# DFG-Schwerpunktprogramm 1324

„Extraktion quantifizierbarer Information aus komplexen Systemen"

## Spatial Besov Regularity for Semilinear Stochastic Partial Differential Equations on Bounded Lipschitz Domains

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Petru A. Cioica, Stephan Dahlke

**Abstract**

We study the spatial regularity of semilinear parabolic stochastic partial differential equations on bounded Lipschitz domains $\mathcal{O} \subseteq \mathbb{R}^d$ in the scale $B^{\alpha}_{\tau,\tau}(\mathcal{O})$, $1/\tau = \alpha/d + 1/p$, $p \geq 2$ fixed. The Besov smoothness in this scale determines the order of convergence that can be achieved by adaptive numerical algorithms and other nonlinear approximation schemes. The proofs are performed by establishing weighted Sobolev estimates and combining them with wavelet characterizations of Besov spaces.

*Keywords:* Semilinear stochastic partial differential equation, Besov space, weighted Sobolev space, nonlinear approximation, wavelet expansions

*Mathematics Subject Classification (2010):* 60H15, *Secondary:* 46E35, 65C30

## 1 Introduction

This paper is concerned with the spatial regularity of the solutions to semilinear stochastic partial differential equations (semilinear SPDEs, for short) measured in specific scales of Besov spaces. The motivation for these studies can be explained as follows.

In recent years, the numerical treatment of operator equations by adaptive numerical algorithms has become a field of increasing importance, with many applications in science and engineering. Especially, adaptive finite element schemes have been very successfully developed and implemented, and innumerable numercial experiments impressively confine their excellent performance. Complementary to this, also adaptive algorithms based on wavelets have become more and more in the center of attention during the last years, for

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the following reason. The strong analytical properties of wavelets can be used to derive adaptive strategies which are guaranteed to converge for a huge class of elliptic operator equations, involving operators of negative order [4, 11]. Moreover, these algorithms are optimal in the sense that they asymptotically realize the convergence order of the optimal (but not directly implementable) approximation scheme, i.e. the order of best $N$-term wavelet approximation. Moreover, the number of arithmetic operations that is needed stays proportional to the number of degrees of freedom [4]. By now, various generalizations to non-elliptic equations [5], saddle point problems [12] and also nonlinear operator equations [6] exist. For finite element schemes, rigorous statements of these forms have been rather rare, even though, inspired by the results for wavelet schemes, the situation has changed during the last years [1, 18].

Although the above mentioned results are quite impressive, in the realm of adaptivity one is always faced with the following question: does adaptivity really pay for the problem under consideration, i.e. does our favourite adaptive scheme really provide a substantial gain of efficiency compared to more conventional nonadaptive schemes which are usually much easier to implement? At least for the case of adaptive wavelet schemes, it is possible to give a quite rigorous answer. A reasonable comparison would be to compare the performance of wavelet algorithms with classical, nonadaptive schemes which consist of approximations by linear spaces that are generated by uniform grid refinements. It is well-known that, under natural assumptions, the approximation order that can be achieved by such a uniform method depends on the smoothness of the exact solution measured in the classical $L_p$-Sobolev scale. More precisely, let $e_N(f)$ denote the uniform approximation error, then

$$f \in W_p^s(\mathcal{O}) \quad \Longrightarrow \quad e_N(f) \le C \cdot N^{-s/d}.$$

On the other hand, as already outlined above, for adaptive wavelet methods the best $N$-term approximation serves as the benchmark scheme. It is well-known that the convergence order that can be achieved by best $N$-term approximations also depends on the smoothness of the object we want to approximate. But, in contrast to the case of nonadaptive schemes, the smoothness has now to be measured in specific Besov spaces, usually corresponding to $L_\tau$-spaces with $0 < \tau < 2$. Indeed, let $\sigma_N(f)$ denote the error of best $N$-term approximation, then

$$f \in B^\alpha_{\tau,\tau}(\mathcal{O}), \quad \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p} \quad \Longrightarrow \quad \sigma_N(f) \le C \cdot N^{-\alpha/d}.$$

For a detailed description of these relationships, we refer, e.g. to [10] and [16], see also [3, Chapters 3 and 4]. Consequently, we can make the following statement: the use of adaptive wavelet schemes is completely justified if the spatial Besov smoothness $\alpha$ of the unknown solution of our SPDE in the scale $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}$ is higher compared to its regularity in the Sobolev scale. The situation is illustrated in Figure 1, where each point $(1/\tau, s)$ represents the smoothness spaces of functions with "$s$ derivatives in $L_\tau(\mathcal{O})$".

For deterministic elliptic partial differential equations, many positive results in this direction already exist, see e.g. [7], [8], [9], [13], [14]. In contrast, little is known concerning

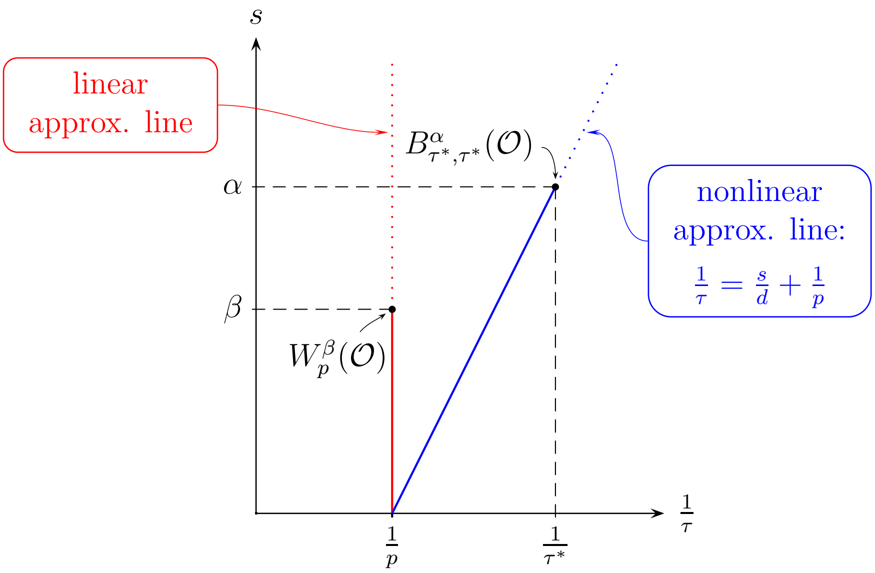


Figure 1: Linear vs. nonlinear approximation illustrated in a DeVore-Triebel diagram.

SPDEs. To our knowledge, the first results were obtained in [2]. In that paper, linear stochastic partial differential equations are considered and sufficiently high Besov smoothness to justify the use of adaptive schemes is established. The aim of this paper is to generalize these basic results to *semilinear* equations.

The stochastic setting is as follows. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Fix $T \in (0,\infty)$ and let $(\mathcal{F}_t)_{t\in[0,T]}$ be a normal filtration on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, let $\big(w^k_t\big)_{t\in[0,T]}$, $k \in \mathbb{N}$, be a sequence of independent $\mathbb{R}$-valued standard Brownian motions w.r.t. $(\mathcal{F}_t)_{t\in[0,T]}$. For arbitrary $\varepsilon = (\varepsilon_f, \varepsilon_g) \in [0,\infty) \times [0,\infty)$ we consider the model equation

$$\mathrm{d}u = \big(Au + \varepsilon_f\, F(u)\big)\mathrm{d}t + \sum_{k=1}^{\infty} \big(\varepsilon_g\, G(u)^k + g^k(t)\big)\mathrm{d}w^k_t, \quad u(0,\cdot) = u_0, \qquad (*_\varepsilon)$$

for $t \in [0,T]$ on the domain $\mathcal{O}$. Here $\mathrm{d}u$ is Itô's stochastic differential with respect to $t$, $A$ denotes the Laplacian and the coefficients $g^k$, $k \in \mathbb{N}$, are random functions depending on $t \in [0,T]$ and $x \in \mathcal{O}$. The functions $F$ and $G$ are Lipschitz continuous on certain classes of stochastic processes taking values in weighted Sobolev spaces. For details see Section 2, especially Assumptions 5.

Equation $(*_\varepsilon)$ is understood in a weak or distributional sense, i.e. $u^*_\varepsilon$ is a solution of $(*_\varepsilon)$, if for all smooth and compactly supported test functions $\varphi \in \mathscr{C}^\infty_0(\mathcal{O})$ the equality

$$\langle u^*_\varepsilon(t,\cdot), \varphi\rangle = \langle u_0, \varphi\rangle + \int_0^t \langle \Delta u^*_\varepsilon(s,\cdot) + \varepsilon_f F(u^*_\varepsilon)(s,\cdot), \varphi\rangle \mathrm{d}s$$

$$+\sum_{k=1}^{\infty}\int_0^t \langle \varepsilon_g G(u^*_\varepsilon)^k(s,\cdot) + g^k(s,\cdot),\varphi\rangle \mathrm{d}w^k_s$$

holds for all $t \in [0,T]$ $\mathbb{P}$-almost surely. Here and throughout the paper we write $\langle u,\varphi\rangle$ for the application of a distribution $u \in \mathcal{D}'(\mathcal{O})$ to a test function $\varphi \in \mathscr{C}_0^\infty(\mathcal{O})$. We refer again to Section 2 for details.

Let us assume that the solution $u^*_\varepsilon = u^*_\varepsilon(t,\omega,x)$, $(t,\omega,x) \in [0,T]\times\Omega\times\mathcal{O}$, vanishes on the boundary $\partial\mathcal{O}$, satisfying a zero Dirichlet boundary condition. It is clear that the smoothness of $x \mapsto u^*_\varepsilon(t,\omega,x)$ depends on the smoothness of the mappings $x \mapsto g^k(t,\omega,x)$, $k \in \mathbb{N}$, as well as on the properties of the nonlinearities $F$ and $G$. However, even if $F$ and $G$ vanish, and the spatial smoothness of the $g^k$ is high, the Sobolev smoothness of $x \mapsto u^*_\varepsilon(t,\omega,x)$ can be additionally limited by singularities of the spatial derivatives of $u^*_\varepsilon$ at the boundary of $\mathcal{O}$, due to the zero Dirichlet boundary condition. Such corner singularities are a typical example that the spatial $L_p$-Sobolev regularity of $u^*_\varepsilon$ is exceeded by the regularity in the scale of Besov spaces $B^\alpha_{\tau,\tau}(\mathcal{O})$, $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}$. In this paper, we present a result on the spatial Besov regularity of the solution $u^*_\varepsilon$ to equation $(*_\varepsilon)$ which has the following structure: If

$$u^*_\varepsilon \in L_p([0,T]\times\Omega, \mathscr{P}_T, \lambda\otimes\mathbb{P};\, W^s_p(\mathcal{O}))$$

and if $\varepsilon_f, \varepsilon_g > 0$ are small enough and the functions $g^k$, $k \in \mathbb{N}$, are sufficiently regular, then

$$u^*_\varepsilon \in L_\tau([0,T]\times\Omega, \mathscr{P}_T, \lambda\otimes\mathbb{P};\, B^\alpha_{\tau,\tau}(\mathcal{O}))$$

for certain $\alpha > s$ and $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}$. Here $\mathscr{P}_T$ is the predictable $\sigma$-algebra w.r.t. the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and $\lambda$ denotes Lebesgue measure on $[0,T]$. This result becomes even more important in the light of the following recently proven result, see [27]. Roughly speaking, the following has been shown: Consider the two-dimensional case and assume that the domain $\mathcal{O} \subseteq \mathbb{R}^2$ has a polygonal boundary $\partial\mathcal{O}$ such that $\mathcal{O}$ lies on one side of $\partial\mathcal{O}$. Denote by $\gamma_{\max}$ the largest interior angle at any corner of the boundary $\partial\mathcal{O}$. Consider equation $(*_\varepsilon)$ with $\varepsilon = 0$ and $g$ being sufficiently smooth (see Section 2 for the concrete setting). Then, the regularity of the solution $u^*_0$ in the scale $L_2(\Omega_T; W^s_2(\mathcal{O}))$, $s > 0$, is limited by $s^* = 1 + \pi/\gamma_{\max}$, which is strictly less than 2 if $\gamma_{\max} > \pi$. However, at the same time $u^*_0 \in L_\tau(\Omega_T; B^\alpha_{\tau,\tau}(\mathcal{O}))$, $1/\tau = \alpha/2 + 1/2$, for each $\alpha < 2$. With the explanations from above in mind, this result is important for the theoretical foundation of adaptive numerical methods for the approximation of $u^*_\varepsilon$.

The proofs of our results heavily rely on regularity estimates in weighted Sobolev spaces. For linear SPDEs, estimates of this form have already been derived, see [23], as well as [24]. In this paper, we generalize these results to the semilinear case. By following the lines of [2], these new weighted Sobolov estimates can then be used to derive the desired Besov results.

There exists an extensive literature on the Besov regularity of SPDEs, although most of the results are obtained for linear equations. Recently, *semilinear* evolution equations gained center stage. In general, however, the assumptions on the domain and the scale of parameters considered do not fit into our setting. To mention an example, in [32] non-autonomous semilinear evolution equations in UMD Banach spaces with M-type 2 have been studied (see also [29] for the autonomous case). As an application, it has been shown that, if the boundary of the domain is smooth enough and certain conditions on the coefficients are fulfilled, the solutions of these equations take values in the Besov space $B^s_{p,p}(\mathcal{O})$ with $p \geq 2$ and some $s \in [0, 2]$. One of the main purposes in these works is to gain better Hölder regularity results. Hence, due to the Sobolev embedding theorem, the main interest is on spaces with summability parameter $p$ as large as possible. However, our focus is different. Firstly, we are explicitly interested in domains with non-smooth boundary. In this situation, one of the main ingredients for the application mentioned above, namely the equality $W^2_2(\mathcal{O}) \cap \mathring{W}^1_2(\mathcal{O}) = D(A)$, where $D(A) := \{u \in \mathring{W}^1_2(\mathcal{O}) : Au \in L_2(\mathcal{O})\}$, $A = \Delta = \sum_{\mu=1}^d \frac{\partial^2}{\partial x_\mu^2}$, does not hold any more (see [19], [20], as well as [22]). Secondly, we are interested in the scale $B^\alpha_{\tau,\tau}(\mathcal{O})$, $1/\tau = \alpha/d + 1/p$, $p \geq 2$ fixed, including spaces which are no Banach spaces but quasi-Banach spaces. The parameter $\tau$ decreases if $\alpha$ increases and $B^\alpha_{\tau,\tau}(\mathcal{O})$ fails to be a Banach space for $\tau < 1$. While our methods work in this setting, any direct approach requires (at least!) a fully-fledged theory of stochastic integration in quasi-Banach spaces which is not yet available.

Let us emphasize that our result can be extended to more general semilinear equations of the type

$$\left.\begin{aligned} \mathrm{d}u &= \left( \sum_{i,j=1}^d a^{ij} u_{x_i x_j} + \sum_{i=1}^d b^i u_{x_i} + cu + \varepsilon_f F(u) + f \right) \mathrm{d}t \\ &\qquad\qquad + \sum_{k=1}^\infty \left( \sum_{i=1}^d \sigma^{ik} u_{x_i} + \eta^k u + \varepsilon_g G(u)^k + g^k \right) \mathrm{d}w^k_t, \\ u(0,\,\cdot\,) &= u_0. \end{aligned}\right\} \qquad (\bullet_\varepsilon)$$

Here the coefficients $a^{ij}$, $b^i$, $c$, $\sigma^{ik}$, $\eta^k$ and the free terms $f$ and $g^k$ are random functions depending on $t$ and $x$, while $F$ and $G$ are as in equation $(*_\varepsilon)$. This extension is possible because our result in Theorem 8 as well as Corollary 10 can be extended to this wider class. Therefore, additionally assumptions on the coefficients, as presented in [23] and [24], have to be fulfilled. Since this mainly adds notational complications, we will focus on equation $(*_\varepsilon)$ and give a short account of how to treat equations of type $(\bullet_\varepsilon)$ in the last section.

The paper is organized as follows: In Section 2 we extend the result from [24] concerning the solvability of linear SPDEs in weighted Sobolev spaces to a class of *semilinear* SPDEs. Furthermore, we will prove a weighted Sobolev estimate for this unique solution. Section 3 contains the Besov regularity result (Theorem 11), its proof and an example for an application of this theorem. General equations of type $(\bullet_\varepsilon)$ will be treated in Section 4. In the appendix the reader can find the definition of Besov spaces $B^s_{p,q}(\mathcal{O})$ as well as some details on wavelet characterizations of $B^s_{p,q}(\mathbb{R}^d)$ for particular parameter constellations.

Before we start let us fix some **Conventions and Notations** (see [2]). If not otherwise stated, throughout this paper, $\mathcal{O} \subseteq \mathbb{R}^d$ denotes a *bounded* Lipschitz domain. A domain is called Lipschitz if each point on the boundary $\partial\mathcal{O}$ has a neighbourhood whose intersection with the boundary—after relabeling and reorienting the coordinate axes if necessary—is the graph of a Lipschitz function.

By $\mathcal{D}'(\mathcal{O})$ we denote the space of Schwartz distributions on $\mathcal{O}$. If not explicitly stated otherwise, all function spaces or spaces of distributions are meant to be spaces of real-valued functions or distributions. If $f \in \mathcal{D}'(\mathcal{O})$ is a generalized function and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ is a multi-index, we write $D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}$ for the corresponding derivative w.r.t. $x = (x_1, \ldots, x_d) \in \mathcal{O}$, where $|\alpha| = \alpha_1 + \ldots + \alpha_d$. For $m \in \mathbb{N}_0$, $D^m f = \{D^\alpha f : |\alpha| = m\}$ is the set of all $m$-th order derivatives of $f$ which is identified with an $\mathbb{R}^{\binom{d+m-1}{m}}$-valued distribution. Given $p \in [1, \infty)$ and $m \in \mathbb{N}_0$, $W_p^m(\mathcal{O})$ denotes the classical Sobolev space consisting of all (equivalence classes of) measurable functions $f : \mathcal{O} \to \mathbb{R}$ such that $\|f\|_{W_p^m(\mathcal{O})} = \|f\|_{L_p(\mathcal{O})} + |f|_{W_p^m(\mathcal{O})} = (\int_{\mathcal{O}} |f(x)|^p \, dx)^{1/p} + \sum_{|\alpha|=m} (\int_{\mathcal{O}} |D^\alpha f(x)|^p \, dx)^{1/p}$ is finite. For $p \in (1, \infty)$ and $s \in (m, m+1)$, $m \in \mathbb{N}_0$, we define the fractional order Sobolev space $W_p^s(\mathcal{O})$ to be the Besov space $B_{p,p}^s(\mathcal{O})$ introduced in the appendix. (This scale of fractional order Sobolev spaces can also be obtained by real interpolation of $W_p^n(\mathcal{O})$, $n \in \mathbb{N}_0$. One can show that $W_2^n(\mathcal{O}) = B_{2,2}^n(\mathcal{O})$ for all $n \in \mathbb{N}$ and $W_p^n(\mathcal{O}) \subset B_{p,p}^n(\mathcal{O})$ for all $n \in \mathbb{N}$, $p > 2$, see, e.g. [31, Remark 2.3.3/4 and Theorem 4.6.1.(b)] together with [17].) Given any countable index set $\mathcal{J}$, the space of $p$-summable sequences indexed by $\mathcal{J}$ is denoted by $\ell_p = \ell_p(\mathcal{J})$ and $|\cdot|_{\ell_p}$ is the respective norm. Usually we have $\ell_p = \ell_p(\mathbb{N})$ but, for instance we may also use the notation $|D^m f(x)|_{\ell_p}^p = \sum_{|\alpha|=m} |D^\alpha f(x)|^p$ for $f \in W_p^m(\mathcal{O})$.

Given a distribution $f \in \mathcal{D}'(\mathcal{O})$ and a smooth and compactly supported test function $\varphi \in \mathscr{C}_0^\infty(\mathcal{O})$, we write $\langle f, \varphi \rangle$ for the application of $f$ to $\varphi$. If $H$ is a Hilbert space, then $\langle \cdot, \cdot \rangle_H$ denotes the inner product in $H$. $\mathcal{M}_T^{2,c}(H, (\mathcal{F}_t))$ is the space of continuous, square integrable, $H$-valued martingales with respect to the filtration $(\mathcal{F}_t)_{t \in [0,T]}$. For $[0,T] \times \Omega$ we use the shorthand notation $\Omega_T$ and

$$\mathscr{P}_T := \sigma\big(\{(s,t] \times F_s : 0 \le s < t \le T,\ F_s \in \mathcal{F}_s\} \cup \{\{0\} \times F_0 : F_0 \in \mathcal{F}_0\}\big)$$

is the predictable $\sigma$-algebra. $\lambda \otimes \mathbb{P}$ is the product measure of Lebesgue measure $\lambda$ on $([0,T], \mathcal{B}([0,T]))$, where $\mathcal{B}([0,T])$ denotes the Borel $\sigma$-algebra on $[0,T]$, and the probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. Given any measure space $(A, \mathcal{A}, m)$, any (quasi-)normed space $B$ with (quasi-)norm $\|\cdot\|_B$ and any summability index $p > 0$, we denote by $L_p(A, \mathcal{A}, m; B)$ the space of all strongly measurable, i.e. Bochner measurable, functions $u : A \to B$ whose (quasi-)norm $\|u\|_{L_p(A,\mathcal{A},m;B)} := \big(\int_A \|u(z)\|_B^p \, m(dz)\big)^{1/p}$ is finite.

All equalities of random variables or random (generalized) functions appearing in this paper are meant to be $\mathbb{P}$-almost sure equalities. Throughout the paper, $C$ denotes a positive constant which may change its value from line to line.

# 2 Existence Results in Weighted Sobolev Spaces

In this chapter we extend the $L_p$-theory of linear SPDEs on Lipshitz domains from [24] to *semilinear* SPDEs of the form $(*_\varepsilon)$.

If we assume that our solution $u_\varepsilon^* = u_\varepsilon^*(t,\omega,x)$, $(t,\omega,x) \in [0,T] \times \Omega \times \mathcal{O}$, vanishes at the boundary $\partial\mathcal{O}$, satisfying a zero Dirichlet boundary condition, we are facing the same problem as in the linear case: Nonsmooth domains $\mathcal{O}$ induce singularities of the (spatial) derivatives at the boundary $\partial\mathcal{O}$, so that our solution fails to have high regularity in the Sobolev scale $W_p^s(\mathcal{O})$, $s > 0$. A general way to deal with smoothness regardless of certain singularities at the boundary is to use weighted Sobolev spaces, where the weight function is a power of the distance to the boundary. The $L_p$-theory in [23, 24] is based on spaces of this type, namely the weighted Sobolev spaces $H^\gamma_{p,\theta}(\mathcal{O})$, $p \in (1,\infty)$, $\theta$, $\gamma \in \mathbb{R}$, introduced in [28]. For the convenience of the reader we give here a short presentation of these spaces, which can also be found in [2]. They are defined in terms of the Bessel-potential spaces

$$H_p^\gamma(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) \,:\, \|u\|_{H_p^\gamma(\mathbb{R}^d)} = \|(1-\Delta)^{\gamma/2}u\|_{L_p(\mathbb{R}^d)} < \infty\}.$$

Here, $\mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$ is the space of (real valued) tempered distributions and $(1-\Delta)^{\gamma/2} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is the pseudo-differential operator with symbol $\mathbb{R}^d \ni \xi \mapsto (1+|\xi|^2)^{\gamma/2}$, i.e. $(1-\Delta)^{\gamma/2}u = \mathcal{F}^{-1}\big((1+|\xi|^2)^{\gamma/2}\mathcal{F}u\big)$, where $\mathcal{F}$ denotes the Fourier transform on the (complex valued) tempered distributions.

For $x \in \mathcal{O}$ we write $\rho(x) := \mathrm{dist}(x, \partial\mathcal{O})$ for the distance between $x$ and the boundary of the domain $\mathcal{O}$. Fix $c > 1$, $k_0 > 0$ and for $n \in \mathbb{Z}$ consider the subsets $\mathcal{O}_n$ of $\mathcal{O}$ given by

$$\mathcal{O}_n := \{x \in \mathcal{O} \,:\, c^{-n-k_0} < \rho(x) < c^{-n+k_0}\}.$$

Let $\zeta_n$, $n \in \mathbb{Z}$, be non-negative functions satisfying $\zeta_n \in \mathscr{C}_0^\infty(\mathcal{O}_n)$, $\sum_{n\in\mathbb{Z}} \zeta_n(x) = 1$ and $|D^m\zeta_n(x)| \leq C \cdot c^{mn}$ for all $n \in \mathbb{Z}$, $m \in \mathbb{N}_0$, $x \in \mathcal{O}$, and a constant $C > 0$ that does not depend on $n$, $m$ and $x$. The functions $\zeta_n$ can be constructed by mollifying the indicator functions of the sets $\mathcal{O}_n$, see, e.g. [21, Section 1.4]. If $\mathcal{O}_n$ is empty we set $\zeta_n \equiv 0$. For $u \in \mathcal{D}'(\mathcal{O})$ $\zeta_n u$ is a distribution on $\mathcal{O}$ with compact support which can be extended by zero to $\mathbb{R}^d$. This extension is a tempered distribution, i.e. $\zeta_n u \in \mathcal{S}'(\mathbb{R}^d)$.

**Definition 1.** Let $\zeta_n$, $n \in \mathbb{Z}$, be as above and $p \in (1,\infty)$, $\theta$, $\gamma \in \mathbb{R}$. Then

$$H^\gamma_{p,\theta}(\mathcal{O}) := \Big\{u \in \mathcal{D}'(\mathcal{O}) \,:\, \|u\|^p_{H^\gamma_{p,\theta}(\mathcal{O})} := \sum_{n\in\mathbb{Z}} c^{n\theta} \|\zeta_{-n}(c^n\,\cdot\,)u(c^n\,\cdot\,)\|^p_{H^\gamma_p(\mathbb{R}^d)} < \infty\Big\}.$$

According to [28] this definition is independent of the specific choice of $c$, $k_0$ and $\zeta_n$, $n \in \mathbb{N}_0$, in the sense that one gets equivalent norms. If $\gamma = m \in \mathbb{N}_0$ then the spaces can be characterized as

$$\begin{aligned} H^0_{p,\theta}(\mathcal{O}) &= L_{p,\theta}(\mathcal{O}) := L_p(\mathcal{O}, \rho(x)^{\theta-d}\mathrm{d}x),\\ H^m_{p,\theta}(\mathcal{O}) &= \left\{u \,:\, \rho^{|\alpha|}D^\alpha u \in L_{p,\theta}(\mathcal{O}) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq m\right\}, \end{aligned}$$

and one has the norm equivalence

$$C^{-1}\|u\|^p_{H^m_{p,\theta}(\mathcal{O})} \;\leq\; \sum_{\alpha\in\mathbb{N}_0^d,\,|\alpha|\leq m} \int_{\mathcal{O}} \left|\rho(x)^{|\alpha|}D^\alpha u(x)\right|^p \rho(x)^{\theta-d}\,\mathrm{d}x \;\leq\; C\|u\|^p_{H^m_{p,\theta}(\mathcal{O})}.$$

Analogous notations are used for $\ell_2 = \ell_2(\mathbb{N})$-valued functions $g = (g^k)_{k\in\mathbb{N}}$. For $p \in (1,\infty)$, $\theta$, $\gamma \in \mathbb{R}$ and $\zeta_n$, $n \in \mathbb{Z}$, as above

$$\begin{aligned}
H^\gamma_p(\mathbb{R}^d;\ell_2) &:= \Big\{ g \in (\mathcal{S}'(\mathbb{R}^d))^{\mathbb{N}} \,:\, (1-\Delta)^{\gamma/2} g^k \in L_p(\mathbb{R}^d) \text{ for all } k \in \mathbb{N} \text{ and} \\
&\qquad\qquad \|g\|_{H^\gamma_p(\mathbb{R}^d;\ell_2)} := \Big\| \big| \big( (1-\Delta)^{\gamma/2} g^k \big)_{k\in\mathbb{N}} \big|_{\ell_2} \Big\|_{L_p(\mathbb{R}^d)} < \infty \Big\}, \\
H^\gamma_{p,\theta}(\mathcal{O};\ell_2) &:= \Big\{ g \in (\mathcal{D}'(\mathcal{O}))^{\mathbb{N}} \,:\, \|g\|^p_{H^\gamma_{p,\theta}(\mathcal{O};\ell_2)} := \sum_{n\in\mathbb{Z}} c^{n\theta} \|\zeta_{-n}(c^n\,\cdot\,) g(c^n\,\cdot\,)\|^p_{H^\gamma_p(\mathbb{R}^d;\ell_2)} < \infty \Big\}.
\end{aligned}$$

**Remark 2** (see Remark 7.(a) in [2])**.** One can consider the spaces $H^\gamma_{p,\theta}(\mathcal{O})$ as generalizations of the classical Sobolev spaces on $\mathcal{O}$ with zero Dirichlet boundary conditions. For $\gamma = m \in \mathbb{N}_0$ we have the identity

$$H^m_{p,d-mp}(\mathcal{O}) = \mathring{W}^m_p(\mathcal{O}),$$

and the norms in both spaces are equivalent, see Theorem 9.7 in [26]. Here $\mathring{W}^m_p(\mathcal{O})$ is the closure of $\mathscr{C}^\infty_0(\mathcal{O})$ in the classical Sobolev space $W^m_p(\mathcal{O})$.

We can now define spaces of stochastic processes and random functions in terms of the weighted Sobolev spaces introduced above.

**Definition 3.** For $\gamma$, $\theta \in \mathbb{R}$ and $p \in (1,\infty)$ we set

$$\begin{aligned}
\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T) &:= L_p\left(\Omega_T, \mathscr{P}_T, \lambda\otimes\mathbb{P};\, H^\gamma_{p,\theta}(\mathcal{O})\right), \\
\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T;\ell_2) &:= L_p\left(\Omega_T, \mathscr{P}_T, \lambda\otimes\mathbb{P};\, H^\gamma_{p,\theta}(\mathcal{O};\ell_2)\right), \\
U^\gamma_{p,\theta}(\mathcal{O}) &:= L_p\big(\Omega, \mathcal{F}_0, \mathbb{P};\, H^{\gamma-2/p}_{p,\,\theta+2-p}(\mathcal{O})\big),
\end{aligned}$$

and for $p \in [2,\infty)$,

$$\begin{aligned}
\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T) := \Big\{ u \in \mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T) \,:\, u(0,\,\cdot\,) \in U^\gamma_{p,\theta}(\mathcal{O}) \text{ and } \mathrm{d}u = f\,\mathrm{d}t + \sum_{k=1}^\infty g^k\,\mathrm{d}w^k_t \\
\text{for some } f \in \mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T),\ g \in \mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2) \Big\},
\end{aligned}$$

equipped with the norm

$$\|u\|_{\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)} := \|u\|_{\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)} + \|f\|_{\mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T)} + \|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u(0,\,\cdot\,)\|_{U^\gamma_{p,\theta}(\mathcal{O})}.$$

The equality $\mathrm{d}u = f\,\mathrm{d}t + \sum_{k=1}^\infty g^k\,\mathrm{d}w^k_t$ above is a shorthand for

$$\langle u(t,\,\cdot\,),\varphi\rangle = \langle u(0,\,\cdot\,),\varphi\rangle + \int_0^t \langle f(s,\,\cdot\,),\varphi\rangle\,\mathrm{d}s + \sum_{k=1}^\infty \int_0^t \langle g^k(s,\,\cdot\,),\varphi\rangle\,\mathrm{d}w^k_s \tag{1}$$

for all $\varphi \in \mathscr{C}^\infty_0(\mathcal{O})$, $t \in [0,T]$.

**Remark 4** (see Remark 9 in [2]). **(a)** If $p \in [2,\infty)$, then the sum of stochastic integrals in (1) converges in the space $\mathcal{M}_T^{2,c}(\mathbb{R},(\mathcal{F}_t))$ of continuous, square integrable, $\mathbb{R}$-valued martingales w.r.t $(\mathcal{F}_t)_{t\in[0,T]}$. A proof can be found in [2, Appendix A].

**(b)** Using the arguments from [25, Remark 3.3], we get the uniqueness (up to indistinguishability) of the pair $(f,g) \in \mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T) \times \mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)$ which fulfils (1). Consequently, the norm in $\mathfrak{H}^{\gamma}_{p,\theta}(\mathcal{O},T)$ is well defined.

We are now ready to get more concrete on our model equation $(*_\varepsilon)$. In the sequel we will fix

$$p \in [2,\infty),\ \gamma \in \mathbb{N} \text{ and } \theta \in \mathbb{R}$$

and assume that the following conditions are fulfilled.

**Assumptions 5.** The differential operator $A$, the noise term $\left(g^k\right)_{k\in\mathbb{N}}$, the nonlinearity $F$, the initial condition $u_0$, and the nonlinearity $G$ in equation $(*_\varepsilon)$ fulfil the following conditions:

[$A1$] The operator

$$\begin{aligned} A : H^{\gamma}_{p,\theta-p}(\mathcal{O}) &\to H^{\gamma-2}_{p,\theta+p}(\mathcal{O}) \\ u &\mapsto Au \end{aligned}$$

is the unique linear and continuous extension of the linear operator

$$\begin{aligned} \Delta : \mathscr{C}_0^\infty(\mathcal{O}) &\to H^{\gamma-2}_{p,\theta+p}(\mathcal{O}) \\ u &\mapsto \Delta u := \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} u \end{aligned}$$

on $H^{\gamma}_{p,\theta-p}(\mathcal{O})$. We will make abuse of notation and denote $A$ by $\Delta$ as well.

[$A2$] The free noise term has the following regularity: $g = \left(g^k\right)_{k\in\mathbb{N}} \in \mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)$.

[$A3$] The function

$$\begin{aligned} F : \mathbb{H}^{\gamma}_{p,\theta-p}(\mathcal{O},T) &\to \mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T) \\ u &\mapsto F(u) \end{aligned}$$

is Lipschitz continuous with Lipschitz constant $C_F \in (0,\infty)$, i.e.

$$\|F(u)-F(v)\|_{\mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T)} \le C_F \|u-v\|_{\mathbb{H}^{\gamma}_{p,\theta-p}(\mathcal{O},T)} \tag{2}$$

for arbitrary $u,v \in \mathbb{H}^{\gamma}_{p,\theta-p}(\mathcal{O},T)$.

[$A4$] The function $F$ is zero at the origin, i.e. $F(0)=0$.

[$A5$] The initial value has the following regularity: $u_0 \in U^\gamma_{p,\theta}(\mathcal{O})$.

[$A6$] The function

$$G : \mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T) \to \mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)$$
$$u \mapsto G(u) = \big(G(u)^k\big)_{k\in\mathbb{N}}$$

is Lipschitz continuous with Lipschitz constant $C_G \in (0,\infty)$, i.e.

$$\|G(u) - G(v)\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} \leq C_G \|u - v\|_{\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)} \tag{3}$$

for arbitrary $u, v \in \mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)$.

[$A7$] The function $G$ is zero at the origin, i.e. $G(0) = 0$.

**Remark 6. (a)** To keep notation simple, we restrict ourself in this section to the case $A = \Delta$, see [$A1$]. In Section 4 we allow more general operators $A$, including for example positive definite second order elliptic differential operators.

**(b)** By making slightly abuse of notation we will also denote by $\Delta$ the operator

$$\tilde{A} : \mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T) \to \mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T)$$
$$u \mapsto \tilde{A}u,$$

where $\tilde{A}u(t,\omega) := \Delta u(t,\omega,\cdot)$ for all $(t,\omega) \in [0,T]\times\Omega$.

Let us now recall what is meant by a *solution* of equation $(*_\varepsilon)$ in the class $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$.

**Definition 7.** A solution of equation $(*_\varepsilon)$ in the class $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$ is a stochastic process $u^*_\varepsilon \in \mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)$, which fulfils the following equality for all $\varphi \in \mathscr{C}^\infty_0(\mathcal{O})$ and all $t \in [0,T]$ $\mathbb{P}$-a.s.

$$\begin{aligned}\langle u^*_\varepsilon(t,\cdot),\varphi\rangle = \langle u_0,\varphi\rangle &+ \int_0^t \langle \Delta u^*_\varepsilon(s,\cdot) + \varepsilon_f F(u^*_\varepsilon)(s,\cdot),\varphi\rangle \mathrm{d}s \\ &+ \sum_{k=1}^\infty \int_0^t \langle \varepsilon_g G(u^*_\varepsilon)^k(s,\cdot) + g^k(s,\cdot),\varphi\rangle \mathrm{d}w^k_s.\end{aligned}$$

We can now prove the following result concerning the existence and uniqueness of a solution to equation $(*_\varepsilon)$.

**Theorem 8.** *Fix $p \in [2,\infty)$ and $\gamma \in \mathbb{N}$. There exists a constant $\kappa = \kappa(d,p,\mathcal{O}) \in (0,1)$, such that the following holds: If assumptions [$A1$]–[$A7$] are satisfied for some $\theta \in (d+p-2-\kappa, d+p-2+\kappa)$, there exist two constants $a > 0$ and $b > 0$ depending on $d$, $p$, $\gamma$, $\theta$, $T$, $\mathcal{O}$, $C_F$ and $C_G$, so that for each*

$$\varepsilon = (\varepsilon_f,\varepsilon_g) \in B = B(a,b) := \big\{(x,y) \in [0,\infty)\times[0,\infty) \,|\, y < a - bx\big\},$$

*equation $(*_\varepsilon)$ has a unique solution $u^*_\varepsilon$ in the class $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$. Moreover, the following inequality holds*

$$\|u^*_\varepsilon\|_{\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)} \leq C\big(\|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U^\gamma_{p,\theta}(\mathcal{O})}\big), \tag{4}$$

*where the constant $C \in (0,\infty)$ depends only on $d$, $p$, $\gamma$, $\theta$, $T$, $\mathcal{O}$, $\varepsilon$, $C_F$ and $C_G$.*

**Remark 9.** **(a)** The existence of a constant $\kappa = \kappa(d, p, \mathcal{O}) \in (0, 1)$, such that the statements of Theorem 8 are true for the special case of linear equations, i.e. $\varepsilon = (\varepsilon_f, \varepsilon_g) = (0, 0)$, has been proven in [24]. The constant $\kappa$ in Theorem 8 is the same as the constant $\beta_0$ in Theorem 2.12 in [24], see also Remark 2.13 therein, while the constant $C = C(d, p, \gamma, \theta, T, \mathcal{O}, \varepsilon_f, \varepsilon_g, C_F, C_G)$ in (4) is given by

$$C = \frac{C_0}{1 - C_0 \left(C_F \varepsilon_f + C_G \varepsilon_g\right)},$$

with $C_0 = C(d, p, \gamma, \theta, T, \mathcal{O})$ from [24, Theorem 2.12].

**(b)** Because Theorem 2.12 in [24] holds true for fractional and negative parameters $\gamma$, see Remark 2.13 therein, the statement of Theorem 8 holds true for arbitrary $\gamma \in \mathbb{R}$.

**(c)** If the nonlinearity in the noise part of equation $(*_\varepsilon)$ vanishes, i.e. $G \equiv 0$, we can choose the Lipschitz constant $C_G \in (0, \infty)$ from assumption $[A6]$ to be as small as we want. Therefore, instead of the set $B$ in Theorem 8, we get the existence of a constant $a > 0$ depending on $d$, $p$, $\gamma$, $\theta$, $T$, $\mathcal{O}$ and $C_F$, such that the rest of the statement is true for each $\varepsilon \in [0, a) \times [0, \infty)$.

*Proof of Theorem 8.* Fix $p \in [2, \infty)$ and $\gamma \in \mathbb{N}$. Set $\kappa = \beta_0(d, p, \mathcal{O}) \in (0, 1)$ as in [24, Theorem 2.12] and choose $\theta \in (d + p - 2 - \kappa, d + p - 2 + \kappa)$. First we prove the existence and uniqueness of the solution. To achieve this goal, we use an appropriate version of the Banach fixed point theorem. For $\varepsilon = (\varepsilon_f, \varepsilon_g) \in [0, \infty) \times [0, \infty)$, let us define the following operator:

$$\begin{aligned} \mathcal{N}_\varepsilon : \mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O}, T) &\to \mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O}, T) \\ u &\mapsto \mathcal{N}_\varepsilon(u), \end{aligned}$$

where $\mathcal{N}_\varepsilon(u)$ is the unique solution of the equation

$$\mathrm{d}v = (\Delta v + \varepsilon_f F(u))\,\mathrm{d}t + \left(\varepsilon_g G(u)^k + g^k\right) \mathrm{d}w^k_t, \quad v(0, \cdot) = u_0$$

in the class $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O}, T) \subseteq \mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O}, T)$. Note that this operator is well defined due to Theorem 2.12 in [24], see also Remark 2.13 therein. Let us fix two processes $u, v \in \mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O}, T)$. Then, we know that, if we fix $\varphi \in \mathscr{C}^\infty_0(\mathcal{O})$, for all $t \in [0, T]$, $\mathbb{P}$-a.s.

$$\begin{aligned} \langle \mathcal{N}_\varepsilon(u)(t, \cdot), \varphi\rangle = \langle u_0, \varphi\rangle &+ \int_0^t \langle \Delta \mathcal{N}_\varepsilon(u)(s, \cdot) + \varepsilon_f F(u)(s, \cdot), \varphi\rangle \mathrm{d}s \\ &+ \sum_{k=1}^\infty \int_0^t \langle \varepsilon_g G(u)^k(s, \cdot) + g^k(s, \cdot), \varphi\rangle \mathrm{d}w^k_s, \end{aligned}$$

as well as

$$\langle \mathcal{N}_\varepsilon(v)(t, \cdot), \varphi\rangle = \langle u_0, \varphi\rangle + \int_0^t \langle \Delta \mathcal{N}_\varepsilon(v)(s, \cdot) + \varepsilon_f F(v)(s, \cdot), \varphi\rangle \mathrm{d}s$$

$$+\sum_{k=1}^{\infty}\int_0^t \langle \varepsilon_g G(v)^k(s,\cdot)+g^k(s,\cdot),\varphi\rangle \mathrm{d}w_s^k.$$

Substracting the two equations gives us

$$\begin{aligned}\langle \mathcal{N}_\varepsilon(u)(t,\cdot)-\mathcal{N}_\varepsilon(v)(t,\cdot),\varphi\rangle &= \int_0^t \langle \Delta\mathcal{N}_\varepsilon(u)(s,\cdot)-\Delta\mathcal{N}_\varepsilon(v)(s,\cdot),\varphi\rangle \mathrm{d}s\\ &\quad+\int_0^t \langle \varepsilon_f F(u)(s,\cdot)-\varepsilon_f F(v)(s,\cdot),\varphi\rangle \mathrm{d}s\\ &\quad+\sum_{k=1}^{\infty}\int_0^t \langle \varepsilon_g G(u)^k(s,\cdot)-\varepsilon_g G(v)^k(s,\cdot),\varphi\rangle \mathrm{d}w_s^k\end{aligned}$$

for all $t\in[0,T]$ $\mathbb{P}$-a.s. Because the Laplacian is a linear operator, we get

$$\begin{aligned}\langle \mathcal{N}_\varepsilon(u)(t,\cdot)-\mathcal{N}_\varepsilon(v)(t,\cdot),\varphi\rangle &= \int_0^t \langle \Delta\big(\mathcal{N}_\varepsilon(u)(s,\cdot)-\mathcal{N}_\varepsilon(v)(s,\cdot)\big),\varphi\rangle \mathrm{d}s\\ &\quad+\int_0^t \langle \varepsilon_f F(u)(s,\cdot)-\varepsilon_f F(v)(s,\cdot),\varphi\rangle \mathrm{d}s\\ &\quad+\sum_{k=1}^{\infty}\int_0^t \langle \varepsilon_g G(u)^k(s,\cdot)-\varepsilon_g G(v)^k(s,\cdot),\varphi\rangle \mathrm{d}w_s^k.\end{aligned}$$

Consequently, this means that $\mathcal{N}_\varepsilon(u)-\mathcal{N}_\varepsilon(v)\in\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)$ is the unique solution in the class $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$ of the linear (stochastic) partial differential equation

$$\mathrm{d}\tilde{v}=\big(\Delta\tilde{v}+\varepsilon_f F(u)-\varepsilon_f F(v)\big)\mathrm{d}t+\big(\varepsilon_g G(u)^k-\varepsilon_g G(v)^k\big)\mathrm{d}w_t^k,\quad \tilde{v}(0,\cdot)=0,$$

and the inequality

$$\begin{aligned}&\|\mathcal{N}_\varepsilon(u)-\mathcal{N}_\varepsilon(v)\|_{\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)}\\ &\qquad\le C_0\left(\varepsilon_f\|F(u)-F(v)\|_{\mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T)}+\varepsilon_g\|G(u)-G(v)\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)}\right)\end{aligned}$$

holds with some constant $C_0\in(0,\infty)$ depending only on $d$, $p$, $\gamma$, $\theta$, $T$ and $\mathcal{O}$. This follows from Theorem 2.12 in [24], see also Remark 2.13 therein for the case $\gamma=1$. The main point is that the constant $C$ there does not depend on the free terms $f$ and $g$, meaning that the constant $C_0$ in our situation does not depend on $F$, $G$, $u$ or $v$. Now, using assumptions [$A3$] and [$A6$], we get

$$\begin{aligned}\|\mathcal{N}_\varepsilon(u)-\mathcal{N}_\varepsilon(v)\|_{\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)} &\le C_0\left(C_F\,\varepsilon_f\,\|u-v\|_{\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)}+C_G\,\varepsilon_g\,\|u-v\|_{\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)}\right)\\ &\le C_0\,(C_F\,\varepsilon_f+C_G\,\varepsilon_g)\,\|u-v\|_{\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)}.\end{aligned}$$

It follows that, for

$$\varepsilon=(\varepsilon_f,\varepsilon_g)\in B:=\left\{(x,y)\in[0,\infty)\times[0,\infty)\,\Big|\quad y<\frac{1}{C_0\,C_G}-\frac{C_F}{C_G}x\right\},$$

the operator $\mathcal{N}_\varepsilon$ is a contraction from $\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)$ to $\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)$. Because the space $\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)$ of stochastic processes is a Banach space, the well-known Banach fixed point theorem gives us the existence of a unique fixed point of $\mathcal{N}_\varepsilon$. Regarding the fact that any solution of the equation $(*_\varepsilon)$ in $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$ is a fixed point of the operator $\mathcal{N}_\varepsilon$ and vice versa, we have just proven that for each $\varepsilon \in B \subseteq [0,\infty)\times[0,\infty)$, equation $(*_\varepsilon)$ has a unique solution $u^*_\varepsilon$ in the class $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$.

Now, let us prove inequality (4). Therefore, fix $\varepsilon \in B$ and denote by $q_\varepsilon \in (0,1)$ the contraction number of $\mathcal{N}_\varepsilon$. That is,

$$\|\mathcal{N}_\varepsilon(u) - \mathcal{N}_\varepsilon(v)\|_{\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)} \le q_\varepsilon \, \|u-v\|_{\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)} \text{ for all } u,v \in \mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T).$$

Let us start our fixed point iteration with $u^{(1)} = 0 \in \mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)$ and set $u^{(j+1)} := \mathcal{N}_\varepsilon(u^{(j)})$ for all $j \in \mathbb{N}$. We know

- $(*)$ that the sequence $\big(u^{(j)}\big)_{j\in\mathbb{N}}$ converges to the solution $u^*_\varepsilon$ in $\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)$, and
- $(*)$ that
$$\|\mathcal{N}_\varepsilon(u^{(1)})\|_{\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O})} \le C_0 \left(\|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U^\gamma_{p,\theta}(\mathcal{O})}\right).$$
This follows from Theorem 2.12 in [24] and assumptions [$A4$] and [$A7$]. See also Remark 2.13 therein for the case $\gamma = 1$.

The a-priori estimate from the Banach fixed point theorem leads to

$$\|u^*_\varepsilon\|_{\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)} \le \frac{1}{1-q_\varepsilon}\, C_0 \left(\|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U^\gamma_{p,\theta}(\mathcal{O})}\right).$$

The proof is finished. □

The following result is a straightforward consequence from Theorem 8. We will need it in the next section where we want to establish Besov regularity of the solution.

**Corollary 10.** *In the situation of Theorem 8, the following inequality holds for every $\varepsilon \in B$ and $\tau \in [0,p]$.*

$$\int_\Omega \int_0^T \|\rho^{\gamma-\left(1+\frac{d-\theta}{p}\right)} |D^\gamma u^*_\varepsilon(t,\omega,\,\cdot\,)|_{\ell_p}\|^\tau_{L_p(\mathcal{O})} \,\mathrm{d}t\,\mathbb{P}(\mathrm{d}\omega) \le C \left(\|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U^\gamma_{p,\theta}(\mathcal{O})}\right)^\tau.$$

*Proof.* Just replace $u$ by $u^*_\varepsilon$ in the proof of Corollary 12 in [2] and use Theorem 8 instead of Theorem 11 in [2]. □

# 3 Besov Regularity of the Solution

In this section we state and prove the main result of this paper. It is formulated in terms of the $L_\tau$-spaces

$$L_\tau(\Omega_T; B^s_{\tau,\tau}(\mathcal{O})) = L_\tau(\Omega_T, \mathscr{P}_T, \lambda\otimes\mathbb{P}; B^s_{\tau,\tau}(\mathcal{O})), \quad \tau \in (0,\infty),\ s \in (0,\infty),$$

and the spaces introduced in the last section. We use the convention "$1/0 := \infty$".

**Theorem 11.** *Given the situation from Theorem 8, fix $\varepsilon \in B$ and let $u_\varepsilon^*$ be the unique solution of equation $(*_\varepsilon)$ in the class $\mathfrak{H}_{p,\theta}^{\gamma}(\mathcal{O},T)$. Assume furthermore that*

$$u_\varepsilon^* \in L_p(\Omega_T; B_{p,p}^s(\mathcal{O})) \quad \text{for some} \quad 0 < s \le \gamma \wedge \left(1 + \frac{d-\theta}{p}\right).$$

*Then, we have*

$$u_\varepsilon^* \in L_\tau(\Omega_T; B_{\tau,\tau}^\alpha(\mathcal{O})), \quad \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}, \quad \text{for all} \quad 0 < \alpha < \gamma \wedge s\frac{d}{d-1},$$

*and the following equation holds*

$$\|u_\varepsilon^*\|_{L_\tau(\Omega_T; B_{\tau,\tau}^\alpha(\mathcal{O}))} \le C \left( \|g\|_{\mathbb{H}_{p,\theta}^{\gamma-1}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U_{p,\theta}^{\gamma}(\mathcal{O})} + \|u_\varepsilon^*\|_{L_p(\Omega_T; B_{p,p}^s(\mathcal{O}))} \right).$$

*Here the constant $C > 0$ depends only on $d$, $p$, $\gamma$, $\theta$, $T$, $\mathcal{O}$, $\varepsilon$, $C_F$, $C_G$ and $\alpha$.*

**Remark 12.** Since for $p \in (1,\infty)$ and $s \in \mathbb{R}\backslash\mathbb{N}_0$ we have $W_p^s(\mathcal{O}) = B_p^s(\mathcal{O})$, and since $\frac{d}{d-1} > 1$, Theorem 11 implies that the Besov regularity of the solution to $(*_\varepsilon)$ is usually higher than the Sobolev regularity, see also Example 13 below. Consequently, the use of adaptive schemes for the numerical treatment of $(*_\varepsilon)$ is completely justified.

*Proof of Theorem 11.* We fix $\varepsilon \in B$ and follow the lines of the proof of Theorem 15 in [2]. I.e. we establish Besov regularity by first using extension operators from $B_{p,p}^s(\mathcal{O})$ to $B_{p,p}^s(\mathbb{R}^d)$ as described in [30], and then employing the wavelet characterization according to Theorem 19 in the appendix. The estimation of the wavelet coefficients, which correspond to the wavelets with support inside of the domain $\mathcal{O}$ is performed by the weighted Sobolev results from the previous section. □

**Example 13.** Let $p = 2$ and $\gamma = 2$ and choose $\theta = d \in (d-c, d+c)$ for all $c \in (0,1)$. Furthermore, let $G \equiv 0$ and set

$$\begin{aligned} F : \mathbb{H}_{2,d-2}^2(\mathcal{O},T) &\to \mathbb{H}_{2,d+2}^0(\mathcal{O},T) \\ u &\mapsto F(u), \end{aligned}$$

where for each $u \in \mathbb{H}_{2,d-2}^2(\mathcal{O},T)$ we set

$$\begin{aligned} F(u) : [0,T] \times \Omega &\to H_{2,d+2}^0(\mathcal{O}) \\ (t,\omega) &\mapsto F(u)(t,\omega) := \rho^{-2} \sin(u(t,\omega,\cdot)). \end{aligned}$$

Then obviously, $F(0) = 0$ and $F$ is Lipschitz continuous with Lipschitz constant $C_F = 1$. Now, Theorem 8 gives us the existence of an $\varepsilon_{0f} > 0$, such that for any $\varepsilon_f \in (0, \varepsilon_{0f})$ the equation

$$\mathrm{d}u = \big(\Delta u + \varepsilon_f\, F(u)\big)\mathrm{d}t + \sum_{k=1}^{\infty} g^k(t)\mathrm{d}w_t^k, \quad u(0,\cdot) = u_0$$

has a unique solution $u_\varepsilon^*$ in the class $\mathfrak{H}^2_{2,d}(\mathcal{O},T)$. This implies that our solution also has the following spatial regularity

$$\begin{aligned} u_\varepsilon^* \in L_2(\Omega_T, \mathscr{P}_T, \lambda\otimes\mathbb{P}; H^2_{2,d-2}(\mathcal{O})) &\subseteq L_2(\Omega_T, \mathscr{P}_T, \lambda\otimes\mathbb{P}; H^1_{2,d-2}(\mathcal{O})) \\ &= L_2(\Omega_T, \mathscr{P}_T, \lambda\otimes\mathbb{P}; \overset{\circ}{W}{}^1_2(\mathcal{O})) \\ &\subseteq L_2(\Omega_T, \mathscr{P}_T, \lambda\otimes\mathbb{P}; B^1_{2,2}(\mathcal{O})). \end{aligned}$$

Hence, we can apply Theorem 11 and get

$$u_\varepsilon^* \in L_\tau(\Omega_T, \mathscr{P}_T, \lambda\otimes\mathbb{P}; B^\alpha_{\tau,\tau}(\mathcal{O})) \text{ for all } \alpha \in (0, \frac{d}{d-1}) \text{ with } \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{2}.$$

In the two-dimensional setting ($d = 2$) we conclude for our solution, that

$$u_\varepsilon^* \in L_\tau([0,T]\times\Omega, \mathscr{P}_T, \lambda\otimes\mathbb{P}; B^\alpha_{\tau,\tau}(\mathcal{O})) \text{ for all } \alpha \in (0,2) \text{ with } \frac{1}{\tau} = \frac{\alpha+1}{2}.$$

# 4 A Wider Class of Equations

In the introduction we have indicated that our main result can be extended to equation $(\bullet_\varepsilon)$. The major reason is, that with the same techniques used in the proof of Theorem 8 we can prove existence and uniqueness of a solution to equation $(\bullet_\varepsilon)$, provided the coefficients $a^{ij}$, $b^i$, $c$, $\sigma^{ik}$ and $\eta^k$, the free terms $f$ and $g^k$ and the initial value $u_0$ fulfil certain conditions. This is done by using the general result from [24, Theorem 2.12] and employing again the Banach fixed point theorem. A similar estimate to the one from Corollary 10 follows immediately. We can use this fact to extend our regularity result to such equations. In this section we want to sketch the basic steps.

For the convenience of the reader we begin by presenting the assumptions on the coefficients used in [24, Theorem 2.12]. Therefore, we need some additional notations. For $x, y \in \mathcal{O}$ we shall write $\rho(x,y) := \rho(x) \wedge \rho(y)$. For $\alpha \in \mathbb{R}$, $\delta \in (0,1]$ and $m \in \mathbb{N}_0$ we set:

$$[f]^{(\alpha)}_m := \sup_{x\in\mathcal{O}} \rho^{m+\alpha}(x)|D^m f(x)| \qquad \text{and} \qquad [f]^{(\alpha)}_{m+\delta} := \sup_{\substack{x,y\in\mathcal{O} \\ |\beta|=m}} \rho^{m+\alpha}(x,y)\frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\delta},$$

$$|f|^{(\alpha)}_m := \sum_{j=0}^m [f]^{(\alpha)}_j \qquad \text{and} \qquad |f|^{(\alpha)}_{m+\delta} := |f|^{(\alpha)}_m + [f]^{(\alpha)}_{m+\delta},$$

whenever it makes sense. We shall use the same notations for $\ell_2$-valued functions (just replace the absolute values in the above definitions by the $\ell_2$-norms). Furthermore, let's fix an arbitrary function

$$\mu_0 : [0,\infty) \to [0,\infty),$$

vanishing only on the set of nonnegative integers, i.e. $\mu_0(m) = 0$ if and only if $m \in \mathbb{N}_0$. We set

$$t_+ := t + \mu_0(t).$$

Now we are able to present the assumptions on the coefficients of equation $(\bullet_\varepsilon)$ (see [23, Assumptions 2.5 and 2.6] as well as [24, Assumptions 2.10]).

$[K1]$ For any fixed $x \in \mathcal{O}$, the coefficients

$$a^{ij}(.,.,x), b^i(.,.,x), c(.,.,x), \sigma^{ik}(.,.,x), \eta^k(.,.,x) : [0,T] \times \Omega \to \mathbb{R}$$

are predictable processes with respect to the given normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$.

$[K2]$ (Stochastic parabolicity) There are constants $\delta_0, K > 0$, such that for all $(t,\omega,x) \in [0,T] \times \Omega \times \mathcal{O}$ and $\lambda \in \mathbb{R}^d$:

$$\delta_0|\lambda|^2 \leq \overline{a^{ij}}(t,\omega,x)\lambda_i\lambda_j \leq K|\lambda|^2,$$

where $\overline{a^{ij}} := a^{ij} - \frac{1}{2}\langle\sigma^i,\sigma^j\rangle_{\ell_2}$ for $i,j \in \{1,\dots,d\}$.

$[K3]$ For all $(t,\omega) \in [0,T] \times \Omega$:

$$\begin{aligned}|a^{ij}(t,\omega,.)|^{(0)}_{|\gamma|_+} + |b^i(t,\omega,.)|^{(1)}_{|\gamma|_+} + |c(t,\omega,.)|^{(2)}_{|\gamma|_+}& \\ + |\sigma^i(t,\omega,.)|^{(0)}_{|\gamma|_+} + |\eta(t,\omega,.)|^{(1)}_{|\gamma+1|_+} &\leq K.\end{aligned}$$

$[K4]$ The coefficients $a^{ij}$ and $\sigma^i$ are uniformly continuous in $x \in \mathcal{O}$, i.e. for any $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$, such that

$$|a^{ij}(t,\omega,x) - a^{ij}(t,\omega,y)| + |\sigma^i(t,\omega,x) - \sigma^i(t,\omega,y)|_{\ell_2} \leq \epsilon,$$

for all $(t,\omega) \in [0,T] \times \Omega$, whenever $x, y \in \mathcal{O}$ with $|x-y| \leq \delta$.

$[K5]$ The behaviour of the coefficients $b^i$, $c$ and $\eta$ can be controlled near the boundary of $\mathcal{O}$ in the following way:

$$\lim_{\substack{\rho(x)\to 0\\ x\in\mathcal{O}}} \sup_{\substack{\omega\in\Omega\\ t\in[0,T]}} \{\rho(x)|b^i(t,\omega,x)| + \rho^2(x)|c(t,\omega,x)| + \rho(x)|\eta(t,\omega,x)|_{\ell_2}\} = 0.$$

Here is the result concerning the solvability of equation $(\bullet_\varepsilon)$ in the classes $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$.

**Theorem 14.** *Fix $p \in [2,\infty)$ and $\gamma \in \mathbb{N}$. Furthermore, let assumptions $[K1]$–$[K5]$ be satisfied for some $K > 0$ and $\delta_0 > 0$. Then there exists a constant $\kappa = \kappa(d,p,\gamma,\delta_0,K,\mathcal{O}) \in (0,1)$, such that the following holds: If assumptions $[A1]$–$[A7]$ are satisfied for some $\theta \in (d+p-2-\kappa, d+p-2+\kappa)$, there exist two constants $a > 0$ and $b > 0$ depending on $d$, $\gamma$, $p$, $\theta$, $\delta_0$, $K$, $T$, $\mathcal{O}$, $C_F$ and $C_G$, so that for each*

$$\varepsilon = (\varepsilon_f, \varepsilon_g) \in B = B(a,b) := \big\{(x,y) \in [0,\infty) \times [0,\infty) \,|\, y < a - bx\big\},$$

*equation $(\bullet_\varepsilon)$ has a unique solution $u^\bullet_\varepsilon$ in the class $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$. Moreover, for this solution we have*

$$\|u^\bullet_\varepsilon\|_{\mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T)} \leq C\left(\|f\|_{\mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T)} + \|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U^\gamma_{p,\theta}(\mathcal{O})}\right), \tag{5}$$

*where $C$ is a constant depending only on $d$, $\gamma$, $p$, $\theta$, $\delta_0$, $K$, $T$, $C_F$, $C_G$, $\varepsilon$ and $\mathcal{O}$.*

**Remark 15.** The statements of Remark 9 (a) and (b) also carry over to Theorem 14.

An immediate consequence of Theorem 14 is the following estimate.

**Corollary 16.** *In the situation of Theorem 14, the following inequality holds for every $\varepsilon \in B$ and $\tau \in [0,p]$.*

$$\begin{aligned}\int_\Omega \int_0^T \|\rho^{\gamma-\left(1+\frac{d-\theta}{p}\right)}|D^\gamma u_\varepsilon^\bullet(t,\omega,\,\cdot\,)|_{\ell_p}\|_{L_p(\mathcal{O})}^\tau \,\mathrm{d}t\,\mathbb{P}(\mathrm{d}\omega) \\ \leq C\left(\|f\|_{\mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T)} + \|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U^\gamma_{p,\theta}(\mathcal{O})}\right)^\tau.\end{aligned}$$

*Proof.* Just repeat the arguments of the proof of Corollary 10 and use estimate (5) instead of (4) at the beginning. ☐

Now we can present our result concerning the Besov regularity of the solution in the generalized setting.

**Theorem 17.** *Given the situation from Theorem 14, fix $\varepsilon \in B$ and let $u_\varepsilon^\bullet$ be the unique solution to equation $(\bullet_\varepsilon)$ in the class $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$. Furthermore, assume that*

$$u_\varepsilon^\bullet \in L_p(\Omega_T; B^s_{p,p}(\mathcal{O})) \quad \textit{for some} \quad 0 < s \leq \gamma \wedge \left(1 + \frac{d-\theta}{p}\right).$$

*Then, we have*

$$u_\varepsilon^\bullet \in L_\tau(\Omega_T; B^\alpha_{\tau,\tau}(\mathcal{O})), \quad \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}, \quad \textit{for all} \quad 0 < \alpha < \gamma \wedge s\frac{d}{d-1},$$

*and the following inequality holds*

$$\|u_\varepsilon^\bullet\|_{L_\tau(\Omega_T;B^\alpha_{\tau,\tau}(\mathcal{O}))} \leq C\left(\|f\|_{\mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T)} + \|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U^\gamma_{p,\theta}(\mathcal{O})} + \|u_\varepsilon^\bullet\|_{L_p(\Omega_T;B^s_{p,p}(\mathcal{O}))}\right).$$

*Proof.* We can argue like we did in the proof of Theorem 11. We just have to use Corollary 16 where we used Corollary 10. ☐

# A Appendix

In this appendix we give a definition of Besov spaces and present a well-known result concerning the wavelet characterization of these function spaces. Our standard reference in this context is the monograph [3].

The most common way to define Besov spaces is by making use of differences. For a function $f : \mathcal{O} \to \mathbb{R}$ and a natural number $n \in \mathbb{N}$ let

$$\Delta_h^n f(x) := \prod_{i=0}^n \mathbb{1}_{\mathcal{O}}(x+ih) \cdot \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} f(x+jh)$$

be the $n$-th difference of $f$ with step $h \in \mathbb{R}^d$. For $p \in (0, \infty)$ the modulus of smoothness is given by

$$\omega^n(t, f)_p := \sup_{|h|<t} \| \Delta_h^n f \|_{L_p(\mathcal{O})}, \qquad t > 0.$$

One approach to introduce Besov spaces is the following (see, e.g. [16, p.91]).

**Definition 18.** Let $s, p, q \in (0, \infty)$ and $n \in \mathbb{N}$ with $n > s$. Then $B^s_{p,q}(\mathcal{O})$ is the collection of all functions $f \in L_p(\mathcal{O})$ such that

$$| f |_{B^s_{p,q}(\mathcal{O})} := \left( \int_0^\infty \left[ t^{-s}\, \omega^n(t, f)_p \right]^q \frac{\mathrm{d}t}{t} \right)^{1/q} < \infty.$$

These classes are equipped with a (quasi-)norm by taking

$$\| f \|_{B^s_{p,q}(\mathcal{O})} := \| f \|_{L_p(\mathcal{O})} + | f |_{B^s_{p,q}(\mathcal{O})}.$$

For certain constellations of the parameters $p$, $q$ and $s$, Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ can be characterized by the decay properties of wavelet coefficients. To this end, fix an arbitrary $r \in \mathbb{N}$, and let $\varphi \in \mathscr{C}^r(\mathbb{R}^d)$ be a compactly supported scaling function as constructed by I. Daubechies, see [15]. We denote by $\psi_i \in \mathscr{C}^r(\mathbb{R}^d)$, $i = 1, \ldots, 2^d - 1$, the corresponding multivariate Daubechies mother wavelets having compact support and vanishing moments up to order $r$. Furthermore, assume that

$$\{\varphi_k, \psi_{i,j,k} \,:\, (i, j, k) \in \{1, \ldots, 2^d - 1\} \times \mathbb{N} \times \mathbb{Z}^d\}$$

is an orthogonal wavelet basis of $L_2(\mathbb{R}^d)$. (We used the standard abbreviations for dyadic shifts and dilations of functions, see e.g. [2, p.8].) Such a basis can be used for the characterization of Besov spaces as follows.

**Theorem 19.** *Let $p, q \in (0, \infty)$ and $s > \max\{0, d(1/p - 1)\}$. Choose $r \in \mathbb{N}$ with $r > s$ and a corresponding Daubechies orthonormal wavelet basis. Then a locally integrable function $f : \mathbb{R}^d \to \mathbb{R}$ is in the Besov space $B^s_{p,q}(\mathbb{R}^d)$ if, and only if,*

$$f = \sum_{k \in \mathbb{Z}^d} \langle f, \varphi_k \rangle \varphi_k + \sum_{i=1}^{2^d-1} \sum_{j \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}^d} \langle f, \psi_{i,j,k} \rangle \psi_{i,j,k}$$

*(convergence in $\mathcal{D}'(\mathbb{R}^d)$) with*

$$\Big( \sum_{k \in \mathbb{Z}^d} |\langle f, \varphi_k \rangle|^p \Big)^{1/p} + \Big( \sum_{i=1}^{2^d-1} \sum_{j \in \mathbb{N}_0} 2^{jsq} 2^{jd(1/2-1/p)q} \big( \sum_{k \in \mathbb{Z}^d} |\langle f, \psi_{i,j,k} \rangle|^p \big)^{q/p} \Big)^{1/q} < \infty, \tag{6}$$

*and* (6) *is an equivalent (quasi-)norm for $B^s_{p,q}(\mathbb{R}^d)$.*

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