \|h\|_k^{2|u|}. \end{aligned}$$

The equality $\|J_u\| = \|J\|^{|u|}$ may also hold for kernels with a change of sign. For instance, if k is of product form $k(x, y) = \kappa(x)\kappa(y)$ with $\kappa : D \to \mathbb{R}$, then $H(k) = \operatorname{span}\{\kappa\}$ and $\|\kappa\|_k = 1$. It follows that

$$\|J_u\|^2 = \left(\int_D \int_D |k(x, y)| \, d\rho(x) \, d\rho(y) \right)^{|u|} = \|J\|^{2|u|}. \tag{7}$$

If (3) is satisfied then

$$\|J_u\| \leq \left(\int_D k^{1/2}(x, x) \, d\rho(x) \right)^{|u|}, \tag{8}$$

which is verified analogously to (4), provides an alternative to the upper bound from Lemma 7.

4.3. **Infinite-Dimensional Integration.** In the sequel we study integration with respect to the probability measure

$$\mu = \rho^{\mathbb{N}}$$

on $D^{\mathbb{N}}$.

Lemma 8. *For every $s \in \mathbb{N}$ we have*

$$\mu(\mathfrak{X}) = \mu(\mathfrak{X}_s) \in \{0, 1\}.$$

Proof. From (A5) we get $\mu(\mathfrak{N}_s) = 0$, so that Lemma 2 implies $\mu(\mathfrak{X}) = \mu(\mathfrak{X}_s)$.

We apply Kolmogorov's 0-1 law to derive $\mu(\mathfrak{X}) \in \{0, 1\}$. To this end we put $Y_j(\mathbf{x}) = k(x_j, x_j)$ for $j \in \mathbb{N}$ to obtain an independent sequence of random variables on $D^{\mathbb{N}}$, equipped with the probability measure μ. Let $\mathcal{A}_\infty$ be the terminal σ-algebra associated to $(Y_j)_{j \in \mathbb{N}}$, i.e., $\mathcal{A}_\infty = \cap_{s \in \mathbb{N}} \mathcal{A}_s$ with $\mathcal{A}_s = \sigma(\{Y_j : j \geq s\})$.

In the unanchored case we have $\mathfrak{X} = \mathfrak{X}_s \in \mathcal{A}_s$ for every $s \in \mathbb{N}$ due to Lemma 2. Hence $\mathfrak{X} \in \mathcal{A}_\infty$, and therefore $\mu(\mathfrak{X}) \in \{0, 1\}$. To deal with the anchored case, we show that $(\mathfrak{X}_s)_{s \in \mathbb{N}}$ is decreasing, which is also true in the unanchored case. In fact, for $s < t$, $\mathbf{x} \in \mathfrak{X}_t$, and $u' \subseteq 1 : s$ we obtain

$$\sum_{w' \subseteq (1:s)^c} \gamma_{u' \cup w'} \prod_{j \in w'} k(x_j, x_j) = \sum_{v \subseteq (1:s)^c \cap (1:t)} \prod_{i \in v} k(x_i, x_i) \sum_{w \subseteq (1:t)^c} \gamma_{(u' \cup v) \cup w} \prod_{j \in w} k(x_j, x_j) < \infty,$$

implying $\mathbf{x} \in \mathfrak{X}_s$. For $\mathfrak{X}_\infty = \cap_{t \in \mathbb{N}} \mathfrak{X}_t$ it follows that $\mathfrak{X}_\infty = \cap_{t \geq s} \mathfrak{X}_t \in \mathcal{A}_s$ for every $s \in \mathbb{N}$, i.e., $\mathfrak{X}_\infty \in \mathcal{A}_\infty$, and therefore $\mu(\mathfrak{X}_\infty) \in \{0, 1\}$. It remains to observe that

$$\mu(\mathfrak{X}_\infty) = \lim_{s \to \infty} \mu(\mathfrak{X}_s) = \mu(\mathfrak{X}).$$

□

Let us introduce the conditions

(C1) $$\sum_u \gamma_u \|h\|_k^{2|u|} < \infty,$$

(C2) $$\sum_u \gamma_u \|J_u\|^2 < \infty,$$

(C3) $$\sum_u \gamma_u \left(\int_D k(x,x)\, d\rho(x)\right)^{|u|} < \infty.$$

Note that (C2) implies (C1) according to Lemma 7, and (C3) implies (C2), see (8). Both implications are strict. For the first implication one may take a non-trivial ANOVA-type kernel and suitable weights, see Remark 5, and for the second implication we refer to Example 2.

Lemma 9. *If* (C3) *is satisfied, then* $\mu(\mathfrak{X}) = 1$*.*

Proof. Put $Y_u(\mathbf{x}) = \gamma_u k_u(\mathbf{x}, \mathbf{x})$ for all u to obtain a family of non-negative random variables on $D^{\mathbb{N}}$, equipped with the probability measure μ. Thus

$$\mathrm{E}\left(\sum_u Y_u\right) = \sum_u \gamma_u \int_{D^{\mathbb{N}}} k_u(\mathbf{x}, \mathbf{x})\, d\mu(\mathbf{x}) = \sum_u \gamma_u \left(\int_D k(x,x)\, d\rho(x)\right)^{|u|}.$$

Hence $\sum_u Y_u$ is finite μ-almost surely, i.e., $\mu(\mathfrak{X}) = 1$, if (C3) is satisfied. $\square$

To define the integral of $f \in H(K)$ with respect to μ we need a proper extension of f from $\mathfrak{X}$ to $D^{\mathbb{N}}$ in the case $\mu(\mathfrak{X}) = 0$. This extension is based on partial sums of the orthogonal decomposition $f = \sum_u f_u$ with $f_u \in H(\gamma_u k_u)$, see Proposition 2, and obviously it works as well in the case $\mu(\mathfrak{X}) = 1$. Recall the definition and properties of the mapping ψ_u from Section 3.1. For $s \in \mathbb{N}$ we define $f^{(s)} : D^{\mathbb{N}} \to \mathbb{R}$ by

$$f^{(s)}(\mathbf{y}) = \sum_{u \subseteq 1:s} \psi_u^{-1} f_u(\mathbf{y}_u), \qquad \mathbf{y} \in D^{\mathbb{N}}.$$

For every $\mathbf{y} \in D^{\mathbb{N}}$ there exists a point $\mathbf{x} \in \mathfrak{X}$ such that $\mathbf{x}_{1:s} = \mathbf{y}_{1:s}$, see Lemma 2, and for any such $\mathbf{x}$ we have

$$f^{(s)}(\mathbf{y}) = \sum_{u \subseteq 1:s} f_u(\mathbf{x}).$$

Clearly

$$\lim_{s \to \infty} f^{(s)}(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \in \mathfrak{X}.$$

Moreover, $f^{(s)} \in L_1(\mu)$, which follows from Lemma 4, Lemma 7, and the fact that ρ^u is the image of μ under $\mathbf{x} \mapsto \mathbf{x}_u$.

We are interested in the following property:

(E) For every $f \in H(K)$ the sequence $(f^{(s)})_s$ converges in $L_1(\mu)$, and

$$Tf = \lim_{s \to \infty} f^{(s)}$$

defines a bounded linear mapping

$$T : H(K) \to L_1(\mu).$$

Lemma 10. *Suppose that $\mu(\mathfrak{X}) = 1$ and that the sequence $(f^{(s)})_s$ converges in $L_1(\mu)$ for every $f \in H(K)$. Then property* (E) *holds true and $H(K) \subseteq L_1(\mu|_{\mathfrak{X}})$, where $\mu|_{\mathfrak{X}}$ denotes the restriction of μ to measurable subsets of $\mathfrak{X}$.*

Proof. It remains to show the continuity of T. Let $f, f_n \in H(K)$ and $g \in L_1(\mu)$ such that $\lim_{n\to\infty} \|f_n - f\|_K = 0$ as well as $\lim_{n\to\infty} \|Tf_n - g\|_{L_1(\mu)} = 0$. Moreover, let $\tilde{f}$ and $\tilde{f}_n$ denote the extensions of f and f_n by zero to the domain $D^{\mathbb{N}}$. Since $\mu(\mathfrak{X}) = 1$, we get $Tf = \tilde{f}$ and $Tf_n = \tilde{f}_n$ μ-a.s. Apply the closed graph theorem. □

Proposition 3. *We have*

$$(\text{C2}) \quad \Rightarrow \quad (\text{E}) \quad \Rightarrow \quad (\text{C1}).$$

If (C1) *is satisfied, then*

$$h^* = \sum_u \gamma_u \psi_u h_u \in H(K)$$

and

$$\langle f, h^* \rangle_K = \sum_u \int_{D^u} \psi_u^{-1} f_u \, d\rho^u$$

for every $f = \sum_u f_u \in H(K)$ with $f_u \in H(\gamma_u k_u)$. If (E) *holds true, then*

$$\langle f, h^* \rangle_K = \int_{D^{\mathbb{N}}} Tf \, d\mu$$

for every $f \in H(K)$.

Proof. Recall that $H(l_u) \subseteq L_1(\rho^u)$ and h_u is the representer for integration with respect to ρ^u on $H(l_u)$, if $u \neq \emptyset$, see Lemma 7. In the sequel, let $f = \sum_u f_u \in H(K)$ with $f_u \in H(\gamma_u k_u)$.

Assume that (C1) is satisfied. Then $\sum_u \gamma_u \psi_u h_u \in H(K)$, see Lemma 4 and (5), and we obtain

$$\langle f, h^* \rangle_K = \sum_u \langle f_u, \psi_u h_u \rangle_{k_u} = \sum_u \int_{D^u} \psi_u^{-1} f_u \, d\rho^u,$$

see Proposition 2 and Lemma 4.

Assume that (E) holds true. Then there exists a function $h^* \in H(K)$ such that

$$\langle f, h^* \rangle_K = \int_{D^{\mathbb{N}}} Tf \, d\mu$$

for every $f \in H(K)$. In particular,

$$\langle f_u, h^* \rangle_K = \int_{D^u} \psi_u^{-1} f_u \, d\rho^u = \langle f_u, \gamma_u \psi_u h_u \rangle_K,$$

see Proposition 2 and Lemma 4, so that $h^* = \sum_u \gamma_u \psi_u h_u$. This implies (C1).

Note that

$$\|f^{(s)}\|_{L_1(\mu)} \leq \sum_{u \subseteq 1:s} \|\psi_u^{-1} f_u\|_{L_1(\rho^u)} \leq \left(\sum_{u \subseteq 1:s} \gamma_u \|J_u\|^2 \right)^{1/2} \cdot \|f\|_K$$

for every $s \in \mathbb{N}$, which follows from Proposition 2 and Lemma 7. Therefore (C2) guarantees (E) to hold true. □

Roughly speaking, Proposition 3 shows the following. If (C1) is satisfied, then the integral of $f \in H(K)$ may be understood as a series of finite dimensional integrals of its components f_u with respect to the product measures ρ^u, and the associated representer is the function h^*. If (E) holds true, then all functions in $H(K)$, properly extended from $\mathfrak{X}$ to $D^{\mathbb{N}}$ if $\mu(X) = 0$, are integrable with respect to μ, and h^* is the representer of the integration functional.

Example 2. Consider the uniform distribution ρ on $D = [0, 1]$, and let $k = \min$. Moreover, fix $\alpha > 0$, and let

$$\gamma_u = \alpha^s$$

if $u = 1 : s$ for any $s \in \mathbb{N}$, while $\gamma_u = 0$ otherwise. See [26, Exmp. 1]. In this case the assumptions (A1)–(A6) are satisfied, and $h(x) = x \cdot (1 - x/2)$ so that $\|h\|_k^2 = 1/3$.

Observe that

$$\text{(C2)} \quad \Leftrightarrow \quad \text{(E)} \quad \Leftrightarrow \quad \text{(C1)} \quad \Leftrightarrow \quad \alpha < 3,$$

see Remark 5 and Proposition 3.

We claim that

$$\mu(\mathfrak{X}) = 1 \quad \Leftrightarrow \quad \alpha < \exp(1).$$

To verify this fact we put $Y_j(\mathbf{x}) = -\ln(x_j)$ for $\mathbf{x} \in D^{\mathbb{N}}$ with $x_j > 0$ to obtain an independent sequence of random variables on $D^{\mathbb{N}}$, equipped with the probability measure μ. Note that Y_j is exponentially distributed with parameter one, so that $\mathrm{E}(Y_j) = 1$ and $\mathrm{Var}(Y_j) = 1$. The Strong Law of Large Numbers yields

$$\lim_{s \to \infty} \frac{1}{s} \sum_{j \in 1:s} Y_j = 1$$

almost surely. Furthermore,

$$\liminf_{s \to \infty} \frac{\sum_{j \in 1:s} Y_j - s}{\sqrt{s \cdot \ln \ln s}} = -\sqrt{2}$$

holds almost surely due to the Law of the Iterated Logarithm. Since

$$\sum_u \gamma_u \prod_{j \in u} k(x_j, x_j) = \sum_{s=1}^{\infty} \alpha^s \exp\left(- \sum_{j \in 1:s} Y_j(\mathbf{x}) \right),$$

the statement follows.

Moreover,

$$\text{(C3)} \quad \Leftrightarrow \quad \alpha < 2,$$

while $\mathfrak{X} = D^{\mathbb{N}}$ is equivalent to $\alpha < 1$. For $\alpha \in [2, 3[$ and $\mathbf{y} = (1, \dots) \notin \mathfrak{X}$ we have $(\gamma_u \psi_u h_u)_u \in H$, but

$$\sum_u \gamma_u |h_u(\mathbf{y}_u)| = \infty,$$

cf. Remark 3.

In particular, for $\alpha \in [\exp(1), 3[$, property (E) holds true, but $\mu(\mathfrak{X}) = 0$.

Example 3. Consider the uniform distribution ρ on $D = [-1, 1]$, and let $k(x, y) = x \cdot y$. Moreover, let γ_u be given as in Example 2. In this case the assumptions (A1)–(A6) are satisfied, and $h = 0$.

Hence (C1) trivially holds for every $\alpha > 0$. We claim that

$$\alpha < 4 \quad \Leftrightarrow \quad \text{(C2)} \quad \Rightarrow \quad \text{(E)} \quad \Rightarrow \quad \alpha \le 4.$$

For a proof note that $\|J_u\| = 2^{-|u|}$ due to (7). In view of Proposition 3 it remains to show that (E) implies $\alpha \leq 4$. To this let $f(\mathbf{y}) = \prod_{j\in 1:s} y_j$ for $\mathbf{y} \in D^{1:s}$. Use $\|f\|_{l_{1:s}} = 1$ and $T\psi_{1:s}f(\mathbf{y}) = f(\mathbf{y}_{1:s})$ for $\mathbf{y} \in D^{\mathbb{N}}$ to derive

$$\int_{D^{\mathbb{N}}} |T\psi_{1:s}f| \, d\mu = \int_{D^{1:s}} |f| \, d\rho^{1:s} = 2^{-s} = \|\psi_{1:s}f\|_K \cdot (\alpha/4)^{s/2}.$$

Continuity of T therefore implies $\alpha \leq 4$.

Furthermore, we claim that

$$\mu(\mathfrak{X}) = 1 \quad \Leftrightarrow \quad \alpha < \exp(2).$$

To verify this fact we put $Y_j(\mathbf{x}) = -\ln(x_j^2)$ for $\mathbf{x} \in D^{\mathbb{N}}$ with $x_j \neq 0$, so that $\mathrm{E}(Y_j) = 2$ with respect to the probability measure μ on $D^{\mathbb{N}}$. It remains to apply the argument from the previous example.

Finally

$$\text{(C3)} \quad \Leftrightarrow \quad \alpha < 3,$$

and $\mathfrak{X} = D^{\mathbb{N}}$ is equivalent to $\alpha < 1$.

In particular, for $\alpha \in \,]4, \exp(2)[$, condition (C1) is satisfied and $\mu(\mathfrak{X}) = 1$, but property (E) does not hold true.

It is open to us whether the implication (C2) $\Rightarrow$ (E) is strict.

Appendix A. Basic Facts

Let $E \neq \emptyset$, and let $K_1, \dots, K_n$ denote reproducing kernels on $E \times E$. Put $K = \sum_{i=1}^n K_i$.

Lemma 11. *The following properties are equivalent:*

(i) *Each $f \in H(K)$ has a unique representation $f = \sum_{i=1}^n f_i$ with $f_i \in H(K_i)$.*
(ii) *The spaces $H(K_1), \dots, H(K_n)$ are pairwise orthogonal in $H(K)$.*
(iii) *For all $f_i \in H(K_i)$ with $\sum_{i=1}^n f_i = 0$ we have $f_1 = \dots = f_n = 0$.*
(iv) *For all $f = \sum_{i=1}^n f_i$ with $f_i \in H(K_i)$ we have $\|f\|_K^2 = \sum_{i=1}^n \|f_i\|_{K_i}^2$.*
(v) *For all orthonormal bases $(d_{i,j})_{j\in J_i}$ of the spaces $H(K_i)$ the family $(d_{i,j})_{i\in 1:n, j\in J_i}$ is an orthonormal base of $H(K)$.*
(vi) *There exist orthonormal bases $(d_{i,j})_{j\in J_i}$ of the spaces $H(K_i)$ such that $(d_{i,j})_{i\in 1:n, j\in J_i}$ is an orthonormal base of $H(K)$.*

Proof. The implication (v) $\Rightarrow$ (vi) is trivial, and (i) obviously implies (iii) as well as (ii) obviously implies (v). Let $f \in H(K)$ and $g_i, h_i \in H(K_i)$ such that $f = \sum_{i=1}^n g_i = \sum_{i=1}^n h_i$. Then it is easy to see that (ii)–(vi) each imply $g_i = h_i$. Hence (i) follows.

For (i) $\Rightarrow$ (iv) $\Rightarrow$ (ii) recall that by definition of $H(K)$ we have

$$\|f\|_K^2 = \min\{\sum_{i=1}^n \|f_i\|_{K_i}^2 : f = \sum_{i=1}^n f_i \text{ and } f_i \in H(K_i)\}$$

for $f \in H(K)$. But then (iv) follows immediately. In particular, $\|f\|_K = \|f\|_{K_i}$ for $f \in H(K_i)$. Let $h \in H(K_i)$ and $g \in H(K_j)$ with $i \neq j$. Now we obtain (ii) from

$$\begin{aligned} \|h\|_{K_i}^2 + \|g\|_{K_j}^2 &\overset{\text{(iv)}}{=} \langle h+g, h+g \rangle_K = \|h\|_K^2 + 2\langle h, g \rangle_K + \|g\|_K^2 \\ &\overset{\text{(iv)}}{=} \|h\|_{K_i}^2 + 2\langle h, g \rangle_K + \|g\|_{K_j}^2. \end{aligned}$$

$\square$

Suppose that $E = E_1 \times E_2$ with $E_1, E_2 \neq \emptyset$. Let L be a reproducing kernel on $E_1 \times E_1$, and let M and M' be reproducing kernels on $E_2 \times E_2$.

Lemma 12. *If $H(M) \subseteq H(M')$ then*

$$H(L \otimes M) \subseteq H(L \otimes M').$$

Proof. By assumption, $c\,M' - M$ is non-negative definite for some $c > 0$. Furthermore, $c\,L \otimes M' - L \otimes M = L \otimes (c\,M' - M)$. □

Let L_i and M_i be reproducing kernels on $E_1 \times E_1$ and $E_2 \times E_2$, respectively, for $i = 1, \ldots, n$. Consider the kernels $L = \sum_{i=1}^n L_i$ on $E_1 \times E_1$ and $K = \sum_{i=1}^n L_i \otimes M_i$ on $E \times E$.

Lemma 13. *Pairwise orthogonality of the spaces $H(L_i)$ in $H(L)$ implies the same property for the spaces $H(L_i \otimes M_i)$ in $H(K)$.*

Proof. At first we assume that $M_1 = \cdots = M_n$, which implies $K = L \otimes M_1$. Take orthonormal bases $(d_j)_{j \in J_i}$ of $H(L_i)$ and $(e_j)_{j \in J^{(2)}}$ of $H(M_1)$. Without loss of generality we assume that $J_1, \ldots, J_n$ are pairwise disjoint. Put $J^{(1)} = \bigcup_{i=1}^n J_i$ as well as $J = J^{(1)} \times J^{(2)}$.

Let $\mathbf{x} \in E_1$, $\mathbf{y} \in E_2$, and $\alpha \in \ell_2(J)$. Since

$$L(\mathbf{x}, \mathbf{x}) = \sum_{j_1 \in J^{(1)}} d_{j_1}(\mathbf{x})^2 \quad \text{and} \quad M_1(\mathbf{y}, \mathbf{y}) = \sum_{j_2 \in J^{(2)}} e_{j_2}(\mathbf{y})^2,$$

we have for all $\mathbf{x} \in E_1$ and $\mathbf{y} \in E_2$ that

$$(d_{j_1}(\mathbf{x}))_{j_1 \in J^{(1)}} \in \ell_2(J^{(1)}) \quad \text{and} \quad (e_{j_2}(\mathbf{x}))_{j_2 \in J^{(2)}} \in \ell_2(J^{(2)}).$$

Hence

$$\Phi(\alpha) = \sum_{j \in J} \alpha_j \cdot d_{j_1} \otimes e_{j_2}$$

yields a linear mapping $\Phi : \ell_2(J) \to \mathbb{R}^E$. Note that

$$\left(\sum_{j_2 \in J^{(2)}} |\alpha_{j_1,j_2} \cdot e_{j_2}(\mathbf{y})| \right)^2 \leq \sum_{j_2 \in J^{(2)}} \alpha_{j_1,j_2}^2 \cdot M_1(\mathbf{y}, \mathbf{y})$$

and that $(d_{j_1})_{j_1 \in J^{(1)}}$ is an orthonormal basis of $H(L)$, which follows from Lemma 11. Thus $\Phi(\alpha) = 0$ implies

$$\sum_{j_2 \in J^{(2)}} \alpha_{j_1,j_2} \cdot e_{j_2}(\mathbf{y}) = 0, \qquad j_1 \in J^{(1)}, \mathbf{y} \in E_2,$$

and hereby $\alpha = 0$.

Consider the Hilbert space $H = \Phi(\ell_2(J))$, equipped with the scalar product

$$\langle f, g \rangle = \langle \Phi^{-1}(f), \Phi^{-1}(g) \rangle_{\ell_2(J)}, \qquad f, g \in H.$$

Choose $\alpha_{j_1,j_2} = d_{j_1}(\mathbf{x}) \cdot e_{j_2}(\mathbf{y})$ to obtain $K(\cdot, (\mathbf{x}, \mathbf{y})) \in H$, and for $\beta \in \ell_2(J)$ we get

$$\langle \Phi(\beta), K(\cdot, (\mathbf{x}, \mathbf{y})) \rangle = \sum_{j \in J} \beta_j \cdot d_{j_1}(\mathbf{x}) \cdot e_{j_2}(\mathbf{y}) = \Phi(\beta)(\mathbf{x}, \mathbf{y}).$$

Therefore $H = H(K)$, and $(d_{j_1} \otimes e_{j_2})_{j \in J}$ is an orthonormal basis of this space. By the same arguments, formally with $n = 1$, $(d_{j_1} \otimes e_{j_2})_{j_1 \in J_i, j_2 \in J^{(2)}}$ is an orthonormal basis of the space $H(L_i \otimes M_1)$. Apply Lemma 11.

We turn to the general case. Assume that $\sum_{i=1}^{n} f_i = 0$ for $f_i \in H(L_i \otimes M_i)$. Put $K_i = L_i \otimes M$ with $M = \sum_{i=1}^{n} M_i$. Use Lemma 12 to conclude that $f_i \in H(K_i)$. The first part of the proof, together with Lemma 11, yields $f_1 = \cdots = f_n = 0$. $\square$

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