DFG-Schwerpunktprogramm 1324

„Extraktion quantifizierbarer Information aus komplexen Systemen"

Primal and Dual Pricing of Multiple Exercise Options in Continuous Time

C. Bender

Preprint 75

Edited by

AG Numerik/Optimierung
Fachbereich 12 - Mathematik und Informatik
Philipps-Universität Marburg
Hans-Meerwein-Str.
35032 Marburg

DFG-Schwerpunktprogramm 1324

„Extraktion quantifizierbarer Information aus komplexen Systemen"

Primal and Dual Pricing of Multiple Exercise Options in Continuous Time

C. Bender

Preprint 75

Primal and dual pricing of multiple exercise options in continuous time

Christian Bender[1]

December 21, 2010

Abstract

In this paper we study the pricing problem of multiple exercise options in continuous time on a finite time horizon. For the corresponding multiple stopping problem, we prove, under quite general assumptions, the existence of the Snell envelope, a reduction principle as nested single stopping problems, and a Doob-Meyer type decomposition for the Snell envelope. The main technical difficulty arises from the fact that the price process of a multiple exercise option typically exhibits discontinuities from the right hand side, even if the payoff process of the option is right-continuous. We also derive a dual minimization problem for the price of the multiple exercise option in terms of martingales and processes of bounded variation. Moreover, we explain how the primal and dual pricing formulas can be applied to compute confidence intervals on the option price via Monte-Carlo methods and present a numerical example.

Keywords: Doob-Meyer decomposition, duality, option pricing, Monte-Carlo methods, multiple exercise options, swing options.

AMS classification: 91B28, 60G40, 65C05, 60G44.

1 Introduction

Options with several early exercise rights are popular in different financial markets, in particular in energy markets. Due the lack of storability and the varying demand for energy, options have been designed which admit flexibilty concerning the timing of delivery. In a simple version of such a swing option, the buyer of the option is entitled to exercise a certain right (e.g. the delivery of some amount of energy) several times until the maturity of the option. The number of exercise rights and possible constraints on how to use the rights are specified in the contract (see e.g. Jaillet et al., 2004). One of such constraints which is typically included in a swing option is

[1]Saarland University, Department of Mathematics, PO Box 151150, D-66041 Saarbrücken, Germany. bender@math.uni-sb.de.

the so called refraction period. The refraction period is the minimal time between exercising two rights. For swing options, which include the delivery of energy, the refraction period is at least the time required for the physical delivery.

The buyer of a multiple exercise option is faced with the problem of exercising the rights optimally while obeying the refraction period. Mathematically, this problem is a multiple stopping problem, which we here treat in a continuous time setting with finite time horizon (i.e. with finite maturity). The aim of the present paper is to study such a multiple stopping problem (which we refer to as the primal problem) and to derive a dual representation as a minimization problem. Primal and dual representation can then be combined to design a Monte-Carlo algorithm for the pricing of multiple exercise options.

Our first main result is Theorem 2.2, which summarizes some structural results on the multiple stopping problem. Under the assumption that the cashflow $Z(t)$, which can be exercised at most n times by the holder of the option, is right-continuous (up to maturity) and satisfies an integrability condition, we prove the existence of the Snell envelope and its Doob-Meyer decomposition. Moreover, we derive a reduction principle of the multiple stopping problem as a nested sequence of single stopping problems. Such reduction principle is well-known in discrete time (see Haggstrom, 1967; Cairoli and Dalang, 1996). In continuous time it was studied by Carmona and Touzi (2008) and Zeghal and Mnif (2006) under stronger conditions than the ones imposed in the present paper. The main technical difficulty related to Theorem 2.2 is that the value process of the multiple stopping problem typically exhibits discontinuities from the right, (although the cash-flow Z is right-continuous). The lack of right-continuity even occurs in very simple examples, e.g. if the cashflow Z is constant 1 until maturity (see Example 4.1 below). It is a consequence of the finite maturity and the refraction period, which are both part of the swing contracts traded in industry practice. The refraction period is also essential from a mathematical point of view, because, without refraction period, all n optimal exercise times for the multiple stopping problem would cluster at an optimal time for the single stopping problem, i.e. the value of the multiple stopping problem would trivially reduce to n times the value of the single stopping problem with the same cashflow.

As a second contribution we derive a dual minimization problem for the price of the multiple exercise option in continuous time. In the classical case of an American option (i.e. a single exercise right) a dual minimization problems in terms of martingales is due to Rogers (2002) and Haugh and Kogan (2004). Extensions to multiple exercise option were only known in a discrete time setting. One such extension by Meinshausen and Hambly (2004) yields a dual minimization problem in terms of a martingale and $(n-1)$ stopping times for the marginal price of an additional nth right. The Meinshausen-

Hambly dual, and its generalization to volume constraints by Bender (2010) (see also Aleksandrov and Hambly, 2010, for an alternative proof), requires that the refraction period equals the discrete time step, and may, thus, not be tranferable to a continuous time setting. Recently, Schoenmakers (2010) came up with an alternative multiple dual in discrete time, which is a minimization problem over martingales only. Theorem 2.3 below can be interpreted as a generalization of the Schoenmakers dual to a continuous time setting. The conceptually new feature of the continuous time multiple dual is that not only the martingale part, but also the increasing part of the Doob-Meyer decomposition of the price process matters. Consequently, we end up with a minimization problem over martingales and processes of bounded variation.

We also discuss how the primal and the dual formulation for the price of a multiple exercise option can be applied to compute the option price numerically by a Monte-Carlo approach. Our algorithm extends the primal-dual approach by Andersen and Broadie (2004) from a single exercise rights to several rights. After a time discretizaton of the option is performed, the algorithm consists of three steps. First, a dynamic programming formulation suggested by the reduction principle is solved approximatively. We apply the least-squares Monte-Carlo approach (Longstaff and Schwartz, 2001) for the conditional expectations, but other estimators e.g. using quantization (Bally and Pagès, 2003) or Malliavin Monte-Carlo (Bouchard et al., 2004; Carmona and Touzi, 2008) are possible. As a second step, based on the approximative continuation values one obtains candidates for 'close-to-optimal' stopping times which are applied to calculate a lower bound on the option price. The final step consists of performing the Doob-Meyer decomposition of the approximative solution of the dynamic program numerically, which is then plugged into the dual formulation and yields an upper bound of the price. We use a nested simulation approach (Andersen and Broadie, 2004) for the numerical Doob-Meyer decomposition, but one could alternatively apply the non-nested procedure by Belomestny et al. (2009) (if the filtration is generated by a Brownian motion).

The paper is organized as follows: The main results are stated in Section 2, which also contains a detailed comparison to the results by Schoenmakers (2010). Section 3 is devoted to the Monte-Carlo algorithm, which are illustrated by some numerical experiments. The proof of the main results are postponed to Sections 5 and 6.

2 Statement of the main results

We consider an option which can be exercised up to n times by the holder of the option. A refraction period $\delta > 0$ imposes the constraint that a minimal time period of δ must lie in between two exercises. The discounted cash-flow of the option is modeled by a stochastic process $Z(t)$, $0 \leq t \leq T$, adapted to a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, where the underlying probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ is assumed to satisfy the usual conditions. In our context P is already a fixed pricing measure (i.e. the discounted tradable and storable primary securities in the market are σ-martingales under P). Throughout the paper, the following conditions are imposed on the discounted cash-flow Z:

(H): $(Z(t),\, 0 \leq t \leq T)$ has right-continuous, nonnegative paths and

$$E[\sup_{0\leq t\leq T} |Z(t)|^p] < \infty$$

for some $p > 1$.

We emphasize that, due to the refraction period, it may be beneficial for the holder not to exercise all of the rights. We incorporate this feature in the problem formulation by exercising some of the rights later than time T where we set the cash-flow equal to zero. Precisely, we extend $Z(t)$ and $\mathcal{F}_t$ on the positive real line by setting

$$Z_t = 0, \quad \mathcal{F}_t = \mathcal{F}_T$$

for $T < t < \infty$. This extension induces a discontinuity from the right at $t = T$, unless $Z(T) = 0$.

Given a stopping time σ, we denote by $\mathcal{S}_\sigma$ the set of $(\mathcal{F}_s)_{0\leq s<\infty}$-stopping times taking value bigger or equal to σ. For $n \in \mathbb{N}$, $\delta > 0$, and $t \geq 0$ we define

$$\begin{aligned}
\Delta^{\delta,n} &= \{(u_1, \ldots, u_n) \in [0,\infty)^n;\; u_i \geq u_{i+1} + \delta \text{ for all } i = 1, \ldots, n-1\}, \\
\Delta^{\delta,n}_t &= \Delta^{\delta,n} \cap \{(u_1, \ldots, u_n) \in [0,\infty)^n;\; u_n \geq t\}
\end{aligned}$$

Finally, for every stopping time σ, $\mathcal{S}^n_{\delta,\sigma}$ contains those n-tuples of stopping times $(\tau_1, \ldots, \tau_n) \subset \mathcal{S}_\sigma$ taking values in $\Delta^{\delta,n}$. Then, the discounted price of the multi-exercise option with n exercise rights and refraction period δ in the price system induced by P at time σ is given by

$$\bar{Y}^{*,n}(\sigma) = \operatorname*{esssup}_{(\tau_1,\ldots,\tau_n)\in\mathcal{S}^n_{\delta,\sigma}} \sum_{\nu=1}^{n} E[Z(\tau_\nu)|\mathcal{F}_\sigma].$$

The aim of this paper is to analyze the structure of the multiple stopping problem $\bar{Y}^{*,n}(\sigma)$, which we consider the primal pricing formula for the multiple exercise option, and to derive an equivalent minimization problem in terms of martingales and processes of bounded variation, which can be interpreted as a dual formulation. The primal and dual pricing formulas then serve as a starting point for designing a Monte Carlo algorithm to compute confidence intervals on the option price of the multiple exercise option.

Our first main result includes a *reduction principle* of the multiple stopping problem in terms of single stopping problems and a Doob-Meyer decomposition for $\bar{Y}^{*,n}$. Both results are valid under condition (H) and are also crucial for the derivation of the dual pricing formula. Before we can properly state the corresponding theorem, we introduce two notions on the path regularity.

Definition 2.1. We call a stochastic process X D_n-*RC*, if its discontinuities from the right are included in the set $D_n := \{T - \nu\delta;\ \nu = 0, \ldots, n-1\}$. It is called D_n-*RCLL*, if it additionally has left limits.

Theorem 2.2. *Suppose (H). Then, for every $n \in \mathbb{N}$, it holds that:*
– (S_n) (Snell envelope) There is an D_n-RCLL supermartingale $Y^{*,n}(t)$ such that for every stopping time σ

$$Y^{*,n}(\sigma) = \bar{Y}^{*,n}(\sigma)$$

and

$$E[\sup_{0\le t<\infty} |Y^{*,n}(t)|^p] < \infty.$$

– (DM_n) (Doob-Meyer-decomposition) There is an RCLL martingale $M^{*,n}$ with $M^{*,n}(0) = 0$ and $E[\sup_{0\le t<\infty} |M(t)|] < \infty$ and a predictable, integrable, D_n-RCLL, nondecreasing process $A^{*,n}$ with $A^{*,n}(0) = 0$ such that for every $0 \le t < \infty$

$$Y^{*,n}(t) = Y^{*,n}(0) + M^{*,n}(t) - A^{*,n}(t).$$

– (R_n) (Reduction principle) There is an adapted, D_n-RC process $Z^n(t)$ such that

$$Y^{*,n}(\sigma) = \operatorname*{esssup}_{\tau\in\mathcal{S}_\sigma} E[Z^n(\tau)|\mathcal{F}_\sigma].$$

and

$$E[\sup_{0\le t<\infty} |Z^n(t)|^p] < \infty.$$

Here $Z^n(t)$ is a modification of $Z(t) + E[Y^{*,n-1}(t+\delta)|\mathcal{F}_t]$ such that

$$Z^n(\tau) = Z(\tau) + E[Y^{*,n-1}(\tau+\delta)|\mathcal{F}_\tau]$$

for every stopping time τ (with the convention $Y^{*,0} \equiv 0$).

The proof will be given in Section 4 below.

The reduction principle follows the intuition that the option with n exercise rights is as good as an option with a single right which promises the cash-flow plus the right to enter a contract with $(n-1)$ rights δ years later, i.e. a single optimal stopping problem with cash-flow $Z(t) + E[Y^{*,n-1}(t+\delta)|\mathcal{F}_t]$. In a discrete time setting it is easily verified and can be traced back to Haggstrom (1967), see also Cairoli and Dalang (1996). In continuous time it is derived by Carmona and Dayanik (2008) on an infinite time horizon ($T = \infty$), and on a finite time horizon by Carmona and Touzi (2008) and Zeghal and Mnif (2006) under more restrictive assumptions than (H). The main technical difficulty related to Theorem 2.2 in the general case is that we cannot expect the processes $Y^{*,n}(t)$ to have a right-continuous modification on $[0, T]$ for $n \geq 2$, see Example 4.1 below. This problem cannot be avoided on a finite time horizon due to the presence of the refraction period, unless the discounted cash-flow vanishes at terminal time, i.e. $Z(T) = 0$. Due to the lack of right-continuity of $Y^{*,n}(t)$ and $Z^n(t)$ neither the standard formulation of the Doob-Meyer decomposition (e.g. Karatzas and Shreve, 1991, Theorem 1.4.10) nor the standard theory of optimal stopping (e.g. Karatzas and Shreve, 1998, Appendix D) are at our disposal.

With the structural results in Theorem 2.2 at hand are now in the position to state the second main result on dual pricing of multiple exercise options in continuous time. For the case of a single exercise $n = 1$ right it coincides with the dual formulation of Rogers (2002) and Haugh and Kogan (2004) in terms of the martingale part of the Doob-Meyer decomposition. The conceptual novelty for several exercise rights is that also the increasing process of the Doob-Meyer decomposition enters the formula.

Theorem 2.3 (Dual pricing formula)**.** *Suppose (H). Then:*
(i) Suppose $M^1, \ldots M^n$ are RCLL martingales with $M^\nu(0) = 0$ and

$$E[\sup_{0 \leq t \leq T} |M^\nu(t)|] < \infty, \quad \nu = 1, \ldots, n$$

and $A^1, \ldots, A^{n-1}$ are integrable adapted processes of bounded variation with $A^\nu(0) = 0$ and such that A^ν is D_ν-RCLL for $\nu = 1, \ldots, n-1$. Then,

$$\begin{aligned} Y^{*,n}(t) \quad \leq \quad & E\Bigg[\sup_{u_1,\ldots,u_n \in \Delta_t^{\delta,n}} \sum_{\nu=1}^{n-1} \Big(Z(u_\nu) - (M^\nu(u_\nu) - M^\nu(u_{\nu+1})) \\ & \qquad + A^\nu(u_{\nu+1} + \delta) - E[A^\nu(u_{\nu+1} + \delta) | \mathcal{F}_{u_{\nu+1}}] \Big) \\ & \qquad + Z(u_n) - (M^n(u_n) - M^n(t)) \Bigg| \mathcal{F}_t \Bigg] \end{aligned}$$

(ii) Moreover, with the Doob-Meyer decomposition in Theorem 2.2,

$$\begin{aligned} Y^{*,n}(t) \quad = \quad & \sup_{u_1,\ldots,u_n \in \Delta_t^{\delta,n}} \Bigg\{ \sum_{\nu=1}^{n-1} \Big(Z(u_\nu) - (M^{*,\nu}(u_\nu) - M^{*,\nu}(u_{\nu+1})) \\ & \qquad + A^{*,\nu}(u_{\nu+1} + \delta) - E[A^{*,\nu}(u_{\nu+1} + \delta) | \mathcal{F}_{u_{\nu+1}}] \Big) \\ & \qquad + Z(u_n) - (M^{*,n}(u_n) - M^{*,n}(t)) \Bigg\}. \end{aligned}$$

The proof is postponed to Section 5.

Remark 2.4. (A technical remark) Thanks to Proposition 4.4 and Remark 4.5 below, the processes $E[A^\nu(t+\delta)|\mathcal{F}_t]$ in part (i) of Theorem 2.3 have a modification $\mathcal{A}^\nu(t)$ such that $\mathcal{A}^\nu(\tau) = E[A^\nu(\tau+\delta)|\mathcal{F}_\tau]$ for every bounded stopping time τ. Expressions such as $E[A^\nu(u_{\nu+1}+\delta)|\mathcal{F}_{u_{\nu+1}}]$ in part (i) of Theorem 2.3 are to be read as the respective modifications.

We close this section by a comparison of Theorem 2.3 with the dual formulation for multiple exercise options obtained by Schoenmakers (2010) in discrete time. Hence, we switch to a discrete time setting with K time steps and consider a discrete time stochastic process $Z(k),\ k = 0,\dots,K$, adapted to some filtration $(\mathcal{F}_k)_{k=1,\dots,K}$. As in the continuous time setting we extend the process and the filtration by setting $Z(k) = 0$, $\mathcal{F}_k = \mathcal{F}_K$ for $k \geq K+1$. We assume that $\delta \in \mathbb{N}$ and consider the discrete time multiple stopping problem

$$Y^{d,n}(k) = \operatorname*{esssup}_{(\tau_1,\dots,\tau_n)\in \mathcal{S}^{d,n}_{\delta,k}} \sum_{\nu=1}^{n} E[Z(\tau_\nu)|\mathcal{F}_k], \quad k \in \mathbb{N},$$

where $\mathcal{S}^{d,n}_{\delta,k}$ is the subset of $\mathcal{S}^{n}_{\delta,k}$ taking values in $\mathbb{N}^n$. (The superscript d indicates that we study the discrete time problem). We embed this discrete time multiple stopping problem in the continuous time setting with maturity $T = K+1$ by defining

$$Z(t) = Z(k), \quad \mathcal{F}_t = \mathcal{F}_k, \quad k \leq t < k+1,\ k \in \mathbb{N}.$$

Moreover, we denote by $Y^{*,n}(t)$ the Snell envelope for the continuous time multiple stopping problem with discounted cash-flow $Z(t)$, cp. Theorem 2.2, (S_n). It is straightforward to verify that $Y^{*,n}(t)$ has RCLL and piecewise constant paths:

$$Y^{*,n}(t) = Y^{d,n}(k), \quad k \leq t < k+1,\ k \in \mathbb{N}.$$

Now, Theorem 2.3 specializes to the discrete time multiple stopping problem as follows.

Corollary 2.5 (Dual pricing formula in discrete time)**.** *In the discrete time setting it holds:*
(i) Suppose $M^1,\dots M^n$ are discrete time martingales starting in zero and and $A^1,\dots,A^{n-1}$ are integrable adapted processes in discrete time starting in zero. Then, for every $k \in \mathbb{N}$

$$\begin{aligned} Y^{d,n}(k) \ \leq\ & E\Bigg[\sup_{u_1,\dots,u_n\in\Delta_k^{\delta,n}\cap\mathbb{N}^n} \sum_{\nu=1}^{n-1}\Big(Z(u_\nu) - (M^\nu(u_\nu) - M^\nu(u_{\nu+1})) \\ & \qquad + A^\nu(u_{\nu+1}+\delta) - E[A^\nu(u_{\nu+1}+\delta)|\mathcal{F}_{u_{\nu+1}}]\Big) \\ & \qquad + Z(u_n) - (M^n(u_n) - M^n(k))\Bigg|\mathcal{F}_k\Bigg] \end{aligned}$$

(ii) Moreover, if $Y^{d,\nu}(k) = Y^{d,\nu}(0) + M^{d,\nu}(k) - A^{d,\nu}(k)$ is the Doob decomposition of the discrete time process $Y^{d,\nu}(k)$ for $\nu = 1, \dots, n$, then

$$\begin{aligned} Y^{d,n}(k) &= \sup_{u_1,\dots,u_n \in \Delta_k^{\delta,n} \cap \mathbb{N}^n} \Bigg\{ \sum_{\nu=1}^{n-1} \Big(Z(u_\nu) - (M^{d,\nu}(u_\nu) - M^{d,\nu}(u_{\nu+1})) \\ &\qquad + A^{d,\nu}(u_{\nu+1} + \delta) - E[A^{d,\nu}(u_{\nu+1} + \delta) | \mathcal{F}_{u_{\nu+1}}] \Big) \\ &\qquad + Z(u_n) - (M^{d,n}(u_n) - M^{d,n}(k)) \Bigg\}. \end{aligned}$$

Schoenmakers (2010) considers a discrete time setting with refraction period $\delta = 1$, while the above corollary covers any refraction period $\delta \in \mathbb{N}$. For $\delta = 1$, we observe that the processes $A^{d,\nu}(k+\delta)$ in (ii) are $\mathcal{F}_k$-measurable by predictability. Hence,

$$A^{d,\nu}(u_{\nu+1} + \delta) - E[A^{d,\nu}(u_{\nu+1} + \delta) | \mathcal{F}_{u_{\nu+1}}] = 0$$

and (ii) simplifies to

$$Y^{d,n}(k) = \sup_{\substack{u_1 \dots, u_n \in \mathbb{N}^n \\ u_1 > u_2 \dots > u_n \geq k}} \Bigg\{ \sum_{\nu=1}^{n} \Big(Z(u_\nu) - (M^{d,\nu}(u_\nu) - M^{d,\nu}(u_{\nu+1})) \Bigg\},$$

with $u_{n+1} := k$. This is exactly the dual formulation of Schoenmakers (2010). We observe, however, that this pure martingale dual for the multiple stopping problem in discrete time only holds, when the time step is equal to the refraction period. If the refraction period is a non-trivial multiple of the time step, the increasing processes are indispensible even in discrete time.

3 A Monte Carlo algorithm and numerical results

In this section we explain how the results of the previous section can be transformed into a viable numerical scheme. We suggest to perform a time discretization and then to compute a confidence interval for the price of the time discretized option. At the end of the section we provide a numerical example.

3.1 Time discretization

As a first step to a viable numerical scheme we discretize the discounted cash-flow process $Z(t)$, $0 \leq t \leq T$. To this end we again extend Z to the whole positive line by setting $Z(t) = 0$ for $t \in (T, \infty)$. Given the refraction period $\delta > 0$, we define

$$T_0 = \inf\{S \geq T;\ \frac{S}{\delta} \in \mathbb{N}\}.$$

We then choose a $\kappa \in \mathbb{N}$ and consider an equidistant partition of $[0, T_0]$ with $K = (\kappa T_0)/\delta$ time points

$$\{t_k := k\frac{\delta}{\kappa};\ k = 0, \dots, K\}.$$

We assume a Markovian setting for the numerical algorithm. Precisely, let $(X(k), \mathcal{F}_{t_k})$, $k = 0, \dots, K$, be a discrete time Markov process such that

$$Z(t_k) = h(k, X(k)), \quad k = 0, \dots, K$$

for some deterministic function $h(k, x)$. By construction, the refraction period δ in the original continuous time setting ($t \in [0, \infty)$) correponds to a refraction period of $\kappa \in \mathbb{N}$ in the time discretized setting ($k = 0, 1, \dots$). Consequently, as an approximation to the option price of the multiple exercise option with n rights,

$$Y^{*,n}(0) = \operatorname*{esssup}_{(\tau_1,\dots,\tau_n)\in\mathcal{S}^n_{\delta,0}} \sum_{\nu=1}^{n} E[Z(\tau_\nu)],$$

we consider the discrete time multiple stopping problem

$$Y^{d,n}(0) = \operatorname*{esssup}_{(\sigma_1,\dots,\sigma_n)\in\mathcal{S}^{d,n}_{\kappa,0}} \sum_{\nu=1}^{n} E[h(\sigma_\nu, X(\sigma_\nu))]. \tag{1}$$

Here $\mathcal{S}^{d,n}_{\kappa,0}$ contains the n-tuples of $\mathbb{N}$-valued $(\mathcal{F}_{t_k})_{k=0,1,\dots}$-stopping times with $\sigma_\nu \geq \sigma_{\nu+1} + \kappa$ for $\nu = 1, \dots, n-1$ and $h(k, x) := 0$ for $k \geq K + 1$.

Note, that the finer the time partition for the continuous time problem, the larger the refraction period κ for the time discretized problem. Hence, we really need to numerically solve a discrete time multiple stopping problem with non-trivial refraction period ($\kappa \geq 2$).

Combining the reduction principle for multiple stopping problems with the dynamic programming principle for discrete time (single) optimal stopping problems and the Markovian setting, we get

$$Y^{d,n}(0) = y^n(0, X(0)),$$

where the functions $y^\nu(k, x)$, $\nu = 1, \dots, n$, can be defined recursively. Indeed, given

$$y^\nu(k, x) = 0, \quad k = K+1, K+2, \dots,\ \nu = 1, \dots, n, \tag{2}$$

we construct, for $\nu = 1, \dots, n$ and $k = K, \dots, 0$

$$\begin{aligned} q^\nu(k, x) &= E[y^\nu(k+1, X(k+1)) | X(k) = x], \\ q^\nu_\kappa(k, x) &= E[y^\nu(k+\kappa, X(k+\delta)) | X(k) = x], \\ y^\nu(k, x) &= \max\{h(k, x) + q^{\nu-1}_\kappa(k, x),\ q^\nu(k, x)\}. \end{aligned} \tag{3}$$

Moreover, it is well-known (e.g. Bender and Schoenmakers, 2006) that optimal stopping times $\sigma_\nu^{*,n}$, $\nu = 1, \ldots, n$ for the discrete multiple stopping problem (1) with n rights exist and can be constructed via the continuation values $q^\nu(k, x)$ as

$$\begin{aligned} \sigma_{n+1}^{*,n} &= -\kappa, \\ \sigma_\nu^{*,n} &= \inf\{k \geq \sigma_{\nu+1}^{*,n} + \kappa;\ h(k, X(k)) + q_\kappa^{\nu-1}(k, X(k)) \geq q^\nu(k, X(k))\}, \\ &\quad \nu = n, \ldots, 1. \end{aligned} \tag{4}$$

3.2 Constructing confidence intervals for the price

We now explain how to construct confidence intervals for the time-discretized problem (1) in the Markovian setting. The first step is to solve the backward dynamic program (2)–(3) approximatively. Although the backward dynamic program is explicit in time, one cannot expect that the nested conditional expactations in (3) are available in closed form. Hence, the conditional expectations have to be replaced by an estimator which can be nested without exploding cost. Several Monte-Carlo methods are discussed in the literature, which also work for problems with high-dimensional state space, for example quantization methods (Bally and Pagès, 2003), Malliavin Monte-Carlo (Bouchard et al., 2004; Carmona and Touzi, 2008), and least-squares Monte-Carlo (Longstaff and Schwartz, 2001). In the numerical example below we shall apply least-squares Monte-Carlo. However, to keep the presentation generically, for the moment we only suppose that approximations

$$\hat{q}^\nu(k, x), \quad \hat{q}_\kappa^\nu(k, x), \quad \nu = 1, \ldots, n,\ k = 0, \ldots, K, \tag{5}$$

of

$$q^\nu(k, x), \quad q_\kappa^\nu(k, x), \quad \nu = 1, \ldots, n,\ k = 0, \ldots, K,$$

are pre-computed by some numerical method. (The approximations are, of course, extended by 0 for $k > K$). We then define

$$\hat{y}^\nu(k, x) = \max\{h(k, x) + \hat{q}_\kappa^{\nu-1}(k, x),\ \hat{q}^\nu(k, x)\}. \tag{6}$$

By construction, $\hat{y}^n(0, X(0))$ is an approximation of the time-discretized option price $Y^{d,n}(0)$. In order to judge the success of the numerical method which was implemented for the computation of $\hat{y}^n(0, X(0))$, we now propose to construct confindence intervals for $Y^{d,n}(0)$ making use of the approximate backward dynamic program (5). This approach may be seen as a multiple exercise generalization of the procedure by Andersen and Broadie (2004).

For the lower bounds we proceed as follows:

- Simulate L sample paths X_λ, $\lambda = 1, \ldots, L$, of X.

- Along the simulated paths X_λ, $\lambda = 1, \ldots, L$, define approximations to the optimal stopping times by

$$\begin{aligned}\hat{\sigma}^{\lambda,n}_{n+1} &= -\kappa, \\ \hat{\sigma}^{\lambda,n}_{\nu} &= \inf\{k \geq \sigma^{\lambda,n}_{\nu+1} + \kappa;\ h(k, X_\lambda(k)) + \hat{q}^{\nu-1}_{\kappa}(k, X_\lambda(k)) \\ &\qquad \geq \hat{q}^{\nu}(k, X_\lambda(k))\}, \quad \nu = n, \ldots, 1. \qquad (7)\end{aligned}$$

  By the structure of the optimal stopping times in (4) we expect that the stopping times $\hat{\sigma}^{\lambda,n}_{\nu}$ are 'close-to-optimal', if the numerical approximation of the backward dynamic program (5) was successful.

- An (unbiased) approximation of the lower bound for $Y^{d,n}(0)$ based on using the stopping times $\hat{\sigma}$ (instead of the optimal ones σ^*) can be obatined by the sample mean

$$Y^{\text{low},n}(0) := \frac{1}{L}\sum_{\lambda=1}^{L}\sum_{\nu=1}^{n} h(\hat{\sigma}^{\lambda,n}_{\nu}, X_\lambda(\hat{\sigma}^{\lambda,n}_{\nu})). \qquad (8)$$

  For the construction of (asymptotic) confidence intervals we also store the corresponding empirical standard deviation $s^{\text{low},n}$.

For the upper bounds we make use of Corollary 2.5, the discrete time version of the dual pricing formula in Theorem 2.3. Corollary 2.5 suggests that we numerically approximate the Doob decomposition of the processes $\hat{y}^\nu(k, X(k))$, $\nu = 1, \ldots, n$, defined via (6). Recall that the Doob decomposition of $\hat{y}^\nu(k, X(k))$ is given by

$$\hat{y}^\nu(k, X(k)) = \hat{y}^\nu(0, X(0)) + \hat{M}^\nu(k) - \hat{A}^\nu(k)$$

where the martingale part $\hat{M}^\nu$ and the predictable part $\hat{A}^\nu$ are given by

$$\begin{aligned}\hat{M}^\nu(0) &= 0, \quad \hat{A}^\nu(0) = 0 \\ \hat{M}^\nu(k) &= \hat{M}^\nu(k-1) + \hat{y}^\nu(k, X(k)) - E[\hat{y}^\nu(k, X(k))|X(k-1)] \\ \hat{A}^\nu(k) &= \hat{A}^\nu(k-1) + \hat{y}^\nu(k-1, X(k-1)) - E[\hat{y}^\nu(k, X(k))|X(k-1)].\end{aligned}$$

Note also that expressions such as $\hat{A}^\nu(k+\kappa) - E[\hat{A}(k+\kappa)|\mathcal{F}_{t_k}]$, which are required in the dual representation, can be rewritten as

$$\begin{aligned}&\hat{A}^\nu(k+\kappa) - E[\hat{A}(k+\kappa)|\mathcal{F}_{t_k}] \\ =\ &\hat{M}(k+\kappa) - M(k) + E[\hat{y}^\nu(k+\kappa, X(k+\kappa))|X(k)] - \hat{y}^\nu(k+\kappa, X(k+\kappa)).\end{aligned}$$

Hence, the essential step for calculating the dual upper bounds by simulation is to estimate the conditional expectations

$$E[\hat{y}^\nu(k+1, X(k+1))|X(k)], \quad E[\hat{y}^\nu(k+\kappa, X(k+\kappa))|X(k)]$$

along simulated paths. This can be done as follows:

- Simulate L' new sample paths of X, which we denote again by X_λ, $\lambda = 1, \dots, L'$. (As the lower bounds are already constructed, no confusion should arise from this ambiguous notation).

- Given pairs (λ, k), we sample M copies $X_{\lambda,\mu}(k+1)$ and $X_{\lambda,\mu}(k+\kappa)$, $\mu = 1, \dots, M$, with the law of $X(k+1)$, resp. $X(k+\kappa)$, conditional on $\{X(k) = X_\lambda(k)\}$. We now define, for $\nu = 1, \dots, n$,

$$\begin{aligned}\hat{Q}^\nu_\lambda(k) &= \frac{1}{M}\sum_{\mu=1}^{M} \hat{y}^\nu(k+1, X_{\lambda,\mu}(k+1)) \\ \hat{Q}^{\nu,\kappa}_\lambda(k) &= \frac{1}{M}\sum_{\mu=1}^{M} \hat{y}^\nu(k+\kappa, X_{\lambda,\mu}(k+\kappa))\end{aligned}$$

as unbiased estimators for $E[\hat{y}^\nu(k+1, X(k+1))|X(k) = X_\lambda(k)]$ and $E[\hat{y}^\nu(k+\kappa, X(k+\kappa))|X(k) = X_\lambda(k)]$.

- We now approximate the martingale part of the Doob decomposition of $\hat{y}^\nu(k, X_\lambda(k))$, $\nu = 1, \dots, n$, by

$$\begin{aligned}\hat{M}^\nu_\lambda(0) &= 0, \\ \hat{M}^\nu_\lambda(k) &= \hat{M}^\nu_\lambda(k-1) + \hat{y}^\nu(k, X_\lambda(k)) - \hat{Q}^\nu_\lambda(k-1)\end{aligned}$$

Moreover, we approximate $\hat{A}^\nu(k+\kappa) - E[\hat{A}(k+\kappa)|\mathcal{F}_{t_k}]$ along the L' simulated 'outer' paths X_λ, $\lambda = 1, \dots, L$, by

$$\hat{M}^\nu_\lambda(k+\kappa) - \hat{M}^\nu_\lambda(k) + \hat{Q}^{\nu,\kappa}_\lambda(k) - \hat{y}^\nu(k+\kappa, X_\lambda(k+\kappa)).$$

- Replacing the expectation in Corollary 2.5, (i), by the sample mean we finally end up with the dual estimator for the time discretized price

$$\begin{aligned}&Y^{\mathrm{up},n}(0) \\ := &\frac{1}{L}\sum_{\lambda=1}^{L}\Bigg(\sup_{\substack{u_1,\dots,u_n \in \mathbb{N}^n \\ u_\nu \geq u_{\nu+1}+\kappa}} \sum_{\nu=1}^{n-1}\Big(h(u_\nu, X_\lambda(u_\nu)) - (\hat{M}^\nu_\lambda(u_\nu) - \hat{M}^\nu_\lambda(u_{\nu+1}+\kappa)) \\ &\qquad + \hat{Q}^{\nu,\kappa}_\lambda(u_{\nu+1}) - \hat{y}^\nu(u_{\nu+1}+\kappa, X_\lambda(u_{\nu+1}+\kappa))\Big) \\ &\quad + h(u_n, X_\lambda(u_n)) - (\hat{M}^n_\lambda(u_n) - \hat{M}^n_\lambda(k))\Bigg)\end{aligned}$$

(with the convention $u_{n+1} = -\kappa$). Again, we also store the empirical standard deviation $s^{\mathrm{up},n}$ for the construction of (asymptotic) confidence intervals.

We emphasize that the use of the unbiased estimators $\hat{Q}^{\nu}_{\lambda}(k)$ and $\hat{Q}^{\nu,\kappa}_{\lambda}(k)$ instead of the true conditional expectations $E[\hat{y}^{\nu}(k+1, X(k+1))|X(k) = X_{\lambda}(k)]$ and $E[\hat{y}^{\nu}(k+\kappa, X(k+\kappa))|X(k) = X_{\lambda}(k)]$ induces an additional bias up. This is due to interchanging supremum and expectation analogously to the American option case discussed by Andersen and Broadie (2004). Consequently, an asymptotic 95% confidence interval for the price of the time discretized multiple exercise option $Y^{d,n}(0)$ is given by

$$[Y^{\text{low},n}(0) - 1.96 s^{\text{low},n},\ Y^{\text{up},n}(0) + 1.96 s^{\text{up},n}].$$

This interval can be shrunken to some extent by increasing the number of sample paths L, L', and M. However, a gap will remain due to solving the backward dynamic program approximatively (cp. (5)).

3.3 A numerical example

We now illustrate the above algorithm by a numerical example. We consider a stylized swing option contract. The spot price for electricity is modeled by an exponential of a Gaussian Ornstein-Uhlenbeck process. This toy model was suggested by Lucia and Schwartz (2002). It is however a main advantage of the proposed algorithm that it can be generically applied to any Markovian model. In particular, it is applicable to the non-Gaussian exponential Ornstein-Uhlenbeck models which have been proposed by Benth et al. (2007) and Hambly et al. (2009). For more information on the modeling of electricity prices we refer to the recent monograph by Benth et al. (2008).

After a time discretization is performed, we assume that the underlying Markovian process in discrete time is given by $X(k)$ where

$$\log(X(k)) = (1-\rho)(\log(X(k-1)) - \mu) + \mu + \sigma\epsilon(k), \quad X(0) = x_0,$$

$\epsilon(k)$ are i.i.d standard Gaussian random variables and the parameters are specified as

$$\sigma = 0.5, \quad \rho = 0.9, \quad \mu = 0, \quad x_0 = 1.$$

The holder of the swing option has the right to buy elecricity for a strike price of S at n times but is subject to a refraction period of $\kappa \in \mathbb{N}$. Correspondingly, the payoff function is given by

$$h(k, x) = (x - S)_+$$

for $k \leq K$ and is extended by 0 for $k > K$. In this example we consider the parameter values

$$S = 1, \quad n = 3, \quad K = 50, \quad \kappa = 1, \ldots, 20.$$

So, in particular we are interested in the effect of the refraction period on the option price, which is the novel feature of the dual upper bound in Corollary 2.5.

refr. period	$Y^{\text{low},2}(0)$	$Y^{\text{up},2}(0)$	95% confidence interval
1	3.311	3.317	[3.309, 3.323]
2	3.278	3.288	[3.276, 3.293]
3	3.254	3.270	[3.252, 3.276]
4	3.231	3.246	[3.229, 3.251]
5	3.209	3.227	[3.207, 3.232]
6	3.186	3.200	[3.160, 3.205]
8	3.139	3.152	[3.137, 3.157]
10	3.089	3.103	[3.086, 3.108]
12	3.038	3.053	[3.036, 3.058]
14	2.984	3.002	[2.982, 3.007]
16	2.929	2.945	[2.927, 2.950]
18	2.873	2.889	[2.871, 2.894]
20	2.813	2.829	[2.811, 2.834]

Table 1: Numerical results for two exercise rights (lower estimator, upper estimator, 95% confidence interval) for different refraction periods.

In order to obtain the numerical results, reported below, we proceed as follows: The approximation of the continuation values within the dynamic program are pre-calculated by a least-squares regression with 1000 paths and the two basis functions $\psi_1(x) = 1$ and $\psi_2(x) = (x - S)_+$. The lower biased estimator is calculated with $\Lambda = 300000$ simulated paths. For the upper estimator we apply $\Lambda' = 2000$ outer paths and $M = 100$ inner paths.

Table 1 reports the values of the lower and upper estimators $Y^{\text{low},n}(0)$ and $Y^{\text{up},n}(0)$ for $n = 2$ exercise rights and the corresponding 95% confidence intervals for various values of the refraction period ranging from 1 to 20. The numerical results for $n = 3$ rights are displayed in Table 2. In both cases and for all refraction periods $\kappa = 1, \dots, 20$, the width of the 95% confidence interval is less than 1% relative to the price. For comparison we also give a 95% confidence interval for the single exercise case. It is

$$[1.856, 1.866].$$

Figure 1 displays the 95% confidence intervals for the discretized option price with two and three exercise rights as a function of the refraction period κ. We observe that the price of the option with three exercise rights decreases faster than the one with two exercise rights as the refraction period decreases. For a refraction period κ as large as 20, the additional third right only yields a very small marginal price. This was to be expected, because the refraction period ($\kappa = 20$) is rather large compared to the number of time

refr. period	$Y^{\text{low},3}(0)$	$Y^{\text{up},3}(0)$	95% confidence interval
1	4.534	4.545	[4.532, 4.552]
2	4.439	4.455	[4.436, 4.461]
3	4.368	4.393	[4.365, 4.399]
4	4.300	4.321	[4.298, 4.327]
5	4.227	4.255	[4.225, 4.261]
6	4.154	4.177	[4.151, 4.183]
8	3.996	4.021	[3.993, 4.028]
10	3.829	3.855	[3.826, 3.861]
12	3.653	3.679	[3.651, 3.685]
14	3.466	3.494	[3.463, 3.500]
16	3.275	3.299	[3.273, 3.305]
18	3.089	3.109	[3.086, 3.114]
20	2.919	2.939	[2.917, 2.945]

Table 2: Numerical results for three exercise rights (lower estimator, upper estimator, 95% confidence interval) for different refraction periods.

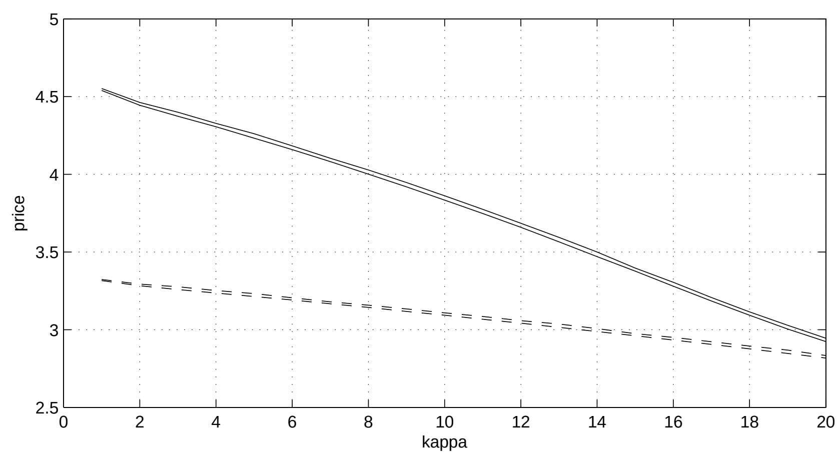

Figure 1: Comparison of the 95% confidence interval for the price of the (discretized) swing option with two exercise rights (dashed lines) and three exercise rights (solid lines) as function of the refraction period κ.

Strike	# of rights	Meinhausen-Hambly upper bound	Schoenmakers upper bound
1	2	3.322 (0.0028)	3.317 (0.0027)
1	3	4.552 (0.0035)	4.545 (0.0033)
0	2	5.305 (0.0028)	5.301 (0.0027)
0	3	7.527 (0.0034)	7.521 (0.0032)

Table 3: Comparison of the upper biased estimators calculated by the Meinshausen-Hambly algorithm and the Schoenmakers algorithm for refraction period $\kappa = 1$. Standard deviation in round brackets.

steps ($K = 50$). So in many scenarios it is profitable to exercise two out of the three rights only.

When the refraction period equals the time step, i.e. $\kappa = 1$, the above algorithm boils down to the one suggested by Schoenmakers (2010). In this case, an upper bound of a discrete time multiple exercise option can be computed alternatively by the Meinshausen-Hambly algorithm (Meinshausen and Hambly, 2004). This algorithm actually computes upper bounds for the marginal price of an additional nth right. The corresponding dual representation can be linked to Roger's dual (Rogers, 2002) by interpreting the marginal price as an optimal stopping problem with a modified cash-flow process, see Bender (2010). Without going into any further details, we compare the numerical upper bounds of the Meinshausen-Hambly algorithm and the Schoenmakers algorithm in Table 3 for the same problem as above with strike $S = 1$ and $S = 0$.

In our simulation study the upper bounds calculated by the Schoenmakers algorithm are slightly smaller in all cases, but the differences to the upper bounds calclulated by the Meinshausen-Hambly algorithm are almost negligible. We also note that the standard deviations resulting from both algorithms (performed with the same number of sample paths) is almost identical. So, when it comes to accuracy, our (admittedly small) simulation study suggests that both upper bound algorithms perform comparably well. For larger number of exercise rights we expect that the Meinshausen-Hambly algorithm is faster, as the pathwise maxima are to be computed over smaller sets.

We emphasize again that, for the time discretization of a continuous time problem, it is essential to treat discrete time multiple stopping problems with refraction period $\kappa \geq 2$. In this respect, an advantage of the Schoenmakers algorithm is that it can be generalized from $\kappa = 1$ to refraction periods $\kappa \in \mathbb{N}$, as discussed in the present paper. Contrarily, no generalization of the Meinshausen-Hambly algorithm to non-trivial refraction periods is known.

4 Proof of Theorem 2.2

In this section we prove the existence of the Snell envelope, the Doob-Meyer decomposition, and the reduction principle for the multiple stopping problem $\bar{Y}^{*,n}$ as stated in Theorem 2.2. As the main technical difficulties arise from the lack of right-continuity, we will first explain by a simple example that this problem is unavoidable.

Example 4.1. Suppose $Z(t) = 1$ for $t \in [0, T]$. Then it is optimal to exercise Z as early and as often as possible up to time T. Hence,

$$\bar{Y}^{*,n}(t) = \inf\{\nu = 0, \dots, n;\ t + \nu\delta > T\}.$$

In particular,

$$\begin{aligned} \bar{Y}^{*,1}(t) &= \begin{cases} 1, & 0 \le t \le T \\ 0, & t > T \end{cases} \\ \bar{Y}^{*,2}(t) &= \begin{cases} 2, & 0 \le t \le T - \delta \\ 1, & T - \delta < t \le T \\ 0, & t > T, \end{cases} \end{aligned}$$

and so on. This rather trivial example illustrates that the discontinuity from the right of the cash-flow Z at time $t = T$, which is forced by the finite time horizon (unless $Z(T) = 0$), propagates backwards in time as the number of exercise rights increases. So the path regularity for $Y^{*,n}$ stated in Theorem 2.2 really is the best one can hope for.

Remark 4.2. Carmona and Touzi (2008) and Zeghal and Mnif (2006) also discuss the reduction principle for multiple stopping problems in continuous time on a finite time horizon for discounted cash-flows Z with continuous paths, respectively with right-continuous paths which are left-continuous in expectation, under stronger conditions than (H). The argumentation in both papers implicitly requires that the right-continuity of the discounted cash-flow $Z(t)$ is inherited by the processes $Z^n(t)$ and $Y^{*,n}(t)$. Then, standard results on the optimal stopping of right-continuous processes can be applied iteratively. Example 4.1 shows that one cannot expect in general that the right-continuity is inherited, but some extra condition such as $Z(T) = 0$ need to be imposed.

We now turn to the proof of Theorem 2.2. As a ramification we first prove two propositions.

Proposition 4.3. *Under condition (H) it holds that, for every $n \in \mathbb{N}$ and every stopping time σ*

$$\bar{Y}^{*,n+1}(\sigma) = \operatorname*{esssup}_{\tau \in \mathcal{S}_\sigma} E[Z(\tau) + E[\bar{Y}^{*,n}(\tau + \delta)|\mathcal{F}_\tau]|\mathcal{F}_\sigma]$$

The *proof* is identical to the proof of Proposition 3.1 in Carmona and Dayanik (2008). We emphasize that Proposition 4.3 is much weaker than the reduction principle in Theorem 2.2, (iii). The reduction principle in the latter theorem states that $\bar{Y}^{*,n+1}(\sigma)$ is obtained as the value of the single stopping problem of the *stochastic process* Z^{n+1}. Contrarily, Proposition 4.3 only characterizes $\bar{Y}^{*,n+1}(\sigma)$ as the value of stopping the *family of random variables* $(Z(\tau) + E[\bar{Y}^{*,n}(\tau + \delta)|\mathcal{F}_\tau])_{\tau \in \mathcal{S}_\sigma}$ optimally.

The next proposition is crucial for constructing the process Z^n in Theorem 2.2, (R_n).

Proposition 4.4. *Suppose A is an adapted, nondecreasing, integrable D_n-RCLL process with $A(0) = 0$. Then the process $E[A(t + \delta)|\mathcal{F}_t]$ has an D_{n+1}-RCLL modification $\mathcal{A}(t)$ which is a submartingale and satisfies $\mathcal{A}(\tau) = E[A(\tau + \delta)|\mathcal{F}_\tau]$ for every bounded stopping time τ.*

Proof. Case 1: A is RCLL: For $s \le t$ we have

$$E[E[A(t + \delta)|\mathcal{F}_t]|\mathcal{F}_s] = E[A(t + \delta)|\mathcal{F}_s] \ge E[A(s + \delta)|\mathcal{F}_s],$$

because A is nondecreasing. Hence, $E[A(t + \delta)|\mathcal{F}_t]$ is a nonnegative submartingale. Due to the right-continuity of A and monotone convergence, we observe that

$$E[E[A(t + \delta)|\mathcal{F}_t]] = E[A(t + \delta)]$$

is a right-continuous function in t. Therefore, the process $E[A(t + \delta)|\mathcal{F}_t]$ has a RCLL modification by Theorem 1.3.13 in Karatzas and Shreve (1991), which we denote by $\mathcal{A}(t)$. Suppose now that τ is a stopping time bounded by a. We denote by $\mathcal{D} = \{k/(2^n);\ k, n \ge 1\}$ the set of dyadic rationals and construct a sequence of $(\mathcal{D} \cap [0, a]) \cup \{a\}$-valued stopping times by

$$\tau_k = \begin{cases} a, & \tau = a \\ \frac{k}{2^n}, & \tau < a \text{ and } \frac{k-1}{2^n} \le \tau < \frac{k}{2^n} \end{cases}$$

Note that τ_k is bounded by a, $\tau_k \downarrow \tau$ and τ_k takes countably many values only. Due to the last property it is straightforward to verify that $E[A(\tau_k + \delta)|\mathcal{F}_{\tau_k}] = \mathcal{A}(\tau_k)$ for each τ_k. Now, fix some $B \in \mathcal{F}_\tau$. As $\mathcal{F}_\tau \subset \mathcal{F}_{\tau_k}$, we get

$$E[\mathbf{1}_B A(\tau_k + \delta)] = E[\mathbf{1}_B E[A(\tau_k + \delta)|\mathcal{F}_{\tau_k}]] = E[\mathbf{1}_B \mathcal{A}(\tau_k)]. \tag{9}$$

By the right-continuity of A and monotone convergence, the left hand side converges to $E[\mathbf{1}_B A(\tau + \delta)]$. For the right-hand side, we recall that $\mathcal{A}(t)$ is a nonnegative RCLL submartingale. By the optional sampling theorem (justified by the boundedness of the sequence τ_k by a), it follows that for every $\lambda > 0$

$$E[\mathbf{1}_{\{\mathcal{A}_{\tau_k} > \lambda\}} \mathcal{A}_{\tau_k}] \le E[\mathbf{1}_{\{\mathcal{A}_{\tau_k} > \lambda\}} \mathcal{A}_a], \quad P(\{\mathcal{A}_{\tau_k} > \lambda\}) \le \frac{E[\mathcal{A}_a]}{\lambda}.$$

Hence, the sequence $(\mathcal{A}(\tau_k))$ is uniformly integrable and so is the sequence $(\mathbf{1}_B\mathcal{A}(\tau_k))$. Interchanging limit and expectation and making use of the right-continuity of $\mathcal{A}$ we see that the right-hand side in (9) converges to $E[\mathbf{1}_B\mathcal{A}(\tau)]$. Thus, for every $B \in \mathcal{F}_\tau$,

$$E[\mathbf{1}_B A(\tau+\delta)] = E[\mathbf{1}_B\mathcal{A}(\tau)],$$

i.e. $E[A(\tau+\delta)|\mathcal{F}_\tau] = \mathcal{A}(\tau)$.

Case 2: A is D_n-RCLL: For $t_i \in D_n$ we define

$$\begin{aligned} \Delta^+ A_{t_i} &= A_{t_i+} - A_{t_i} \\ A^{RC}(t) &= A(t) - \sum_{t_i \in D_n} \mathbf{1}_{\{t>t_i\}} \Delta^+ A_{t_i}, \end{aligned}$$

i.e. all jumps from the right-hand side of A are removed in the process A^{RC}, which is thus RCLL and nondecreasing. By the first case we find an RCLL modification $\mathcal{A}^{RC}$ of $E[A^{RC}(t+\delta)|\mathcal{F}_t]$ such that $\mathcal{A}^{RC}(\tau) = E[A^{RC}(\tau+\delta)|\mathcal{F}_\tau]$ for every bounded stopping time τ. However, for every $t_i \in D_n$ the process $E[\Delta^+ A_{t_i}|\mathcal{F}_t]$ is a nonnegative martingale. Therefore, an analogous argument as in Case 1 yields an RCLL modification $\mathcal{A}^i$ such that $\mathcal{A}^i(\tau) = E[\Delta^+ A_{t_i}|\mathcal{F}_\tau]$. Plugging these processes together we observe that

$$\mathcal{A}(t) = \mathcal{A}^{RC}(t) + \sum_{t_i \in D_n} \mathbf{1}_{\{t+\delta>t_i\}} \mathcal{A}^i(t)$$

is a D_{n+1}-RCLL modification of

$$E\left[A^{RC}(t+\delta) + \sum_{t_i \in D_n} \mathbf{1}_{\{t+\delta>t_i\}} \Delta^+ A_{t_i} \middle| \mathcal{F}_t \right] = E[A(t+\delta)|\mathcal{F}_t]$$

which fulfills $\mathcal{A}(\tau) = E[A(t+\delta)|\mathcal{F}_\tau]$ for every bounded stopping time τ. $\mathcal{A}$ clearly is a submartingale as a sum of finitely many submartingales. □

Remark 4.5. If the process A in the previous proposition is only of bounded variation (instead of nondecreasing), we can decompose $A(t) = A_1(t) - A_2(t)$, where A_1, A_2 are adapted, nondecreasing, D_n-RCLL and start in 0. Hence, applying the previous proposition to A_1 and A_2 we observe that the process $E[A(t+\delta)|\mathcal{F}_t]$ still has an D_{n+1}-RCLL modification $\mathcal{A}(t)$ which satisfies $\mathcal{A}(\tau) = E[A(\tau+\delta)|\mathcal{F}_\tau]$ for every bounded stopping time τ.

We now simultaneously give the *proof* the three assertions in Theorem 2.2 by induction on n. Hence, for the remainder of the section we assume that (H) is in force.

For $n = 1$ we can make use of well-known results for the single optimal stopping problem with right-continuous cash-flow Z on $[0, T]$.

(S_1) The existence of the Snell envelope $Y^{*,1}(t)$ on $[0,T]$ is shown e.g. in Karatzas and Shreve (1998), Proposition D7. For $t \in (T,\infty)$ we can simply let $Y^{*,1}(t) = 0$, because $Z(t) = 0$ for $t > T$. For the integrability note that

$$E[\sup_{0\le t<\infty} |Y^{*,1}(t)|^p] = E[\sup_{0\le t\le T} |Y^{*,1}(t)|^p]$$

and $|Y^{*,1}(t)|$ is dominated by the martingale $E[\sup_{0\le u\le T} |Z(u)||\mathcal{F}_t]$, which is p-integrable on $[0,T]$ by Doob's maximal inequality and condition (H).
(DM_1) The standard Doob-Meyer decomposition applies on $[0,T]$ because of the integrability of $Y^{*,1}(t)$. For $t > T$ we set

$$M^{*,1}(t) = M^{*,1}(T), \quad A^{*,1}(t) = A^{*,1}(T) + Z(T)$$

(R_1) is trivial.

We now suppose that the assertions (S_n), (DM_n), and (R_n) are already proved for some $n \in \mathbb{N}$. We can and shall assume w.l.og. that $n\delta \le T$. Otherwise it is not possible to exercise the $(n+1)$ rights up to time T and the stopping problem is equivalent to the one with n rights.

As a first step we show that $\bar{Y}^{*,n+1}(\sigma)$ is the value of stopping a stochastic process $Z^{n+1}(t)$ optimally. This optimal stopping problem is, however, non-standard because of the discontinuities of Z^{n+1} from the right hand side.

Lemma 4.6. *The process $Z(t) + E[Y^{*,n}(t+\delta)|\mathcal{F}_t]$ has an D_{n+1}-RC modification $Z^{n+1}(t)$ such that for every stopping time*

$$Z^{n+1}(\tau) = Z(\tau) + E[Y^{*,n-1}(\tau+\delta)|\mathcal{F}_\tau].$$

Moreover,

$$E[\sup_{0\le t<\infty} |Z^{n+1}(t)|^p] < \infty$$

and, for every stopping time σ,

$$\bar{Y}^{*,n+1}(\sigma) = \operatorname*{esssup}_{\tau\in\mathcal{S}_\sigma} E[Z^{n+1}(\sigma)|\mathcal{F}_\tau].$$

Proof. Applying the inductive hypothesis (DM_n) for the Doob decomposition we get, for every bounded stopping time τ,

$$E[Y^{*,n}(\tau+\delta)|\mathcal{F}_\tau] = Y^{*,n}(0) + M^{*,n}(\tau) - E[A^{*,n}(\tau+\delta)|\mathcal{F}_\tau].$$

By Proposition 4.4 the process $E[A^{*,n}(t+\delta)|\mathcal{F}_t]$ has an D_{n+1}-RCLL modification $\mathcal{A}^{*,n}$ satisfying

$$E[A^{*,n}(\tau+\delta)|\mathcal{F}_\tau] = \mathcal{A}^{*,n}(\tau).$$

Hence, for every bounded stopping time τ,

$$Z(\tau) + E[Y^{*,n}(\tau+\delta)|\mathcal{F}_\tau] = Z^{n+1}(\tau), \tag{10}$$

where the D_{n+1}-RC process Z^{n+1} is defined by

$$Z^{n+1}(t) := Z(t) + Y^{*,n}(0) + M^{*,n}(t) - \mathcal{A}^{*,n}(t).$$

In particular, Z^{n+1} is a D_{n+1}-RC modification of $Z(t) + E[Y^{*,n}(t+\delta)|\mathcal{F}_t]$ satisfying (10) for bounded stopping times. As $Z(t) = 0$ and $\mathcal{F}_t = \mathcal{F}_T$ for $t > T$, all processes involved in the above considerations stay constant for $t > T$. Hence (10) also holds for unbounded stopping times. We now prove the integrability condition on Z^{n+1}. As Z^{n+1} is a modification of $Z(t)+E[Y^{*,n}(t+\delta)|\mathcal{F}_t]$, we obtain that the D_{n+1}-RCLL process $|Z^{n+1}(t) - Z(t)|$ is bounded by the RCLL martingale $E[\sup_{0\le u<\infty}|Y^{*,n}(u)||\mathcal{F}_t]$. By the inductive hypothesis (S_n) this martingale is p-integrable and, hence, by Doob's maximal inequality

$$E[\sup_{0\le t<\infty}|Z^{n+1}(t) - Z(t)|^p] < \infty.$$

The triangle inequality yields the asserted integrability of Z^{n+1}, thanks to assumption (H) and the fact that Z vanishes on (T,∞).

We now combine (10) with the inductive hypothesis (S_n) on the existence of the Snell envelope and Proposition 4.3 and get for every stopping time σ,

$$\begin{aligned}
\bar{Y}^{*,n+1}(\sigma) &= \operatorname*{esssup}_{\tau\in\mathcal{S}_\sigma} E[Z(\tau) + E[\bar{Y}^{*,n}(\tau+\delta)|\mathcal{F}_\tau]|\mathcal{F}_\sigma] \\
&= \operatorname*{esssup}_{\tau\in\mathcal{S}_\sigma} E[Z(\tau) + E[Y^{*,n}(\tau+\delta)|\mathcal{F}_\tau]|\mathcal{F}_\sigma] \\
&= \operatorname*{esssup}_{\tau\in\mathcal{S}_\sigma} E[Z^{n+1}(\tau)|\mathcal{F}_\sigma]
\end{aligned}$$

□

As a next step we introduce the set of stopping times $\mathcal{S}_{\sigma+}$ taking values strictly larger than a stopping time σ and define the auxillary stopping problem

$$\bar{X}^{*,n+1}(\sigma) = \operatorname*{esssup}_{\tau\in\mathcal{S}_{\sigma+}} E[Z^{n+1}(\tau)|\mathcal{F}_\sigma].$$

By definition, we have

$$\bar{X}^{*,n+1}(\sigma) \le \bar{Y}^{*,n+1}(\sigma).$$

Lemma 4.7. *For every stopping time σ, it holds that*

$$\bar{Y}^{*,n+1}(\sigma) = \begin{cases} \bar{X}^{*,n+1}(\sigma), & \sigma\notin D_{n+1} \\ \max\{Z^{n+1}(\sigma), \bar{X}^{*,n+1}(\sigma)\}, & \sigma\in D_{n+1} \end{cases}$$

Proof. It is obvious that $\bar{Y}^{*,n+1}(\sigma) = \max\{Z^{n+1}(\sigma), \bar{X}^{*,n+1}(\sigma)\}$ for every stopping time σ. So it suffices to show that $\bar{X}^{*,n+1}(\sigma) \ge \bar{Y}^{*,n+1}(\sigma)$ on

$\{\sigma \notin D_{n+1}\}$. To this end, suppose $\tau \in \mathcal{S}_\sigma$. We define a sequence $(\tau_N) \subset \mathcal{S}_{\sigma+}$ by

$$\tau_N = \begin{cases} \tau, & \tau > \sigma \\ \sigma + 1/N, & \tau = \sigma \end{cases}$$

Recall that Z^{n+1} is right-continuous at σ on $\{\sigma \notin D_{n+1}\}$. Therefore, $Z^{n+1}(\tau_N)$ converges to $Z^{n+1}(\tau)$ on $\{\sigma \notin D_{n+1}\}$. In view of the previous lemma we can apply dominated convergence in order to obtain

$$E[Z^{n+1}(\tau)|\mathcal{F}_\sigma] = \lim_{N\to\infty} E[Z^{n+1}(\tau_N)|\mathcal{F}_\sigma] \leq X^{*,n+1}(\sigma) \quad \text{on } \{\sigma \notin D_{n+1}\}.$$

Taking the supremum over all stopping times $\tau \in \mathcal{S}_\sigma$, we get $X^{*,n+1}(\sigma) \geq Y^{*,n+1}(\sigma)$ on $\{\sigma \notin D_{n+1}\}$, hence the assertion. □

Lemma 4.8. *(i) The process $\bar{X}^{*,n+1}(t)$ has an RCLL modification $X^{*,n+1}(t)$ which is a supermartingale and satisfies $\bar{X}^{*,n+1}(\tau) = X^{*,n+1}(\tau)$ for every stopping time τ.*
(ii) The process $\bar{Y}^{*,n+1}(t)$ has a D_{n+1}-RCLL modification $Y^{*,n+1}(t)$ which is a supermartingale and satisfies $\bar{Y}^{*,n+1}(\tau) = Y^{*,n+1}(\tau)$ for every stopping time τ.

Proof. (i) We fix a stopping time σ. The same argumentation as in Lemma D.1 of Karatzas and Shreve (1998) shows that the random variables of the form $E[Z^{n+1}(\tau)|\mathcal{F}_\sigma]$, $\tau \in \mathcal{S}_{\sigma+}$, are closed under pairwise maximization. Thus, there is a sequence $(\rho_k^*)_{k\in\mathbb{N}} \subset \mathcal{S}_{\sigma+}$ such that $E[Z^{n+1}(\rho_k^*)|\mathcal{F}_\sigma]$ nondecreasingly converges to $\bar{X}^{*,n+1}(\sigma)$. Consequently, for $\tau \leq \sigma$,

$$E[\bar{X}^{*,n+1}(\sigma)|\mathcal{F}_\tau] = \lim_{k\to\infty} E[Z^{n+1}(\rho_k^*)|\mathcal{F}_\tau] \leq \bar{X}^{*,n+1}(\tau) \tag{11}$$

In particular, the process $\bar{X}^{*,n+1}(t)$ is a supermartingale. We will now show that it has an RCLL modification $X^{*,n+1}(t)$. In view of Theorem 1.3.13 in Karatzas and Shreve (1991) it suffices to show that the mapping $t \mapsto E[\bar{X}^{*,n+1}(t)]$ is right-continuous. We fix some $t \geq 0$. As explained above there is a sequence $(\rho_k^*) \subset \mathcal{S}_{t+}$ such that $E[Z^{n+1}(\rho_k^*)|\mathcal{F}_t]$ nondecreasingly converges to $\bar{X}^{*,n+1}(t)$. In particular, given an $\epsilon > 0$, we can choose k sufficiently large and get

$$E[\bar{X}^{*,n+1}(t)] \leq E[Z^{n+1}(\rho_k^*)] + \epsilon/2.$$

For $h > 0$ we define the stopping times

$$\rho_{k,h} = \begin{cases} \rho_k^*, & \rho_k^* > t+h \\ t+2h, & \rho_k^* \leq t+h \end{cases},$$

and observe that $\rho_{k,h} \in \mathcal{S}_{(t+h)+}$. As $\rho_k^* > t$, it holds that $\rho_{k,h}(\omega) = \rho_k^*(\omega)$ if h is sufficiently small (depending on ω). In particular $Z^{n+1}(\rho_{k,h})$ converges to $Z^{n+1}(\rho_k^*)$ almost surely, and by dominated convergence,

$$\lim_{h\to 0} E[Z^{n+1}(\rho_{k,h})] = E[Z^{n+1}(\rho_k^*)] \geq E[\bar{X}^{*,n+1}(t)] - \epsilon/2.$$

Hence, for sufficiently small h

$$E[\bar{X}^{*,n+1}(t+h)] \geq E[Z^{n+1}(\rho_{k,h})] \geq E[\bar{X}^{*,n+1}(t)] - \epsilon.$$

Since $\bar{X}^{*,n+1}$ is a supermartingale, we have $E[\bar{X}^{*,n+1}(t)] \geq E[\bar{X}^{*,n+1}(t+h)]$ for $h > 0$. Therefore, the map $t \mapsto E[\bar{X}^{*,n+1}(t)]$ is right-continuous.

We denote the RCLL modification of $\bar{X}^{*,n+1}(t)$ by $X^{*,n+1}(t)$. Finally we show that $\bar{X}^{*,n+1}(\tau) = X^{*,n+1}(\tau)$ for every stopping time τ. This is certainly true for stopping times taking vaules in the countable set $\mathcal{D} = \{k/(2^n);\ n, k \geq 1\}$ of dyadic rationals. Given a stopping time τ we can choose a sequence τ_k with vaules in $\mathcal{D}$ such that $\tau_k \downarrow \tau$ (analogously to the construction in the proof of Proposition 4.4). By right-continuity of $X^{*,n+1}$ and dominated convergence (which is justified by the integrability of Z^{n+1} derived in Lemma 4.6), we, thus, obtain

$$X^{*,n+1}(\tau) = \lim_{k\to\infty} E[X^{*,n+1}(\tau_k)|\mathcal{F}_\tau] = \lim_{k\to\infty} E[\bar{X}^{*,n+1}(\tau_k)|\mathcal{F}_\tau].$$

Hence it remains to prove that

$$\lim_{k\to\infty} E[\bar{X}^{*,n+1}(\tau_k)||\mathcal{F}_\tau] = \bar{X}^{*,n+1}(\tau) \tag{12}$$

To this end we fix some stopping time $\rho \in \mathcal{S}_{\tau+}$ and define a sequence of stopping times

$$\rho_k = \begin{cases} \rho, & \tau_k < \rho \\ \tau_k + 1, & \tau_k \geq \rho \end{cases}$$

Then, $\rho_k \in \mathcal{S}_{\tau_k+}$ and $\rho_k(\omega) = \rho(\omega)$ for sufficiently large k depending on ω, because $\rho > \tau$ and $\tau_k \downarrow \tau$. Consequently, $Z^{n+1}(\rho_k)$ converges to $Z^{n+1}(\rho)$, as k tends to infinity. By dominated convergence (making again use of Lemma 4.6) and (11) we obtain

$$\begin{aligned} E[Z^{n+1}(\rho)|\mathcal{F}_\tau] &= \lim_{k\to\infty} E[Z^{n+1}(\rho_k)|\mathcal{F}_\tau] \leq \liminf_{k\to\infty} E[\bar{X}^{*,n+1}(\tau_k)|\mathcal{F}_\tau] \\ &\leq \limsup_{k\to\infty} E[\bar{X}^{*,n+1}(\tau_k)|\mathcal{F}_\tau] \leq \bar{X}^{*,n+1}(\tau) \end{aligned}$$

Taking the supremum over $\rho \in \mathcal{S}_{\tau+}$ yields (12) and completes the proof of part (i).

(ii) An analogous argumentation as in part (i) shows that for $\tau \leq \sigma$,

$$E[\bar{Y}^{*,n+1}(\sigma)|\mathcal{F}_\tau] \leq \bar{Y}^{*,n+1}(\tau).$$

Hence $\bar{Y}^{*,n+1}(t)$ is a supermartingale. We define

$$Y^{*,n+1}(t) = \begin{cases} \bar{Y}^{*,n+1}(t), & t \in D_{n+1} \\ X^{*,n+1}(t), & t \notin D_{n+1}. \end{cases}$$

By part (i) and Lemma 4.7, $Y^{*,n+1}(t)$ is a D_{n+1}-RCLL modification of $\bar{Y}^{*,n+1}(t)$ and, for every stopping time τ,

$$\begin{aligned} Y^{*,n+1}(\tau) &= \mathbf{1}_{\{\tau\notin D_{n+1}\}} X^{*,n+1}(\tau) + \mathbf{1}_{\{\tau\in D_{n+1}\}} \bar{Y}^{*,n+1}(\tau) \\ &= \mathbf{1}_{\{\tau\notin D_{n+1}\}} \bar{X}^{*,n+1}(\tau) + \mathbf{1}_{\{\tau\in D_{n+1}\}} \bar{Y}^{*,n+1}(\tau) = \bar{Y}^{*,n+1}(\tau). \end{aligned}$$

□

We can now complete the induction step for Theorem 2.2.

Proof of (S_{n+1}). Existence is shown in Lemma 4.8. The integrability condition is satisfied, because

$$\begin{aligned} E[\sup_{0\le t<\infty} |Y^{*,n+1}(t)|^p] &= E[\sup_{0\le t\le T} |Y^{*,n+1}(t)|^p] \\ &\le E[|E[\sup_{0\le u\le T} Z^{n+1}(u)|\mathcal{F}_t]|^p] < \infty \end{aligned} \quad (13)$$

by Doob's maximal inequality and Lemma 4.6. (Recall that $Y^{*,n+1}(t) = 0$ for $t > T$.) □

Proof of (R_{n+1}). This follows by combining Lemmas 4.6 and 4.8. □

Proof of (DM_{n+1}). On an interval $[T-\nu\delta, T-(\nu-1)\delta]$, $\nu = 1, \ldots, n$, we consider the process

$$Y^\nu(t) = \begin{cases} Y^{*,n+1}(t), & t \in (T-\nu\delta, T-(\nu-1)\delta] \\ \lim_{u\downarrow T-\nu\delta} Y^{*,n+1}(u), & t = T-\nu\delta \end{cases},$$

which is a well-defined RCLL process on $[T-\nu\delta, T-(\nu-1)\delta]$. It is a supermartingale, because it coincides with the supermartingale $X^{*,n+1}$ on $[T-\nu\delta, T-(\nu-1)\delta)$ and with the supermartingale $Y^{*,n+1}$ on $(T-\nu\delta, T-(\nu-1)\delta]$. By (13),

$$E[\sup_{T-\nu\delta\le t\le T-(\nu-1)\delta} |Y^\nu(t)|^p] < \infty.$$

Thus, the RCLL supermartingale $Y^\nu(t)$ belongs to class D and admits a Doob-Meyer decomposition

$$Y^\nu(t) = Y^\nu(T-\nu\delta) + M^\nu(t) - M^\nu(T-\nu\delta) - (A^\nu(t) - A^\nu(T-\nu\delta))$$

for $t \in [T-\nu\delta, T-(\nu-1)\delta]$, where M^ν is an RCLL martingale and A^ν is a nondecreasing predictable and integrable RCLL process. The p-integrability of $\sup_t |Y^\nu(t)|$ and the integrability of A^ν ensure that

$$E[\sup_{T-\nu\delta\le t\le T-(\nu-1)\delta} |M^\nu(t) - M^\nu(T-\nu\delta)|] < \infty. \quad (14)$$

Finally, on the interval $[0, T - n\delta]$ we have the Doob-Meyer decomposition of the RCLL (restricted to this interval!) supermartingale $Y^{*,n+1}$

$$Y^{*,n+1}(t) = Y^{*,n+1}(0) + M(t) - A(t).$$

We now define recursively, $M^{*,n+1}(t) = M(t)$, $A^{*,n+1}(t) = A(t)$ on $[0, T - n\delta]$ and, for $t \in (T - \nu\delta, T - (\nu - 1)\delta]$, $\nu = 1, \ldots, n$,

$$\begin{aligned} M^{*,n+1}(t) &= M^{*,n+1}(T - \nu\delta) + M^{\nu}(t) - M^{\nu}(T - \nu\delta), \\ A^{*,n+1}(t) &= A^{*,n+1}(T - \nu\delta) + A^{\nu}(t) - A^{\nu}(T - \nu\delta) \\ &\quad + Y^{*,n+1}(T - \nu\delta) - X^{*,n+1}(T - \nu\delta). \end{aligned}$$

Then, $M^{*,n+1}$ is an RCLL martingale and, by (14),

$$E[\sup_{0 \le t \le T} |M^{*,n+1}(t)|] < \infty.$$

Moreover, $A^{*,n+1}$ is D_{n+1}-RCLL and nondecreasing, because $Y^{*,n+1}(T - \nu\delta) \ge X^{*,n+1}(T - \nu\delta)$. It is predictable as the sum of a predictable and a left-continuous adapted process. Finally, we obtain, for $t \in (T - \nu\delta, T - (\nu - 1)\delta]$,

$$\begin{aligned} & Y^{*,n+1}(t) - Y^{*,n+1}(T - \nu\delta) \\ = \; & Y^{\nu}(t) - Y^{\nu}(T - \nu\delta) - (Y^{*,n+1}(T - \nu\delta) - X^{*,n+1}(T - \nu\delta)) \\ = \; & M^{\nu}(t) - M^{\nu}(T - \nu\delta) - (A^{\nu}(t) - A^{\nu}(T - \nu\delta)) \\ & -(Y^{*,n+1}(T - \nu\delta) - X^{*,n+1}(T - \nu\delta)) \\ = \; & M^{*,n+1}(t) - M^{*,n+1}(T - \nu\delta) - (A^{*,n+1}(t) - A^{*,n+1}(T - \nu\delta)). \end{aligned}$$

Recursively we now observe that we have constructed a decomposition as required in Theorem 2.2, (ii), on the interval $[0, T]$. For $t > T$ we set

$$M^{*,n+1}(t) = M^{*,n+1}(T), \quad A^{*,n+1}(t) = A^{*,n+1}(T) + Z(T).$$

□

The proof of Theorem 2.2 is now complete.

5 Proof of the dual representation

This final section is devoted to the proof of the dual pricing formula.

Proof of Theorem 2.3. (i) Suppose $(\tau_1, \ldots, \tau_n) \in \mathcal{S}^n_{\delta,t}$ and M^{ν}, A^{ν} are as stated in the assertion. As $Z(t) = 0$ for $t \ge T$, we can assume w.l.o.g that

τ_1 is bounded from above by $T + (n+1)\delta$. Then, by optional sampling and the tower property of conditional expectations, we get,

$$\sum_{\nu=1}^{n} E[Z(\tau_\nu)|\mathcal{F}_t] = E\Bigg[\sum_{\nu=1}^{n-1}\Big(Z(\tau_\nu) - (M^\nu(\tau_\nu) - M^\nu(\tau_{\nu+1})) + A^\nu(\tau_{\nu+1}+\delta) - E[A^\nu(\tau_{\nu+1}+\delta)|\mathcal{F}_{\tau_{\nu+1}}]\Big) + Z(\tau_n) - (M^n(\tau_n) - M^n(t))\Bigg|\mathcal{F}_t\Bigg]$$

By Remark 4.5 and the boundedness of τ_1, we obtain,

$$\sum_{\nu=1}^{n} E[Z(\tau_\nu)|\mathcal{F}_t] \leq E\Bigg[\sup_{u_1,\ldots,u_n\in\Delta_t^{\delta,n}} \sum_{\nu=1}^{n-1}\Big(Z(u_\nu) - (M^\nu(u_\nu) - M^\nu(u_{\nu+1})) + A^\nu(u_{\nu+1}+\delta) - E[A^\nu(u_{\nu+1}+\delta)|\mathcal{F}_{u_{\nu+1}}]\Big) + Z(u_n) - (M^n(u_n) - M^n(t))\Bigg|\mathcal{F}_t\Bigg]$$

The assertion now follows by taking the supremum over all $(\tau_1,\ldots,\tau_n)\in\mathcal{S}^n_{\delta,t}$.

(ii) The proof goes by induction on n with the case $n=1$ covered in Rogers (2002). Suppose now that the claim is already proved for some $n\in\mathbb{N}$. By Theorem 2.2, (R_{n+1}), we observe that $Y^{*,n+1}$ is the Snell envelope of a single optimal stopping problem with cash-flow

$$Z^{n+1}(t) = Z(t) + E[Y^{*,n}(t+\delta)|\mathcal{F}_t] \quad \text{(up to modification).}$$

By the Doob-Meyer type decomposition in Theorem 2.2, (DM_n), we obtain

$$\begin{aligned}
& E[Y^{*,n}(t+\delta)|\mathcal{F}_t] \\
= \;& Y^{*,n}(0) + M^{*,n}(t) - E[A^{*,n}(t+\delta)|\mathcal{F}_t] \\
= \;& Y^{*,n}(0) + M^{*,n}(t+\delta) - A^{*,n}(t+\delta) \\
& -(M^{*,n}(t+\delta) - M^{*,n}(t)) + A^{*,n}(t+\delta) - E[A^{*,n}(t+\delta)|\mathcal{F}_t] \\
= \;& Y^{*,n}(t+\delta) - (M^{*,n}(t+\delta) - M^{*,n}(t)) + A^{*,n}(t+\delta) - E[A^{*,n}(t+\delta)|\mathcal{F}_t].
\end{aligned}$$

We now apply the inductive hypothesis on $Y^{*,n}(t+\delta)$ in order to obtain, for $0 \le u_{n+1} \le T$,

$$\begin{aligned}
& Z^{n+1}(u_{n+1}) \\
= \; & Z(u_{n+1}) + \sup_{u_1,\dots,u_n \in \Delta^{\delta,n}_{u_{n+1}+\delta}} \Bigg\{ \sum_{\nu=1}^{n-1} \Big(Z(u_\nu) - (M^{*,\nu}(u_\nu) - M^{*,\nu}(u_{\nu+1})) \\
& \quad + A^{*,\nu}(u_{\nu+1}+\delta) - E[A^{*,\nu}(u_{\nu+1}+\delta)|\mathcal{F}_{u_{\nu+1}}] \Big) \\
& \quad + Z(u_n) - (M^{*,n}(u_n) - M^{*,n}(u_{n+1}+\delta)) \Bigg\} \\
& - (M^{*,n}(u_{n+1}+\delta) - M^{*,n}(u_{n+1})) \\
& + A^{*,n}(u_{n+1}+\delta) - E[A^{*,n}(u_{n+1}+\delta)|\mathcal{F}_{u_{n+1}}] \\
= \; & Z(u_{n+1}) + \sup_{u_1,\dots,u_n \in \Delta^{\delta,n}_{u_{n+1}+\delta}} \Bigg\{ \sum_{\nu=1}^{n} \Big(Z(u_\nu) - (M^{*,\nu}(u_\nu) - M^{*,\nu}(u_{\nu+1})) \\
& \quad + A^{*,\nu}(u_{\nu+1}+\delta) - E[A^{*,\nu}(u_{\nu+1}+\delta)|\mathcal{F}_{u_{\nu+1}}] \Big) \Bigg\}
\end{aligned}$$

Now, thanks to Theorem 2.2 we can follow the argumentation in Rogers (2002) for the cash-flow Z^{n+1}. Indeed, by Theorem 2.2, (R_{n+1}), we have, $Z^{n+1}(u_{n+1}) \le Y^{*,n+1}(u_{n+1})$. Hence, by Theorem 2.2, (DM_{n+1}), and since $A^{*,n+1}$ is nondecreasing,

$$\begin{aligned}
Y^{*,n+1}_t \; & \ge \sup_{u_{n+1} \ge t} \Big\{ Y^{*,n+1}_t - (A^{*,n+1}(u_{n+1}) - A^{*,n+1}(t)) \Big\} \\
& = \sup_{u_{n+1} \ge t} \Big\{ Y^{*,n+1}_{u_{n+1}} - (M^{*,n+1}(u_{n+1}) - M^{*,n+1}(t)) \Big\} \\
& \ge \sup_{u_{n+1} \ge t} \Big\{ Z^{n+1}_{u_{n+1}} - (M^{*,n+1}(u_{n+1}) - M^{*,n+1}(t)) \Big\} \\
& = \sup_{u_1,\dots,u_{n+1} \in \Delta^{\delta,n+1}_t} \Bigg\{ \sum_{\nu=1}^{n} \Big(Z(u_\nu) - (M^{*,\nu}(u_\nu) - M^{*,\nu}(u_{\nu+1})) \\
& \qquad + A^{*,\nu}(u_{\nu+1}+\delta) - E[A^{*,\nu}(u_{\nu+1}+\delta)|\mathcal{F}_{u_{\nu+1}}] \Big) \\
& \qquad + Z(u_{n+1}) - (M^{*,n+1}(u_{n+1}) - M^{*,n+1}(t)) \Bigg\}.
\end{aligned}$$

Here we plugged in the alternative representation for Z^{n+1} in the last step. The converse inequality now follows immediately from (i). □

Acknowledgements

The author wishes to thank René Carmona, Chris Rogers, John Schoenmakers, and Nizar Touzi for interesting discussions on the topic. Financial support by the Deutsche Forschungsgemeinschaft under grant BE3933/3-1 is gratefully acknowledged.

References

Aleksandrov, N., Hambly, B.: A dual approach to multiple exercise option problems under constraints. Math. Meth. in Oper. Res. **71**, 503–533 (2010)

Andersen, L., M. Broadie: A primal-dual simulation algorithm for pricing multidimensional American options. Management Sciences **50**, 1222–1234 (2004)

Bally, V., Pagès, G.: A quantization algorithm for solving multi-dimensional discrete-time optimal stopping problems. Bernoulli. **9**, 1003–1049 (2003)

Belomestny, D., Bender, C., Schoenmakers, J.: True upper bounds for Bermudan products via non-nested Monte Carlo. Math. Finance **19**, 53–71 (2009)

Bender, C.: Dual pricing of multi-exercise options under volume constraints. Finance Stoch., online first, DOI 10.1007/s00780-010-0134-8 (2010).

Bender, C., Schoenmakers, J.: An iterative method for multiple stopping: convergence and stability. Adv. Appl. Probab. **38**, 729–749 (2006)

Benth, F. E., Benth, J. S., Koekebakker, S.: *Stochastic Modelling of Electricity and Related Markets*. World Scientific, Singapore (2008)

Benth, F. E., Kallsen, J., Meyer-Brandis, T.: A non-Gaussian Ornstein-Uhlenbeck process for electricity spot price modeling and derivatives pricing. Appl. Math. Finance **14**, 153–169 (2007)

Bouchard, B., Ekeland, I., Touzi, N.: On the Malliavin approach to Monte Carlo approximation of conditional expectations. Finance Stoch. **8**, 45–71 (2004)

Cairoli, R., Dalang, R.: *Sequential Stochastic Optimization*. Wiley, New York (1996)

Carmona, R., Dayanik, S.: Optimal multiple stopping of linear diffusions. Math. Operations Res. **33**, 446–460 (2008)

Carmona, R., Touzi, N.: Optimal multiple stopping and the valuation of Swing options. Math. Finance **18**, 239–268 (2008)

Haggstrom, G.: Optimal sequential procedures when more than one stop is required. Ann. Math. Stat. **38**, 1618–1626 (1967)

Hambly, B., Howison, S., Kluge, T.: Modeling spikes and pricing swing options in electricity markets. Quant. Finance **9**, 937–949 (2009)

Haugh, M., Kogan, L.: Pricing American options: a duality approach. Operations Research **52**, 258–270 (2004)

Jaillet, P., Ronn, E. I., Tompaidis, S.: Valuation of commodity based swing options. Management Science **50**, 909–921 (2004)

Karatzas, I., Shreve, S.: *Brownian Motion and Stochastic Calculus.* Springer, New York (1991)

Karatzas, I., Shreve, S.: *Methods of Mathematical Finance.* Springer, New York (1998)

Longstaff, F. A., Schwartz, E. S.: Valuing American options by simulation: A simple least-squares approach. Review of Financial Studies **14**, 113–147 (2001)

Lucia, J., Schwartz, E.: Electricity prices and power derivatives: Evidence from the nordic power exchange. Rev. Derivatives Research **5**, 5–50 (2002).

Meinshausen, N., Hambly, B. M.: Monte Carlo methods for the valuation of multiple-exercise options. Math. Finance **14**, 557–583 (2004)

Rogers, L.C.G.: Monte Carlo valuation of American options. Math. Finance **12**, 271–286 (2002)

Schoenmakers, J.: A pure martingale dual for multiple stopping. Finance Stoch., forthcoming (2010).

Zeghal, A., Mnif, M.: Optimal multiple stopping and the valuation of swing options in Lévy models. Int. J. Theor. Appl. Finance **9**, 1267–1297 (2006)

Preprint Series DFG-SPP 1324

http://www.dfg-spp1324.de

Reports

[1] R. Ramlau, G. Teschke, and M. Zhariy. A Compressive Landweber Iteration for Solving Ill-Posed Inverse Problems. Preprint 1, DFG-SPP 1324, September 2008.

[2] G. Plonka. The Easy Path Wavelet Transform: A New Adaptive Wavelet Transform for Sparse Representation of Two-dimensional Data. Preprint 2, DFG-SPP 1324, September 2008.

[3] E. Novak and H. Woźniakowski. Optimal Order of Convergence and (In-) Tractability of Multivariate Approximation of Smooth Functions. Preprint 3, DFG-SPP 1324, October 2008.

[4] M. Espig, L. Grasedyck, and W. Hackbusch. Black Box Low Tensor Rank Approximation Using Fibre-Crosses. Preprint 4, DFG-SPP 1324, October 2008.

[5] T. Bonesky, S. Dahlke, P. Maass, and T. Raasch. Adaptive Wavelet Methods and Sparsity Reconstruction for Inverse Heat Conduction Problems. Preprint 5, DFG-SPP 1324, January 2009.

[6] E. Novak and H. Woźniakowski. Approximation of Infinitely Differentiable Multivariate Functions Is Intractable. Preprint 6, DFG-SPP 1324, January 2009.

[7] J. Ma and G. Plonka. A Review of Curvelets and Recent Applications. Preprint 7, DFG-SPP 1324, February 2009.

[8] L. Denis, D. A. Lorenz, and D. Trede. Greedy Solution of Ill-Posed Problems: Error Bounds and Exact Inversion. Preprint 8, DFG-SPP 1324, April 2009.

[9] U. Friedrich. A Two Parameter Generalization of Lions' Nonoverlapping Domain Decomposition Method for Linear Elliptic PDEs. Preprint 9, DFG-SPP 1324, April 2009.

[10] K. Bredies and D. A. Lorenz. Minimization of Non-smooth, Non-convex Functionals by Iterative Thresholding. Preprint 10, DFG-SPP 1324, April 2009.

[11] K. Bredies and D. A. Lorenz. Regularization with Non-convex Separable Constraints. Preprint 11, DFG-SPP 1324, April 2009.

[12] M. Döhler, S. Kunis, and D. Potts. Nonequispaced Hyperbolic Cross Fast Fourier Transform. Preprint 12, DFG-SPP 1324, April 2009.

[13] C. Bender. Dual Pricing of Multi-Exercise Options under Volume Constraints. Preprint 13, DFG-SPP 1324, April 2009.

[14] T. Müller-Gronbach and K. Ritter. Variable Subspace Sampling and Multi-level Algorithms. Preprint 14, DFG-SPP 1324, May 2009.

[15] G. Plonka, S. Tenorth, and A. Iske. Optimally Sparse Image Representation by the Easy Path Wavelet Transform. Preprint 15, DFG-SPP 1324, May 2009.

[16] S. Dahlke, E. Novak, and W. Sickel. Optimal Approximation of Elliptic Problems by Linear and Nonlinear Mappings IV: Errors in L_2 and Other Norms. Preprint 16, DFG-SPP 1324, June 2009.

[17] B. Jin, T. Khan, P. Maass, and M. Pidcock. Function Spaces and Optimal Currents in Impedance Tomography. Preprint 17, DFG-SPP 1324, June 2009.

[18] G. Plonka and J. Ma. Curvelet-Wavelet Regularized Split Bregman Iteration for Compressed Sensing. Preprint 18, DFG-SPP 1324, June 2009.

[19] G. Teschke and C. Borries. Accelerated Projected Steepest Descent Method for Nonlinear Inverse Problems with Sparsity Constraints. Preprint 19, DFG-SPP 1324, July 2009.

[20] L. Grasedyck. Hierarchical Singular Value Decomposition of Tensors. Preprint 20, DFG-SPP 1324, July 2009.

[21] D. Rudolf. Error Bounds for Computing the Expectation by Markov Chain Monte Carlo. Preprint 21, DFG-SPP 1324, July 2009.

[22] M. Hansen and W. Sickel. Best m-term Approximation and Lizorkin-Triebel Spaces. Preprint 22, DFG-SPP 1324, August 2009.

[23] F.J. Hickernell, T. Müller-Gronbach, B. Niu, and K. Ritter. Multi-level Monte Carlo Algorithms for Infinite-dimensional Integration on $\mathbb{R}^{\mathbb{N}}$. Preprint 23, DFG-SPP 1324, August 2009.

[24] S. Dereich and F. Heidenreich. A Multilevel Monte Carlo Algorithm for Lévy Driven Stochastic Differential Equations. Preprint 24, DFG-SPP 1324, August 2009.

[25] S. Dahlke, M. Fornasier, and T. Raasch. Multilevel Preconditioning for Adaptive Sparse Optimization. Preprint 25, DFG-SPP 1324, August 2009.

[26] S. Dereich. Multilevel Monte Carlo Algorithms for Lévy-driven SDEs with Gaussian Correction. Preprint 26, DFG-SPP 1324, August 2009.

[27] G. Plonka, S. Tenorth, and D. Roşca. A New Hybrid Method for Image Approximation using the Easy Path Wavelet Transform. Preprint 27, DFG-SPP 1324, October 2009.

[28] O. Koch and C. Lubich. Dynamical Low-rank Approximation of Tensors. Preprint 28, DFG-SPP 1324, November 2009.

[29] E. Faou, V. Gradinaru, and C. Lubich. Computing Semi-classical Quantum Dynamics with Hagedorn Wavepackets. Preprint 29, DFG-SPP 1324, November 2009.

[30] D. Conte and C. Lubich. An Error Analysis of the Multi-configuration Time-dependent Hartree Method of Quantum Dynamics. Preprint 30, DFG-SPP 1324, November 2009.

[31] C. E. Powell and E. Ullmann. Preconditioning Stochastic Galerkin Saddle Point Problems. Preprint 31, DFG-SPP 1324, November 2009.

[32] O. G. Ernst and E. Ullmann. Stochastic Galerkin Matrices. Preprint 32, DFG-SPP 1324, November 2009.

[33] F. Lindner and R. L. Schilling. Weak Order for the Discretization of the Stochastic Heat Equation Driven by Impulsive Noise. Preprint 33, DFG-SPP 1324, November 2009.

[34] L. Kämmerer and S. Kunis. On the Stability of the Hyperbolic Cross Discrete Fourier Transform. Preprint 34, DFG-SPP 1324, December 2009.

[35] P. Cerejeiras, M. Ferreira, U. Kähler, and G. Teschke. Inversion of the noisy Radon transform on $SO(3)$ by Gabor frames and sparse recovery principles. Preprint 35, DFG-SPP 1324, January 2010.

[36] T. Jahnke and T. Udrescu. Solving Chemical Master Equations by Adaptive Wavelet Compression. Preprint 36, DFG-SPP 1324, January 2010.

[37] P. Kittipoom, G. Kutyniok, and W.-Q Lim. Irregular Shearlet Frames: Geometry and Approximation Properties. Preprint 37, DFG-SPP 1324, February 2010.

[38] G. Kutyniok and W.-Q Lim. Compactly Supported Shearlets are Optimally Sparse. Preprint 38, DFG-SPP 1324, February 2010.

[39] M. Hansen and W. Sickel. Best m-Term Approximation and Tensor Products of Sobolev and Besov Spaces – the Case of Non-compact Embeddings. Preprint 39, DFG-SPP 1324, March 2010.

[40] B. Niu, F.J. Hickernell, T. Müller-Gronbach, and K. Ritter. Deterministic Multi-level Algorithms for Infinite-dimensional Integration on $\mathbb{R}^{\mathbb{N}}$. Preprint 40, DFG-SPP 1324, March 2010.

[41] P. Kittipoom, G. Kutyniok, and W.-Q Lim. Construction of Compactly Supported Shearlet Frames. Preprint 41, DFG-SPP 1324, March 2010.

[42] C. Bender and J. Steiner. Error Criteria for Numerical Solutions of Backward SDEs. Preprint 42, DFG-SPP 1324, April 2010.

[43] L. Grasedyck. Polynomial Approximation in Hierarchical Tucker Format by Vector-Tensorization. Preprint 43, DFG-SPP 1324, April 2010.

[44] M. Hansen und W. Sickel. Best m-Term Approximation and Sobolev-Besov Spaces of Dominating Mixed Smoothness - the Case of Compact Embeddings. Preprint 44, DFG-SPP 1324, April 2010.

[45] P. Binev, W. Dahmen, and P. Lamby. Fast High-Dimensional Approximation with Sparse Occupancy Trees. Preprint 45, DFG-SPP 1324, May 2010.

[46] J. Ballani and L. Grasedyck. A Projection Method to Solve Linear Systems in Tensor Format. Preprint 46, DFG-SPP 1324, May 2010.

[47] P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrova, and P. Wojtaszczyk. Convergence Rates for Greedy Algorithms in Reduced Basis Methods. Preprint 47, DFG-SPP 1324, May 2010.

[48] S. Kestler and K. Urban. Adaptive Wavelet Methods on Unbounded Domains. Preprint 48, DFG-SPP 1324, June 2010.

[49] H. Yserentant. The Mixed Regularity of Electronic Wave Functions Multiplied by Explicit Correlation Factors. Preprint 49, DFG-SPP 1324, June 2010.

[50] H. Yserentant. On the Complexity of the Electronic Schrödinger Equation. Preprint 50, DFG-SPP 1324, June 2010.

[51] M. Guillemard and A. Iske. Curvature Analysis of Frequency Modulated Manifolds in Dimensionality Reduction. Preprint 51, DFG-SPP 1324, June 2010.

[52] E. Herrholz and G. Teschke. Compressive Sensing Principles and Iterative Sparse Recovery for Inverse and Ill-Posed Problems. Preprint 52, DFG-SPP 1324, July 2010.

[53] L. Kämmerer, S. Kunis, and D. Potts. Interpolation Lattices for Hyperbolic Cross Trigonometric Polynomials. Preprint 53, DFG-SPP 1324, July 2010.

[54] G. Kutyniok and W.-Q Lim. Shearlets on Bounded Domains. Preprint 54, DFG-SPP 1324, July 2010.

[55] A. Zeiser. Wavelet Approximation in Weighted Sobolev Spaces of Mixed Order with Applications to the Electronic Schrödinger Equation. Preprint 55, DFG-SPP 1324, July 2010.

[56] G. Kutyniok, J. Lemvig, and W.-Q Lim. Compactly Supported Shearlets. Preprint 56, DFG-SPP 1324, July 2010.

[57] A. Zeiser. On the Optimality of the Inexact Inverse Iteration Coupled with Adaptive Finite Element Methods. Preprint 57, DFG-SPP 1324, July 2010.

[58] S. Jokar. Sparse Recovery and Kronecker Products. Preprint 58, DFG-SPP 1324, August 2010.

[59] T. Aboiyar, E. H. Georgoulis, and A. Iske. Adaptive ADER Methods Using Kernel-Based Polyharmonic Spline WENO Reconstruction. Preprint 59, DFG-SPP 1324, August 2010.

[60] O. G. Ernst, A. Mugler, H.-J. Starkloff, and E. Ullmann. On the Convergence of Generalized Polynomial Chaos Expansions. Preprint 60, DFG-SPP 1324, August 2010.

[61] S. Holtz, T. Rohwedder, and R. Schneider. On Manifolds of Tensors of Fixed TT-Rank. Preprint 61, DFG-SPP 1324, September 2010.

[62] J. Ballani, L. Grasedyck, and M. Kluge. Black Box Approximation of Tensors in Hierarchical Tucker Format. Preprint 62, DFG-SPP 1324, October 2010.

[63] M. Hansen. On Tensor Products of Quasi-Banach Spaces. Preprint 63, DFG-SPP 1324, October 2010.

[64] S. Dahlke, G. Steidl, and G. Teschke. Shearlet Coorbit Spaces: Compactly Supported Analyzing Shearlets, Traces and Embeddings. Preprint 64, DFG-SPP 1324, October 2010.

[65] W. Hackbusch. Tensorisation of Vectors and their Efficient Convolution. Preprint 65, DFG-SPP 1324, November 2010.

[66] P. A. Cioica, S. Dahlke, S. Kinzel, F. Lindner, T. Raasch, K. Ritter, and R. L. Schilling. Spatial Besov Regularity for Stochastic Partial Differential Equations on Lipschitz Domains. Preprint 66, DFG-SPP 1324, November 2010.

[67] E. Novak and H. Woźniakowski. On the Power of Function Values for the Approximation Problem in Various Settings. Preprint 67, DFG-SPP 1324, November 2010.

[68] A. Hinrichs, E. Novak, and H. Woźniakowski. The Curse of Dimensionality for Monotone and Convex Functions of Many Variables. Preprint 68, DFG-SPP 1324, November 2010.

[69] G. Kutyniok and W.-Q Lim. Image Separation Using Shearlets. Preprint 69, DFG-SPP 1324, November 2010.

[70] B. Jin and P. Maass. An Analysis of Electrical Impedance Tomography with Applications to Tikhonov Regularization. Preprint 70, DFG-SPP 1324, December 2010.

[71] S. Holtz, T. Rohwedder, and R. Schneider. The Alternating Linear Scheme for Tensor Optimisation in the TT Format. Preprint 71, DFG-SPP 1324, December 2010.

[72] T. Müller-Gronbach and K. Ritter. A Local Refinement Strategy for Constructive Quantization of Scalar SDEs. Preprint 72, DFG-SPP 1324, December 2010.

[73] T. Rohwedder and R. Schneider. An Analysis for the DIIS Acceleration Method used in Quantum Chemistry Calculations. Preprint 73, DFG-SPP 1324, December 2010.

[74] C. Bender and J. Steiner. Least-Squares Monte Carlo for Backward SDEs. Preprint 74, DFG-SPP 1324, December 2010.

[75] C. Bender. Primal and Dual Pricing of Multiple Exercise Options in Continuous Time. Preprint 75, DFG-SPP 1324, December 2010.