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„Extraktion quantifizierbarer Information aus komplexen Systemen"

A Local Refinement Strategy for Constructive Quantization of Scalar SDEs

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A LOCAL REFINEMENT STRATEGY FOR CONSTRUCTIVE QUANTIZATION OF SCALAR SDEs

THOMAS MÜLLER-GRONBACH AND KLAUS RITTER

Abstract. We present a fully constructive method for quantization of the solution X of a scalar SDE in the path space $L_p[0,1]$ or $C[0,1]$. The construction relies on a refinement strategy, which takes into account the local regularity of X and uses Brownian motion (bridge) quantization as a building block. Our algorithm is easy to implement, its computational cost is close to the size of the quantization, and it achieves strong asymptotic optimality provided this property holds for the Brownian motion (bridge) quantization.

1. Introduction

Consider a random element X with values in a separable Banach space $\mathfrak{B}$. The quantization problem for X consists in approximation of X by $\mathfrak{B}$-valued random elements $\widetilde{X}$ with finite range $\operatorname{ran}(\widetilde{X})$, where X and $\widetilde{X}$ are defined on the same probability space. For any such $\widetilde{X}$ the error of order $s \in [1, \infty[$ is defined by

$$e^{(s)}(X, \widetilde{X}, \mathfrak{B}) = \left(E\|X - \widetilde{X}\|_{\mathfrak{B}}^s\right)^{1/s},$$

and $\widetilde{X}$ is called an N-quantization of X if $|\operatorname{ran}(\widetilde{X})| \leq N$. The N-th quantization error

$$e_N^{(s)}(X, \mathfrak{B}) = \inf\{e^{(s)}(X, \widetilde{X}, \mathfrak{B}) : |\operatorname{ran}(\widetilde{X})| \leq N\}$$

of order s is the minimal error that can be achieved by any N-quantization of X. For solving this minimization problem it suffices to consider N-quantizations of the form $\widetilde{X} = T(X)$, where $T : \mathfrak{B} \to \mathfrak{B}$ is measurable with $|\operatorname{ran}(T)| \leq N$. Any such mapping T is called an N-quantizer. We refer to the monograph [7] and to the survey [19].

In an asymptotic analysis of the quantization problem one wants to determine the sharp rate of convergence of $e_N^{(s)}(X, \mathfrak{B})$ and to construct a sequence of N-quantizations $\widetilde{X}_N$ such that

$$\limsup_{N\to\infty} \frac{e^{(s)}(X, \widetilde{X}_N, \mathfrak{B})}{e_N^{(s)}(X, \mathfrak{B})} \leq \delta$$

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for a given $\delta \geq 1$. Ideally, $\delta = 1$, and in this case the sequence of quantizers is called strongly asymptotically optimal.

The present paper is devoted to the construction problem for X being the solution process of a scalar autonomous stochastic differential equation (SDE), and we consider the spaces $\mathfrak{B}_p = L_p[0,1]$ with $1 \leq p < \infty$ and $\mathfrak{B}_\infty = C[0,1]$. The sharp asymptotics of the corresponding N-th quantization errors is determined in [2, 3], and the first approach to constructive quantization is due to [12] with subsequent work in [13, 20]. Note that the distribution of X on $\mathfrak{B}_p$ is typically only given implicitly, which constitutes a major challenge for the construction task.

In this paper we present a fully constructive method for quantization of SDEs that employs results and techniques for strong approximation of SDEs. At first we apply a coarse level quantization that is based on the Milstein scheme and on quantization of the one-dimensional standard normal distribution. Secondly, we construct a local refinement in the path space, which takes into account the local regularity of X. This step is crucial for the overall performance of the quantization, and the refinement strategy is similar to asymptotically optimal step-size control for strong approximation of SDEs, see [8, 9] and [15, 16]. The construction uses a sequence of quantizations $\widetilde{W}_N$ of a Brownian motion W (or Brownian bridge) as a building block, see [1, 5, 11] as well as the surveys [4, 19] and the web site [18] for downloads.

Altogether we achieve

$$\limsup_{N\to\infty} \frac{e^{(s)}(X, \widetilde{X}_N, \mathfrak{B}_p)}{e_N^{(s)}(X, \mathfrak{B}_p)} \leq \limsup_{N\to\infty} \frac{e^{(s)}(W, \widetilde{W}_N, \mathfrak{B}_p)}{e_N^{(s)}(W, \mathfrak{B}_p)}$$

with $s = p$ in the case $p < \infty$ and $1 \leq s < \infty$ for $p = \infty$, see Theorems 1 and 2. In particular, we obtain strong asymptotic optimality for the quantizations $\widetilde{X}_N$ of X, i.e.,

$$\lim_{N\to\infty} \frac{e^{(s)}(X, \widetilde{X}_N, \mathfrak{B}_p)}{e_N^{(s)}(X, \mathfrak{B}_p)} = 1,$$

provided that this property holds for the quantizations $\widetilde{W}_N$ of W. We stress that these results hold uniformly over all scalar autonomous SDEs, whose coefficients satisfy standard smoothness assumptions, and that the computational cost is close to the size of the quantization. Our construction can be extended to systems of SDEs, and we expect the same asymptotic results to hold in this case, too.

We briefly describe the content of the paper. In Section 2 we present the construction and the results for our quantization method. We first study quantization of marginal distributions of X in Section 2.1, and we then treat quantization in $L_p[0,1]$ and $C[0,1]$ in Sections 2.2 and 2.3. Section 2 concludes with a discussion and an example. Proofs are postponed to Section 3.

2. Constructions and Results

We consider a scalar autonomous SDE

$$
\begin{aligned}
dX(t) &= a(X(t))\,dt + b(X(t))\,dW(t), \qquad t \in [0,1],\\
X(0) &= x_0
\end{aligned} \tag{1}
$$

with deterministic initial value x_0 and scalar driving Brownian motion W. We assume that the drift and diffusion coefficients a and b have the following properties. Both, a and b are differentiable. Moreover, there exists a constant $K > 0$ such that $f = a$ and $f = b$ satisfy

$$\sup_{x\in\mathbb{R}} |f'(x)| \le K$$

and

$$|f'(x) - f'(y)| \le K \cdot |x - y|$$

for all $x, y \in \mathbb{R}$.

The quantization of the solution X of the SDE (1) will be done in two steps. At first we consider a uniform time-discretization and we construct a quantization of the corresponding marginal distribution. Secondly, we construct a local refinement in the path space, which takes into account the local regularity of X. Actually, the second step turns out to be crucial for the overall performance of the quantization, and we will use different refinement strategies for quantization in $L_p[0,1]$ with $1 \le p < \infty$ and for quantization in $C[0,1]$.

Throughout the paper we use c to denote unspecified positive constants that may only depend on the parameters x_0, $a(0)$, $b(0)$, and K from equation (1) as well as on the moment parameters $p, q, r, s \ge 1$. The value of c may be different at every occurrence. Moreover, ln denotes the natural logarithm, and we use use $\approx$ to denote the strong asymptotic equivalence of sequences of positive reals, i.e., $a_n \approx b_n$ means $\lim_{n\to\infty} a_n/b_n = 1$.

2.1. **Quantization of marginal distributions.** Consider an equidistant discretization

$$t_\ell = \ell/m, \qquad \ell = 0, \dots, m, \tag{2}$$

of $[0,1]$. We construct a sufficiently good quantization of $(X(t_1), \dots, X(t_m))$ in a simple way. To this end we employ a product quantizer for the random vector $Y = (Y_1, \dots, Y_m)$ of normalized increments

$$Y_\ell = m^{1/2} \cdot (W(t_\ell) - W(t_{\ell-1})) \tag{3}$$

of the Brownian motion W as well as the Milstein scheme.

Let $q \ge 1$. For quantization of the increments we take a sequence of n-quantizers $T_n^{(q)} : \mathbb{R} \to \mathbb{R}$, and we put

$$\widetilde{Z}_n^{(q)} = T_n^{(q)}(Z)$$

with $Z \sim \mathrm{N}(0,1)$. We assume that

(A1) $E(\widetilde{Z}_n^{(q)}) = 0$ for every $n \in \mathbb{N}$,
(A2) $\sup_{n\in\mathbb{N}} E|\widetilde{Z}_n^{(q)}|^r < \infty$ for every $r \geq 1$,
(A3) $e^{(q)}(Z, \widetilde{Z}_n^{(q)}, \mathbb{R}) \leq c \cdot n^{-1}$ for every $n \in \mathbb{N}$.

See Section 3.1 for a construction.

The Milstein scheme $\widehat{X}_m$ for approximation of X at the discretization (2) is given by $\widehat{X}_m(t_0) = x_0$ and

$$\widehat{X}_m(t_\ell) = \widehat{X}_m(t_{\ell-1}) + a(\widehat{X}_m(t_{\ell-1})) \cdot m^{-1} + b(\widehat{X}_m(t_{\ell-1})) \cdot m^{-1/2} \cdot Y_\ell + 1/2 \cdot (b \cdot b')(\widehat{X}_m(t_{\ell-1})) \cdot m^{-1} \cdot \big(Y_\ell^2 - 1\big)$$

for $\ell = 1, \dots, m$. Replacing the increments Y_ℓ by their n-quantizations

$$\widetilde{Y}_{\ell,n}^{(q)} = T_n^{(q)}(Y_\ell) \tag{4}$$

we get an n^m-quantization of $(X(t_1), \dots, X(t_m))$ via $\widetilde{X}_{m,n}^{(q)}(t_0) = x_0$ and

$$\widetilde{X}_{m,n}^{(q)}(t_\ell) = \widetilde{X}_{m,n}^{(q)}(t_{\ell-1}) + a(\widetilde{X}_{m,n}^{(q)}(t_{\ell-1})) \cdot m^{-1} + b(\widetilde{X}_{m,n}^{(q)}(t_{\ell-1})) \cdot m^{-1/2} \cdot \widetilde{Y}_{\ell,n}^{(q)} + 1/2 \cdot (b \cdot b')(\widetilde{X}_{m,n}^{(q)}(t_{\ell-1})) \cdot m^{-1} \cdot \big((\widetilde{Y}_{\ell,n}^{(q)})^2 - 1\big) \tag{5}$$

for $\ell = 1, \dots, m$.

Proposition 1. *For all $m, n \in \mathbb{N}$*

$$E\Big(\max_{\ell=1,\dots,m} |X(t_\ell) - \widetilde{X}_{m,n}^{(q)}(t_\ell)|^q\Big) \leq c \cdot (m^{-1} + n^{-1})^q.$$

Remark 1. We recall a main result on quantization in finite-dimensional spaces. Suppose that a random vector Z takes values in the space $\mathfrak{B} = \mathbb{R}^m$. Then, under mild assumptions on the distribution P_Z of Z,

$$e_N^{(q)}(Z, \mathbb{R}^m) \approx c_1 \cdot N^{-1/m} \tag{6}$$

with a constant $c_1 > 0$ that depends on the dimension m, the underlying norm, the moment parameter q, and on the distribution P_Z. See [7, Thm. 6.2]. In particular, this result is valid for Z being normally distributed, so that we require optimality, up to a constant, in assumption (A3).

Proposition 1 yields an upper bound for the quantization errors of marginal distributions on the spaces $\mathfrak{B} = \mathbb{R}^m$, equipped with the supremum norm, which simultaneously holds in every dimension m. More precisely, put $Z_m = (X(t_1), \dots, X(t_m))$, and choose $n_N = \lfloor \ln N \rfloor$ as well as $m_N = \lfloor \ln N / \ln\ln N \rfloor$ to conclude that

$$e_N^{(q)}(Z_{m_N}, \mathbb{R}^{m_N}) \leq c \cdot \ln\ln N / \ln N.$$

Consider suitable subsequences and use projections onto the corresponding coordinates to derive

$$\sup_{m\in\mathbb{N}} e_N^{(q)}(Z_m,\mathbb{R}^m) \leq c\cdot \ln\ln N/\ln N.$$

The corresponding quantizations are obtained in a fully constructive way for distributions that are only given implicitly via the SDE. Note that an upper bound of any polynomial order fails to hold, since

$$\sup_{N,m\in\mathbb{N}} \Big(N^{\varepsilon}\cdot e_N^{(q)}(Z_m,\mathbb{R}^m)\Big) = \infty$$

for every $\varepsilon > 0$, see (6). Moreover,

$$\sup_{m\in\mathbb{N}} e_N^{(q)}(Z_m,\mathbb{R}^m) \geq c\cdot (\ln N)^{-1/2},$$

see Section 2.4 below. We add that constructive quantization of a fixed marginal distribution is studied in [17].

As a basic ingredient for quantization on the path space we extend $\widetilde{X}_{m,n}^{(q)}$ to a $C[0,1]$-valued random element by piecewise linear interpolation, i.e., we define

$$\widetilde{X}_{m,n}^{(q)}(t) = (t-t_{\ell-1})\cdot m\cdot \widetilde{X}_{m,n}^{(q)}(t_\ell) + (t_\ell - t)\cdot m\cdot \widetilde{X}_{m,n}^{(q)}(t_{\ell-1})$$

for $t\in[t_{\ell-1},t_\ell]$.

2.2. **Quantization in $L_p[0,1]$.** Throughout this section we consider the Banach space $\mathfrak{B} = L_p = L_p[0,1]$ with $1\leq p<\infty$, and we employ a sequence of k-quantizers $S_k : L_p\to L_p$ for a Brownian bridge B on $[0,1]$.

Let $m\in\mathbb{N}$. For the discretization (2) with step-size $1/m$ and Y_ℓ given by (3) we define

$$B_\ell(t) = W(t) - W(t_{\ell-1}) - (t-t_{\ell-1})\cdot m^{1/2}\cdot Y_\ell, \tag{7}$$

where $t\in[t_{\ell-1},t_\ell]$ and $\ell = 1,\dots,m$. The processes $B_\ell = (B_\ell(t))_{t\in[t_{\ell-1},t_\ell]}$ form independent Brownian bridges on the corresponding subintervals, and B_ℓ will be considered as a random element with values in $L_p[t_{\ell-1},t_\ell]$. In order to obtain a sequence of k-quantizations $\widetilde{B}_{\ell,k}$ of B_ℓ we use the bijection $\psi : L_p[t_{\ell-1},t_\ell]\to L_p[0,1]$ given by

$$\psi h(t) = m^{1/2}\cdot h(t_{\ell-1} + m^{-1}\cdot t), \qquad t\in[0,1],$$

and we define

$$\widetilde{B}_{\ell,k} = \psi^{-1}(S_k(\psi(B_\ell))). \tag{8}$$

We combine these quantizations of Brownian bridges with the marginal quantization according to Section 2.1 in order to get a quantization of X, which roughly is of size N for a given $N\in\mathbb{N}$. The construction properly takes into account the local regularity of X, which is given by the local Hölder constant $|b(X(t))|$ of X in mean-square sense. Let $q \geq 1$ and $n \in \mathbb{N}$. Put

$$\widetilde{b}_\ell = \widetilde{b}^{(q)}_{\ell,m,n} = b(\widetilde{X}^{(q)}_{m,n}(t_\ell)),$$

where $(\widetilde{X}^{(q)}_{m,n}(t_0), \ldots, \widetilde{X}^{(q)}_{m,n}(t_m))$ denotes the n^m quantization defined by (5). We define the random variables η_ℓ by

$$\eta_\ell = \eta^{(p,q)}_{\ell,m,n} = \max\left(\frac{|\widetilde{b}_{\ell-1}|^{2p/(p+2)}}{\sum_{j=0}^{m-1}|\widetilde{b}_j|^{2p/(p+2)}},\ \frac{1}{m\cdot\ln m}\right)$$

if $\max_{j=0,\ldots,m-1}|\widetilde{b}_j| > 0$, and $\eta_\ell = 1/m$ otherwise.

For each Brownian bridge B_ℓ we choose k in (8) dependent on N and η_ℓ by taking

$$K_{\ell,N} = K^{(p,q)}_{\ell,m,n,N} = \lceil N^{\eta_\ell}\rceil, \tag{9}$$

and we define a quantization $\widetilde{X}^{(p,q)}_{m,n,N}$ of X by

$$\widetilde{X}^{(p,q)}_{m,n,N}(t) = \widetilde{X}^{(q)}_{m,n}(t) + \widetilde{b}^{(q)}_{\ell-1,m,n}\cdot\widetilde{B}_{\ell,K_{\ell,N}}(t)$$

for $t \in [t_{\ell-1}, t_\ell]$. It is easy to see that

$$\operatorname{ran}(\widetilde{X}^{(p,q)}_{m,n,N}) \leq (e\cdot n)^m \cdot N^{1+1/\ln m}. \tag{10}$$

Finally, we choose a sequence of integers $m_N \in \mathbb{N}$ such that

$$\lim_{N\to\infty}\frac{m_N\cdot\ln m_N}{\ln N} = 0 \qquad \text{and} \qquad \lim_{N\to\infty}\frac{m_N^2}{\ln N} = \infty, \tag{11}$$

e.g., $m_N = (\ln N)^{2/3}$, and we consider the sequence of quantizations

$$\widetilde{X}^{(p,q)}_N = \widetilde{X}^{(p,q)}_{m_N,m_N,N}$$

of X.

The asymptotic behavior of the corresponding errors of order p for quantization in $L_p[0,1]$ depends on the SDE via the constant

$$C^{(p)} = \left(E\left(\int_0^1 |b(X(t))|^{2p/(p+2)}\,dt\right)^{(p+2)/2}\right)^{1/p} \tag{12}$$

and on the k-quantizers S_k for the Brownian bridge B on $[0,1]$ via

$$\kappa^{(p)} = \limsup_{k\to\infty}\left((\ln k)^{1/2}\cdot e^{(p)}(B, S_k(B), L_p)\right).$$

Theorem 1. *Let $q \geq \min\{r \in 2\mathbb{N} : r \geq p\}$. Then*

$$\limsup_{N\to\infty}\left((\ln N)^{-1}\cdot\ln|\operatorname{ran}(\widetilde{X}^{(p,q)}_N)|\right) \leq 1$$

and

$$\limsup_{N\to\infty}\left((\ln N)^{1/2}\cdot e^{(p)}(X, \widetilde{X}^{(p,q)}_N, L_p)\right) \leq \kappa^{(p)}\cdot C^{(p)}.$$

2.3. **Quantization in $C[0,1]$.** Throughout this section we consider the Banach space $\mathfrak{B} = C = C[0,1]$ equipped with the supremum norm, and we employ a sequence of N-quantizers $S_N : C \to C$ for a Brownian motion on $[0,1]$.

Let $m \in \mathbb{N}$, consider the equidistant discretization (2) with step-size $1/m$, and recall the definition (7) of the independent Brownian bridges $B_1, \dots, B_m$ on the corresponding subintervals. For technical reasons we do not apply quantization to these processes separately, which would lead to a product quantization as it was used in the L_p-case. Instead we provide a quantization, as a whole, of a weighted combination B^γ of the Brownian bridges, given by

$$B^{(\gamma)}(t) = \gamma_\ell \cdot B_\ell(t) \tag{13}$$

for $t \in [t_{\ell-1}, t_\ell]$. At first we consider deterministic weights $\gamma \in \mathbb{R}^m \setminus \{0\}$ that satisfy

$$\sum_{\ell=1}^{m} \gamma_\ell^2 = m. \tag{14}$$

Instead of the normalized increments $Y_1, \dots, Y_m$ of the driving Brownian motion, we consider an independent sequence of standard normally distributed random variables $Z_1, \dots, Z_m$ that is independent of W. Put $W_0^* = 0$ and $W_\ell^* = m^{-1/2} \cdot \sum_{j=1}^{\ell} Z_j$ for $\ell = 1, \dots, m$, and define a Brownian motion W^* on $[0,1]$ by

$$W^*(t) = B_\ell(t) + (t_\ell - t) \cdot m \cdot W_{\ell-1}^* + (t - t_{\ell-1}) \cdot m \cdot W_\ell^*$$

for $t \in [t_{\ell-1}, t_\ell]$. We introduce a new time discretization by $s_0 = 0$ and

$$s_\ell = \frac{1}{m} \cdot \sum_{j=1}^{\ell} \gamma_j^2$$

for $\ell = 1, \dots, m$. The corresponding piecewise linear time transformation τ on $[0,1]$ is given by

$$\tau(s) = t_{\ell-1} + (s - s_{\ell-1})/\gamma_\ell^2$$

for $s \in [s_{\ell-1}, s_\ell]$. We obtain a Brownian motion $W^{(\gamma)}$ on $[0,1]$ by

$$W^{(\gamma)}(s) = \sum_{j=1}^{\ell-1} \gamma_j \cdot (W^*(\tau(s_j)) - W^*(\tau(s_{j-1}))) + \gamma_\ell \cdot (W^*(\tau(s)) - W^*(\tau(s_{\ell-1})))$$

for $s \in [s_{\ell-1}, s_\ell]$. The Brownian bridges of $W^{(\gamma)}$ on the subintervals $[s_{\ell-1}, s_\ell]$ coincide with the weighted Brownian bridges of W on the subintervals $[t_{\ell-1}, t_\ell]$, up to the time transformation τ, i.e.,

$$B^{(\gamma)}(t) = W^{(\gamma)}(\tau^{-1}(t)) - (t_\ell - t) \cdot m \cdot W^{(\gamma)}(s_{\ell-1}) - (t - t_{\ell-1}) \cdot m \cdot W^{(\gamma)}(s_\ell) \tag{15}$$

for $t \in [t_{\ell-1}, t_\ell]$. Moreover, $W^{(\gamma)}$ is independent of $(Y_1, \dots, Y_m)$, and we have

$$W^{(\gamma)}(s_\ell) = m^{-1/2} \cdot \sum_{j=1}^{\ell} \gamma_j \cdot Z_j$$

Let $N, n \in \mathbb{N}$ and $q \geq 1$. We use

$$\widetilde{W}^{(\gamma,q)}_{m,n}(s_\ell) = m^{-1/2} \cdot \sum_{j=1}^{\ell} \gamma_j \cdot \widetilde{Z}^{(q)}_{j,n}$$

with

$$\widetilde{Z}^{(q)}_{j,n} = T^{(q)}_n(Z_j)$$

to obtain a quantization of $W^{(\gamma)}$ at the discrete points s_ℓ, and we define a quantization $\widetilde{B}^{(\gamma,q)}_{m,n,N}$ of $B^{(\gamma)}$ by

$$\widetilde{B}^{(\gamma,q)}_{m,n,N}(t) = S_N(W^{(\gamma)})(\tau^{-1}(t)) - (t_\ell - t) \cdot m \cdot \widetilde{W}^{(\gamma,q)}_{m,n}(s_{\ell-1}) - (t - t_{\ell-1}) \cdot m \cdot \widetilde{W}^{(\gamma,q)}_{m,n}(s_\ell) \tag{16}$$

for $t \in [t_{\ell-1}, t_\ell]$. Put

$$b_m(x) = \operatorname{sgn}(b(x)) \cdot \max(|b(x)|, m^{-1})$$

for $x \in \mathbb{R}$. Similar to the case of L_p-quantization we adjust the weights γ_ℓ in (16) to the local Hölder constant $|b(X(t))|$ by taking the random weights

$$\Gamma_\ell = \Gamma^{(q)}_{\ell,m,n} = b_m\big(\widetilde{X}^{(q)}_{m,n}(t_{\ell-1})\big) \cdot \left(\frac{1}{m} \sum_{j=0}^{m-1} b_m^2\big(\widetilde{X}^{(q)}_{m,n}(t_j)\big) \right)^{-1/2}.$$

We define a quantization $\widetilde{X}^{(q)}_{m,n,N}$ of X by

$$\widetilde{X}^{(q)}_{m,n,N}(t) = \widetilde{X}^{(q)}_{m,n}(t) + \left(\frac{1}{m} \sum_{j=0}^{m-1} b_m^2\big(\widetilde{X}^{(q)}_{m,n}(t_j)\big) \right)^{1/2} \cdot \widetilde{B}^{(\Gamma,q)}_{m,n,N}(t)$$

for $t \in [t_{\ell-1}, t_\ell]$. Since $\big|\operatorname{ran}\big(\widetilde{B}^{(\gamma,q)}_{m,n,N}\big)\big| \leq n^m \cdot N$ for every weight γ, we obtain

$$\big|\operatorname{ran}\big(\widetilde{X}^{(q)}_{m,n,N}\big)\big| \leq n^{2m} \cdot N. \tag{17}$$

Finally, we choose a sequence of integers $m_N \in \mathbb{N}$ such that

$$\lim_{N\to\infty} \frac{m_N \cdot \ln m_N}{\ln N} = 0 \qquad \text{and} \qquad \lim_{N\to\infty} \frac{(m_N / \ln m_N)^2}{\ln N} = \infty, \tag{18}$$

e.g., $m_N = (\ln N)^{2/3}$, and we consider the sequence of quantizations

$$\widetilde{X}^{(q)}_N = \widetilde{X}^{(q)}_{m_N, m_N, N}$$

of X.

The asymptotic behavior of the corresponding errors of order s for quantization in $C[0,1]$ depends on the SDE via the constant

$$C^{(\infty,s)} = \left(E\left(\int_0^1 |b(X(t))|^2 \, dt \right)^{s/2} \right)^{1/s} \tag{19}$$

and on the N-quantizers S_N for a Brownian motion on $[0,1]$ via

$$\kappa^{(\infty,s)} = \limsup_{N\to\infty} \left((\ln N)^{1/2} \cdot e^{(s)}(W, S_N(W), C) \right).$$

Theorem 2. *Let $q \geq \min\{r \in 2\mathbb{N} : r \geq s\}$. Then*

$$\limsup_{N\to\infty} \left((\ln N)^{-1} \cdot \ln |\operatorname{ran}(\widetilde{X}_N^{(q)})| \right) \leq 1$$

and

$$\limsup_{N\to\infty} \left((\ln N)^{1/2} \cdot e^{(s)}(X, \widetilde{X}_N^{(q)}, C) \right) \leq \kappa^{(\infty,s)} \cdot C^{(\infty,s)}.$$

2.4. **Discussion.** Quantization of Gaussian processes is intensively studied in the literature, see, e.g., [19] for a survey of results and applications in computational finance. In particular, for the Brownian motion W and for the Brownian bridge B on $[0,1]$ the following is known. Let $\mathfrak{B}_p = L_p$ for $p < \infty$ and $\mathfrak{B}_p = C$ for $p = \infty$. For every $1 \leq p \leq \infty$ there exists a constant $\kappa_*^{(p)}$ such that the N-th quantization errors of every order $1 \leq s < \infty$ satisfy

$$e_N^{(s)}(W, \mathfrak{B}_p) \approx e_N^{(s)}(B, \mathfrak{B}_p) \approx \kappa_*^{(p)} \cdot (\ln N)^{-1/2}.$$

See [1, 11] for the case $p = 2$ and [5] for the general case. Moreover, $\kappa_*^{(2)} = \sqrt{2}/\pi$, while only upper and lower bounds for $\kappa_*^{(p)}$ are known in the case $p \neq 2$.

Due to [2, 3] the N-th quantization errors of order s for W and for the solution X of the SDE (1) are related by

$$e_N^{(s)}(X, \mathfrak{B}_p) \approx C^{(p,s)} \cdot e_N^{(s)}(W, \mathfrak{B}_p),$$

where in particular $C^{(p.p)} = C^{(p)}$ is given by (12) for $p < \infty$ and $C^{(\infty,s)}$ is given by (19). Therefore

$$e_N^{(s)}(X, \mathfrak{B}_p) \approx \kappa_*^{(p)} \cdot C^{(p,s)} \cdot (\ln N)^{-1/2}.$$

This means the following for the conclusions from Theorem 1, where we consider the case $s = p < \infty$. If the sequence of quantizations $S_k(B)$ of B is strongly asymptotically optimal, i.e., if $e^{(p)}(B, S_k(B), L_p) \approx e_k^{(p)}(B, L_p)$, then the sequence of quantizations $\widetilde{X}_N^{(p,q)}$ of X is strongly asymptotically optimal, too, i.e.,

$$e^{(p)}(X, \widetilde{X}_N^{(p,q)}, L_p) \approx e_N^{(p)}(X, L_p).$$

Analogously, weak asymptotic optimality for quantization of the Brownian bridge, i.e., $\kappa^{(p)} < \infty$, leads to weak asymptotic optimality for quantization of X with

$$\limsup_{N\to\infty} \frac{e^{(p)}(X, \widetilde{X}_N^{(p,q)}, L_p)}{e_N^{(p)}(X, L_p)} \le \frac{\kappa^{(p)}}{\kappa_*^{(p)}}.$$

We stress that the deviation $\kappa^{(p)}/\kappa_*^{(p)}$ from strong asymptotic optimality does not depend on the underlying SDE. Likewise we obtain strong asymptotic optimality or weak asymptotic optimality with a deviation $\kappa^{(\infty,s)}/\kappa_*^{(\infty,s)}$ in the case $p = \infty$ and $1 \le s < \infty$ according to Theorem 2.

For constructive approaches to quantization of stochastic processes we again refer to [19] as a survey. Most of the results deal with Gaussian processes and $p = 2$, too, and in this case the constructions are based on the Karhunen-Loève expansion and on quantization of normal distributions. In order to achieve strong asymptotic optimality suitable quantizations of multi-dimensional centered normal distributions with covariance matrices of diagonal form are used for quantization of blocks of coefficients of the Karhunen-Loève expansion, and this approach involves large scale numerical optimization. Alternatively, constants $\kappa^{(p)}/\kappa_*^{(p)}$ close to one can already be achieved by product quantizers, which merely rely on quantizations of the one-dimensional standard normal distribution.

Specifically, for the Brownian bridge B the eigenfunctions and eigenvalues in the Karhunen-Loève expansion are given by $e_k(t) = \sqrt{2}\sin(k\pi t)$ and $\lambda_k = (k\pi)^{-2}$, and we refer to [14, 19] for constructions of quantizers as well as for numerical optimization approaches.

Constructive quantization of diffusion processes was initiated in [12], where suitable quantizers for the driving Brownian motion are used as a building block, which is similar to our approach. The approaches differ, however, with respect to the numerical treatment of the SDE. A key assumption in [12] is strict positivity of the diffusion coefficient, which permits to use the Lamperti transform. Along this way one has to solve N deterministic ODEs, in general, to get an N-point quantization of X, which is weakly asymptotically optimal. This work is extended to systems of of SDEs in [20], where rough path theory is used to establish convergence rates in p-variation and in the Hölder metric for ODE-based quantizations.

A different approach is developed in [13], where the mean regularity of stochastic processes is exploited. In particular, they obtain weakly asymptotically optimal quantizations for possibly degenerate systems of SDEs. The construction is based on the expansion of X in terms of the Haar basis, and on the availability of optimal quantizations of the corresponding coefficients.

Our approach relies on results and techniques for asymptotically optimal step-size control for SDEs, see [8, 9] and [15, 16]. The construction is easily implemented, it provides an explicit control on the asymptotic constants of the quantization error, and the overall computational cost is close to the size of the quantization. Our construction can be extended to systems of SDEs, and using sufficiently good quantizations of Lévy areas as a building block for the Milstein scheme we expect the same asymptotic results to hold. For the latter quantization problem we refer to [6], where a probabilistic method for quantization of implicitly given distributions is introduced.

2.5. **Example.** Our method is efficiently implemented in [21] for L_2-quantization of general scalar equations. The implementation is based on the sequence of product quantizers for the Brownian bridge that are provided in [18], together with their probability weights.

Here we consider a square-root diffusion

$$dX(t) = \alpha(\mu - X(t))\,dt + \sigma\sqrt{X(t)}\,dW(t),$$

although the smoothness assumptions imposed in our analysis are not met. The process X is used, e.g., to model the squared volatility in the Heston model. Specifically we take the parameters

$$\alpha = 1.0, \qquad \mu = 0.02, \qquad \sigma = 0.2, \qquad X(0) = 0.04,$$

which means that we are at the critical value for positivity. We construct both, the paths and the probability weights, of a quantization in L_2, which is of a small size for the purpose of illustration.

For the quantization of a marginal distribution we choose

$$q = 2, \qquad m = 3, \qquad n = 2,$$

and the 8 different paths of the coarse level quantization $\widetilde{X}^{(2)}_{3,2}$ are shown in Figure 1. In general, the probability weight of a coarse level path is the product of the corresponding weights of the quantizations $\widetilde{Y}^{(2)}_{\ell,n}$, see (4). For $n = 2$ the latter weights are $1/2$, so that the weight of each coarse level path is $1/8$ in our example. We add that the transition densities are known explicitly for the square-root diffusion, and quantizations thereof could be used instead of employing the Milstein scheme and $N(0,1)$-quantizations.

The local refinement of a coarse level path is controlled by its values $\widetilde{X}(t_\ell) = \widetilde{X}^{(2)}_{3,2}(t_\ell)$ at the grid points $t_0, \dots, t_{m-1}$ via the quantities $\widetilde{b}_\ell = \widetilde{b}^{(2)}_{\ell,m,n}$ and $\eta_\ell = \eta^{(2,2)}_{\ell,m,n}$ for $\ell = 0, \dots, m-1$. See Table 1 for the numerical values in our example. For the local refinement we choose

$$N = 16,$$

which determines the size $K_{\ell,N} = K^{(2,2)}_{\ell,m,n,N}$ of the quantizations of the Brownian bridges on the subinterval $[t_{\ell-1}, t_\ell]$ via (9). See Table 2 for the resulting values in our example.

The product quantizers used for the Brownian bridge are based on truncated Karhunen-Loéve expansions of the Brownian bridge and thus yield finite superpositions of

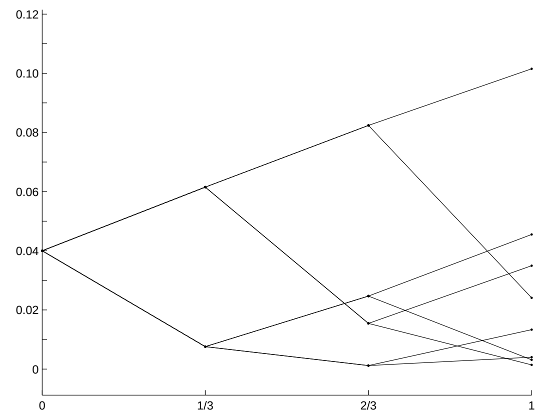

FIGURE 1. Coarse level quantization $\widetilde{X}^{(2)}_{3,2}$

$\widetilde{X}(1/3)$	$\widetilde{X}(2/3)$	$\widetilde{b}_0$	$\widetilde{b}_1$	$\widetilde{b}_2$	η_0	η_1	η_2
0.0615	0.0824	0.04	0.0496	0.0574	0.303	0.337	0.391
0.0615	0.0154	0.04	0.0496	0.0249	0.349	0.433	0.303
0.0076	0.0247	0.04	0.0174	0.0314	0.451	0.303	0.354
0.0076	0.0012	0.04	0.0174	0.0069	0.622	0.303	0.303

TABLE 1. Local regularity in the coarse level quantization

$\widetilde{X}(1/3)$	$\widetilde{X}(2/3)$	$K_{1,16}$	$K_{2,16}$	$K_{3,16}$
0.0615	0.0824	3	3	3
0.0615	0.0154	3	4	3
0.0076	0.0247	4	3	3
0.0076	0.0012	6	3	3

TABLE 2. Quantization size of the Brownian bridges

the functions $e_k(t) = \sqrt{2}\sin(k\pi t)$ on each subinterval. Actually, in the example only the first few quantizers are used. Figure 2 shows the resulting paths of the overall quantization $\widetilde{X}^{(2)}_{3,2,16}$ corresponding to the four cases from Table 2. The respective numbers of paths are given by 54, 72, 72, and 108, leading to a total of 306 trajectories. For the quantization size $K_{\ell,16} = 3, 4$ only multiples of e_1 are used, while e_1 and e_2 are employed in the case $K_{1,16} = 6$.

In general, a collection of fine level paths corresponds to every coarse level path, and the coarse level probability weight is distributed to the fine level paths proportional to

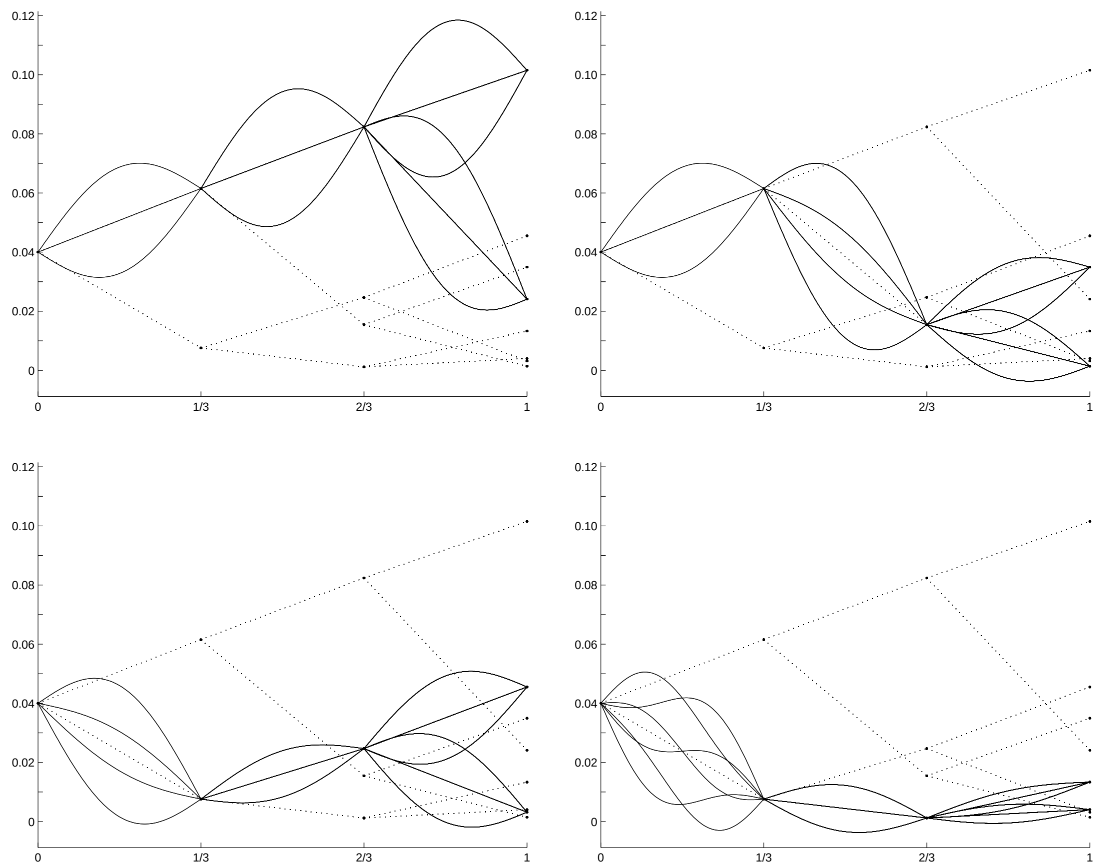

FIGURE 2. Overall quantization $\widetilde{X}^{(2)}_{3,2,16}$

the product of the probability weights of the quantizations of the Brownian bridges. In our example, there are 14 different weights ranging from the minimum value 0.001233 for 32 paths to the maximum value 0.012125 for 2 paths, and the corresponding probability weights for the Brownian bridge quantizations on the successive subintervals are given by $0.135, 0.27, 0.27$ and $0.459, 0.459, 0.459$, respectively. The paths with minimal and maximal weight are shown in Figure 3.

3. PROOFS

3.1. **Quantization of a standard normal distribution.** In dimension one a general construction of asymptotically optimal quantizers is known, which is presented here in the particular case $Z \sim \mathrm{N}(0,1)$. Let $z_{n,i}$ denote the $(2i-1)/(2n)$-quantile of P_Z, and put

$$v_{n,i} = (q+1)^{1/2} \cdot z_{n,i}$$

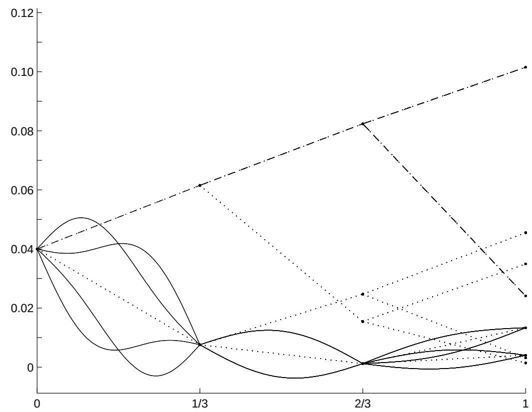

FIGURE 3. Minimum (solid) and maximum (dashed) weight paths

for $i = 1, \dots, n$. Furthermore, let

$$u_{n,i} = 1/2 \cdot (v_{n,i} + v_{n,i+1})$$

for $i = 1, \dots, n-1$, and put $u_{n,0} = -\infty$ and $u_{n,n} = \infty$. The sequence of n-quantizers

$$T_n^{(q)} = \sum_{i=1}^n v_{n,i} \cdot 1_{]u_{n,i-1}, u_{n,i}]}$$

then satisfies

$$e^{(q)}(Z, T_n^{(q)}(Z), \mathbb{R}) \approx e_n^{(q)}(Z) \approx c_1 \cdot n^{-1}$$

with a constant c_1 that only depends on q, see [7, Sec. 7.3]. Clearly (A3) is valid for this sequence of quantizers, and (A1) holds by symmetry of $T_n^{(q)}$.

We present a proof of property (A2). Let φ denote the density function of $\mathrm{N}(0,1)$, and let $n \in \mathbb{N}$ with $n \geq 2$. Then

$$(z_{n,n} - z_{n,n-1}) \cdot \varphi(z_{n,n}) \leq \int_{z_{n,n-1}}^{z_{n,n}} \varphi(x)\, dx = \frac{1}{n} = 2 \cdot \int_{z_{n,n}}^{\infty} \varphi(x)\, dx \leq \frac{2}{z_{n,n}} \cdot \varphi(z_{n,n}).$$

Hence

$$\max_{i=2,\dots,n} (v_{n,i} - v_{n,i-1}) \leq (q+1)^{1/2} \cdot (z_{n,n} - z_{n,n-1}) \leq c \tag{20}$$

and

$$v_{n,n} = (q+1)^{1/2} \cdot z_{n,n} \leq c \cdot (\ln n)^{1/2}. \tag{21}$$

Let $r \geq 1$ and $Z \sim \mathrm{N}(0,1)$. By definition of the quantizers $T_n^{(q)}$ we have

$$E|T_n^{(q)}(Z)|^r = \sum_{i=1}^n |v_{n,i}|^r \cdot \int_{u_{n,i-1}}^{u_{n,i}} \varphi(x)\, dx.$$

Assume that $i \in \{2, \dots, n-1\}$. Then

$$|v_{n,i}|^r \le c \cdot (|u_{n,i-1}|^r + |v_{n,i} - v_{n,i-1}|^r) \le c \cdot (|u_{n,i-1}|^r + 1)$$

due to (20), and consequently,

$$|v_{n,i}|^r \cdot \int_{u_{n,i-1}}^{u_{n,i}} \varphi(x)\,dx \le c \cdot \int_{u_{n,i-1}}^{u_{n,i}} (|x|^r + 1) \cdot \varphi(x)\,dx.$$

Next, let $i \in \{1, n\}$. Use (21) and observe $v_{n,n-1} > z_{n,n-1}$ to obtain

$$|v_{n,i}|^r \cdot \int_{u_{n,i-1}}^{u_{n,i}} \varphi(x)\,dx \le |v_{n,n}|^r \cdot \int_{v_{n,n-1}}^{\infty} \varphi(x)\,dx \le c \cdot (\ln n)^{r/2} \cdot n^{-1} \le c.$$

Hence

$$E|T_n^{(q)}(Y)|^r \le c \cdot \left(1 + \int_{-\infty}^{\infty} (|x|^r + 1) \cdot \varphi(x)\,dx\right) \le c,$$

which completes the proof of property (A2).

3.2. **Properties of the Milstein scheme.** Let $\widehat{X}_m = (\widehat{X}_m(t))_{t\in[0,1]}$ denote the time-continuous Milstein scheme that is based on the discretization (2) with step-size $1/m$, i.e., $\widehat{X}_m(t_0) = x_0$ and

$$\begin{aligned}\widehat{X}_m(t) = \widehat{X}_m(t_{\ell-1}) &+ a(\widehat{X}_m(t_{\ell-1})) \cdot (t - t_{\ell-1}) + b(\widehat{X}_m(t_{\ell-1})) \cdot (W(t) - W(t_{\ell-1})) \\ &+ 1/2 \cdot (b \cdot b')(\widehat{X}_m(t_{\ell-1})) \cdot \big((W(t) - W(t_{\ell-1}))^2 - (t - t_{\ell-1})\big)\end{aligned}$$

for $t \in [t_{\ell-1}, t_\ell]$. The following facts are well known under stronger assumptions on the drift and diffusion coefficients than those formulated in Section 1. See, e.g., [10].

Lemma 1. *Let $r \ge 1$. Then, for every $m \in \mathbb{N}$,*

$$E\big(\sup_{t\in[0,1]} |\widehat{X}_m(t)|^r\big) \le c$$

as well as

$$E\big(\sup_{t\in[0,1]} |X(t) - \widehat{X}_m(t)|^r\big) \le c \cdot m^{-r}.$$

Proof. Put

$$A = \sum_{\ell=1}^{m} a(\widehat{X}_m(t_{\ell-1})) \cdot 1_{]t_{\ell-1}, t_\ell]}$$

and

$$B = \sum_{\ell=1}^{m} \big(b(\widehat{X}_m(t_{\ell-1})) + (b \cdot b')(\widehat{X}_m(t_{\ell-1})) \cdot (W - W(t_{\ell-1}))\big) \cdot 1_{]t_{\ell-1}, t_\ell]}.$$

Then

$$\widehat{X}_m(t) = \int_0^t A(s)\,ds + \int_0^t B(s)\,dW(s).$$

We define

$$\widehat{X}^*_m(t) = \sup_{s\in[0,t]} |\widehat{X}_m(s)|.$$

Suppose that $t \in [t_{\ell-1}, t_\ell]$. Then, due to the properties of a and b,

$$E|A(t)|^r \le c \cdot E(1 + |\widehat{X}_m(t_{\ell-1})|)^r \le c \cdot \Big(1 + E|\widehat{X}^*_m(t)|^r\Big)$$

and

$$\begin{aligned} E|B(t)|^r &\le c \cdot E\big((1 + |\widehat{X}_m(t_{\ell-1})|) \cdot (1 + |W(t) - W(t_{\ell-1})|)\big)^r \\ &= c \cdot E(1 + |\widehat{X}_m(t_{\ell-1})|)^r \cdot E(1 + |W(t) - W(t_{\ell-1})|)^r \\ &\le c \cdot \Big(1 + E|\widehat{X}^*_m(t)|^r\Big). \end{aligned}$$

In particular,

$$E|\widehat{X}_m(t_\ell)|^r \le c \cdot \Big(1 + E|\widehat{X}_m(t_{\ell-1})|^r\Big),$$

which implies $\max_{\ell=1,\dots,m} E|\widehat{X}_m(t_\ell)|^r < \infty$, and hereby

$$\sup_{t\in[0,1]} (E|A(s)|^r + E|B(s)|^r) < \infty.$$

Moreover,

$$E|\widehat{X}^*_m(t)|^r \le c \cdot \int_0^t (E|A(s)|^r + E|B(s)|^r)\, ds \le c \cdot \left(1 + \int_0^t E|\widehat{X}^*_m(s)|^r\, ds\right).$$

Now, apply Gronwall's inequality to obtain the first statement in the lemma.

For the proof of the second statement we define

$$C = \sum_{\ell=1}^m (a' \cdot b)(\widehat{X}_m(t_{\ell-1})) \cdot (W - W(t_{\ell-1})) \cdot 1_{]t_{\ell-1},t_\ell]},$$

and we put

$$\Delta_1 = a \circ X - A - C, \qquad \Delta_2 = b \circ X - B$$

as well as

$$\Delta = X - \widehat{X}_m.$$

Then

$$\Delta(t) = \int_0^t C(s)\, ds + \int_0^t \Delta_1(s)\, ds + \int_0^t \Delta_2(s)\, dW(s).$$

Suppose that $t \in]t_{\ell-1}, t_\ell]$. By the properties of a and b we obtain

$$\begin{aligned} &|\widehat{X}_m(t) - \widehat{X}_m(t_{\ell-1})| \\ &\quad\le c \cdot (1 + |\widehat{X}_m(t_{\ell-1})|) \cdot \big(m^{-1} + |W(t) - W(t_{\ell-1})| + |W(t) - W(t_{\ell-1})|^2\big) \\ &\quad\le c \cdot (1 + |\widehat{X}_m(t_{\ell-1})|) \cdot (m^{-1/2} + m^{1/2} \cdot |W(t) - W(t_{\ell-1})|^2) \end{aligned}$$

and

$$|\widehat{X}_m(t) - \widehat{X}_m(t_{\ell-1}) - b(\widehat{X}_m(t_{\ell-1})) \cdot (W(t) - W(t_{\ell-1}))| \\ \leq c \cdot (1 + |\widehat{X}_m(t_{\ell-1})|) \cdot (m^{-1} + |W(t) - W(t_{\ell-1})|^2).$$

Hence

$$\begin{aligned} |\Delta_2(t)| &\leq |b(X(t)) - b(\widehat{X}_m(t))| \\ &\quad + |b(\widehat{X}_m(t)) - b(\widehat{X}_m(t_{\ell-1})) - b'(\widehat{X}_m(t_{\ell-1})) \cdot (\widehat{X}_m(t) - \widehat{X}_m(t_{\ell-1}))| \\ &\quad + \left|b'(\widehat{X}_m(t_{\ell-1})) \cdot \big(\widehat{X}_m(t) - \widehat{X}_m(t_{\ell-1}) - b(\widehat{X}_m(t_{\ell-1})) \cdot (W(t) - W(t_{\ell-1}))\big)\right| \\ &\leq c \cdot \Big(|\Delta(t)| + (1 + |\widehat{X}_m(t_{\ell-1})|^2) \cdot (m^{-1} + m \cdot |W(t) - W(t_{\ell-1})|^4)\Big), \end{aligned}$$

and the same estimate holds for $|\Delta_1(t)|$, too.

Using the first statement in the lemma we conclude that

$$E|\Delta_1(t)|^r + E|\Delta_2(t)|^r \leq c \cdot \left(E|\Delta(t)|^r + m^{-r}\right),$$

and, consequently, we have

$$E\left(\sup_{s\in[0,t]}\left|\int_0^s \Delta_1(u)\,du + \int_0^s \Delta_2(u)\,dW(u)\right|^r\right) \leq c \cdot \left(\int_0^t E|\Delta(s)|^r\,ds + \cdot m^{-r}\right). \tag{22}$$

Next, put

$$V(t) = \int_0^t C(s)\,ds,$$

and note that $(V(t))_{t\in[0,1]}$ is a continuous martingale. Hence

$$E\big(\sup_{t\in[0,1]} |V(t)|^r\big) \leq c \cdot E|V(t_m)|^r. \tag{23}$$

In order to estimate $E|V(t_m)|^r$ we assume $r \in 2\mathbb{N}$, and we put

$$U_\ell = \int_{t_\ell}^{t_{\ell+1}} (W(t) - W(t_\ell))\,dt.$$

Then

$$|V(t_{\ell+1})|^r = \sum_{j=0}^r \binom{r}{j} \cdot (V(t_\ell))^{r-j} \cdot \big((a' \cdot b)(\widehat{X}_m(t_\ell)\big)^j \cdot U_\ell^j.$$

Clearly

$$E(U_\ell^j) \leq c \cdot m^{-3j/2}$$

if j is even, and $E(U_\ell^j) = 0$ otherwise. Hence, by the independence of $(V(t_\ell), \widehat{X}_m(t_\ell))$ and U_ℓ,

$$E\big((V(t_\ell))^{r-1} \cdot (a' \cdot b)(\widehat{X}_m(t_\ell)) \cdot U_\ell\big) = 0,$$

and

$$\left|E\big((V(t_\ell))^{r-j}\cdot\big((a'\cdot b)(\widehat{X}_m(t_\ell)\big)^j\cdot U_\ell^j\big)\right|\le c\cdot E\big(|V(t_\ell)|^{r-j}\cdot(1+|\widehat{X}_m(t_\ell)|)^j\big)\cdot m^{-(j+1)}$$

for $j\in\{2,\ldots,r\}$. Using the first statement in the lemma, we thus obtain

$$\begin{aligned}E|V(t_{\ell+1})|^r &\le E|V(t_\ell)|^r + c\cdot m^{-1}\cdot E(|V(t_\ell)|+m^{-1}\cdot(1+|\widehat{X}_m(t_\ell)|))^r\\ &\le E|V(t_\ell)|^r\cdot(1+c\cdot m^{-1})+c\cdot m^{-(r+1)},\end{aligned}$$

so that

$$E|V(t_m)|^r\le c\cdot m^{-r} \tag{24}$$

follows from Gronwall's inequality.

Now, combine (23) with (24), and observe (22) to derive

$$E\big(\sup_{s\in[0,t]}|\Delta(t)|^r\big)\le c\cdot\left(m^{-r}+\int_0^t E\big(\sup_{u\in[0,s]}|\Delta(u)|^r\big)\,ds\right),$$

which yields the second statement in the lemma due to Gronwall's inequality. □

3.3. **Proof of Proposition 1.** First, we provide a moment bound for the quantization $(\widetilde{X}_{m,n}^{(q)}(t_1),\ldots,\widetilde{X}_{m,n}^{(q)}(t_m))$, see (5).

Lemma 2. *Let $q,r\ge 1$. Then, for all $m,n\in\mathbb{N}$,*

$$E\big(\max_{\ell=0,\ldots,m}|\widetilde{X}_{m,n}^{(q)}(t_\ell)|^r\big)\le c.$$

Proof. In the following we write $\widetilde{X}_\ell$ and $\widetilde{Y}_\ell$ for $\widetilde{X}_{m,n}^{(q)}(t_\ell)$ and $\widetilde{Y}_{\ell,n}^{(q)}$, respectively. Moreover, we put

$$\widetilde{\alpha}_\ell=a(\widetilde{X}_\ell),\qquad \widetilde{\beta}_\ell=b(\widetilde{X}_\ell),\qquad \widetilde{\gamma}_\ell=1/2\cdot(b\cdot b')(\widetilde{X}_\ell),$$

as well as

$$\widetilde{\vartheta}_\ell=m^{-1}\cdot\big(\widetilde{\alpha}_\ell+\widetilde{\gamma}_\ell\cdot(E(\widetilde{Y}_{\ell+1}^2)-1)\big)$$

and

$$\widetilde{\xi}_\ell=\widetilde{\beta}_\ell\cdot m^{-1/2}\cdot\widetilde{Y}_{\ell+1}+m^{-1}\cdot\widetilde{\gamma}_\ell\cdot(\widetilde{Y}_{\ell+1}^2-E(\widetilde{Y}_{\ell+1}^2)).$$

Then

$$\widetilde{X}_\ell=\widetilde{U}_\ell+\widetilde{V}_\ell,$$

where

$$\widetilde{U}_\ell=x_0+\sum_{j=0}^{\ell-1}\widetilde{\vartheta}_j,\qquad \widetilde{V}_\ell=\sum_{j=0}^{\ell-1}\widetilde{\xi}_j.$$

Put

$$\widetilde{U}_\ell^*=\max_{j=0,\ldots,\ell}|U_\ell|,\qquad \widetilde{V}_\ell^*=\max_{j=0,\ldots,\ell}|V_\ell|,$$

and use the properties of a and b as well as (A2) to derive

$$|\widetilde{\vartheta}_\ell| \leq c \cdot m^{-1} \cdot (1 + |\widetilde{X}_\ell|) \cdot (E(\widetilde{Y}^2_{\ell+1}) + 1) \leq c \cdot m^{-1} \cdot (1 + |\widetilde{U}_\ell| + \widetilde{V}^*_\ell).$$

Hence

$$|\widetilde{U}_{\ell+1}| \leq |\widetilde{U}_\ell| \cdot (1 + c \cdot m^{-1}) + c \cdot m^{-1} \cdot (1 + \widetilde{V}^*_\ell),$$

and

$$\widetilde{U}^*_\ell \leq c \cdot (1 + \widetilde{V}^*_\ell) \tag{25}$$

follows from Gronwall's inequality.

Let $\mathfrak{F}_\ell$ denote the σ-algebra that is generated by $\{W(s) : s \leq t_\ell\}$. Use (A1) to conclude that $(\widetilde{V}_\ell)_{\ell=0,\dots,m}$ is a martingale w.r.t. $(\mathfrak{F}_\ell)_{\ell=0,\dots,m}$, which implies

$$E(\widetilde{V}^*_\ell)^r \leq c \cdot E|\widetilde{V}_\ell|^r. \tag{26}$$

In view of (25) it therefore remains to show that

$$E|\widetilde{V}_m|^r \leq c. \tag{27}$$

Assume $r \in 2\mathbb{N}$ without loss of generality. Then

$$|\widetilde{V}_{\ell+1}|^r = \sum_{j=0}^{r} \binom{r}{j} \cdot \widetilde{V}^{r-j}_\ell \cdot \widetilde{\xi}^j_\ell.$$

Clearly,

$$E(\widetilde{V}^{r-1}_\ell \cdot \widetilde{\xi}_\ell) = 0.$$

Next, let $j \in \{2, \dots, r\}$. Then

$$\begin{aligned} |\widetilde{\xi}_\ell|^j &\leq c \cdot \big(m^{-1/2} \cdot (1 + |\widetilde{X}_\ell|) \cdot (1 + E|\widetilde{Y}_{\ell+1}|^2 + |\widetilde{Y}_{\ell+1}|^2)\big)^j \\ &\leq c \cdot m^{-1} \cdot (1 + |\widetilde{X}_\ell|)^j \cdot (1 + E|\widetilde{Y}_{\ell+1}|^{2j} + |\widetilde{Y}_{\ell+1}|^{2j}), \end{aligned}$$

and consequently, by (A2) and the independence of $(\widetilde{V}_\ell, \widetilde{X}_\ell)$ and $\widetilde{Y}_{\ell+1}$,

$$\big|E(\widetilde{V}^{r-j}_\ell \cdot \widetilde{\xi}^j_\ell)\big| \leq c \cdot m^{-1} \cdot E\big(|\widetilde{V}_\ell|^{r-j} \cdot (1 + |\widetilde{X}_\ell|)^j\big).$$

Use (25) and (26) to conclude that

$$\begin{aligned} E|\widetilde{V}_{\ell+1}|^r &\leq E|\widetilde{V}_\ell|^r + c \cdot m^{-1} \cdot E(|\widetilde{V}_\ell| + 1 + |\widetilde{X}_\ell|)^r \\ &\leq E|\widetilde{V}_\ell|^r + c \cdot m^{-1} \cdot (E|\widetilde{V}_\ell|^r + 1 + E(\widetilde{V}^*_\ell)^r) \\ &\leq E|\widetilde{V}_\ell|^r \cdot (1 + c \cdot m^{-1}) + c \cdot m^{-1}, \end{aligned}$$

and apply Gronwall's inequality to obtain (27). □

Next, we compare $\widetilde{X}^{(q)}_{m,n}(t_\ell)$ with the Milstein approximation $\widehat{X}_m(t_\ell)$.

Lemma 3. *Assume that $q \in 2\mathbb{N}$. Then, for all $m, n \in \mathbb{N}$,*

$$E\big(\max_{\ell=1,\dots,m} |\widehat{X}_m(t_\ell) - \widetilde{X}^{(q)}_{m,n}(t_\ell)|^q\big) \le c \cdot (m^{-1} + n^{-1})^q.$$

Proof. Recall the definition of $\widetilde{X}_\ell$, $\widetilde{Y}_\ell$, $\widetilde{\alpha}_\ell$, $\widetilde{\beta}_\ell$, and $\widetilde{\gamma}_\ell$ from the proof of Lemma 2. Similarly, we write $\widehat{X}_\ell$ for $\widehat{X}_m(t_\ell)$, and we put

$$\widehat{\alpha}_\ell = a(\widehat{X}_\ell), \qquad \widehat{\beta}_\ell = b(\widehat{X}_\ell), \qquad \widehat{\gamma}_\ell = 1/2 \cdot (b \cdot b')(\widehat{X}_\ell),$$

as well as

$$\vartheta_\ell = (\widehat{\alpha}_\ell - \widetilde{\alpha}_\ell) \cdot m^{-1} + \widetilde{\gamma}_\ell \cdot (1 - E(\widetilde{Y}^2_{\ell+1})) \cdot m^{-1}$$

and

$$\xi_\ell = (\widehat{\beta}_\ell \cdot Y_{\ell+1} - \widetilde{\beta}_\ell \cdot \widetilde{Y}_{\ell+1}) \cdot m^{-1/2} + (\widehat{\gamma}_\ell \cdot (Y^2_{\ell+1} - 1) - \widetilde{\gamma}_\ell \cdot (\widetilde{Y}^2_{\ell+1} - E(\widetilde{Y}^2_{\ell+1}))) \cdot m^{-1}.$$

Then

$$\widehat{X}_\ell - \widetilde{X}_\ell = U_\ell + V_\ell,$$

where

$$U_\ell = \sum_{j=0}^{\ell-1} \vartheta_j, \qquad V_\ell = \sum_{j=0}^{\ell-1} \xi_j.$$

Put

$$U^*_\ell = \max_{j=1,\dots,\ell} |U_\ell|, \qquad V^*_\ell = \max_{j=1,\dots,\ell} |V_\ell|, \qquad \widehat{X}^*_\ell = \max_{j=0,\dots,\ell} |\widehat{X}_\ell|, \qquad \widetilde{X}^*_\ell = \max_{j=0,\dots,\ell} |\widetilde{X}_\ell|.$$

Due to (A2) and (A3) we have

$$|1 - E(\widetilde{Y}^2_{\ell+1})| = |E(Y^2_{\ell+1} - \widetilde{Y}^2_{\ell+1})| \le c \cdot n^{-1}. \tag{28}$$

Hence

$$|\vartheta_\ell| \le c \cdot m^{-1} \cdot \Big(|\widehat{X}_\ell - \widetilde{X}_\ell| + (1 + |\widetilde{X}_\ell|) \cdot n^{-1}\Big),$$

which yields

$$|U_{\ell+1}| \le |U_\ell| \cdot (1 + c \cdot m^{-1}) + c \cdot m^{-1} \cdot \big(V^*_\ell + (1 + \widetilde{X}^*_\ell) \cdot n^{-1}\big),$$

and therefore

$$U^*_\ell \le c \cdot \big(V^*_\ell + (1 + \widetilde{X}^*_\ell) \cdot n^{-1}\big) \tag{29}$$

follows from Lemma 2 and Gronwall's inequality.

We derive

$$E(V^*_\ell)^q \le c \cdot E|V_\ell|^q \tag{30}$$

by the same argument that leads to (26), and in view of (29) and Lemma 2 it therefore remains to show that

$$E|V_m|^q \le c \cdot (m^{-1} + n^{-1})^q. \tag{31}$$

Clearly, $E(V_\ell^{r-1} \cdot \xi_\ell) = 0$, which yields

$$E|V_{\ell+1}|^q = E|V_\ell|^q + \sum_{j=2}^{q} \binom{q}{j} \cdot E\big(V_\ell^{q-j} \cdot \xi_\ell^j\big).$$

Due to the properties of a and b we have

$$\begin{aligned}|\widehat{\beta}_\ell \cdot Y_{\ell+1} - \widetilde{\beta}_\ell \cdot \widetilde{Y}_{\ell+1}| &\leq |\widehat{\beta}_\ell - \widetilde{\beta}_\ell| \cdot |Y_{\ell+1}| + |\widetilde{\beta}_\ell| \cdot |Y_{\ell+1} - \widetilde{Y}_{\ell+1}| \\ &\leq c \cdot \Big(|\widehat{X}_\ell - \widetilde{X}_\ell| \cdot |Y_{\ell+1}| + |\widetilde{\beta}_\ell| \cdot |Y_{\ell+1} - \widetilde{Y}_{\ell+1}|\Big)\end{aligned}$$

and, observing (28),

$$\begin{aligned}&|\widehat{\gamma}_\ell \cdot (Y_{\ell+1}^2 - 1) - \widetilde{\gamma}_\ell \cdot (\widetilde{Y}_{\ell+1}^2 - E(\widetilde{Y}_{\ell+1}^2))| \\ &\quad\leq |\widehat{\gamma}_\ell - \widetilde{\gamma}_\ell| \cdot |Y_{\ell+1}^2 - 1| + |\widetilde{\gamma}_\ell| \cdot |Y_{\ell+1}^2 - \widetilde{Y}_{\ell+1}^2 - (1 - E(\widetilde{Y}_{\ell+1}^2))| \\ &\quad\leq c \cdot \Big(|\widehat{X}_\ell - \widetilde{X}_\ell| \cdot (1 + |\widetilde{\beta}_\ell|) \cdot |Y_{\ell+1}^2 - 1| + |\widetilde{\beta}_\ell| \cdot (|Y_{\ell+1}^2 - \widetilde{Y}_{\ell+1}^2| + n^{-1})\Big).\end{aligned}$$

Therefore

$$\begin{aligned}|\xi_\ell| \leq c \cdot \Big(&|\widehat{X}_\ell - \widetilde{X}_\ell| \cdot (1 + Y_{\ell+1}^2) \cdot m^{-1/2} + |\widetilde{\beta}_\ell| \cdot |Y_{\ell+1} - \widetilde{Y}_{\ell+1}| \cdot m^{-1/2} \\ &+ |\widehat{X}_\ell - \widetilde{X}_\ell| \cdot (1 + |\widetilde{\beta}_\ell|) \cdot |Y_{\ell+1}^2 - 1| \cdot m^{-1} + |\widetilde{\beta}_\ell| \cdot (|Y_{\ell+1}^2 - \widetilde{Y}_{\ell+1}^2| + n^{-1}) \cdot m^{-1}\Big).\end{aligned}$$

Furthermore

$$\begin{aligned}E|Y_{\ell+1}^2 - \widetilde{Y}_{\ell+1}^2|^j &\leq E\big(|Y_{\ell+1} - \widetilde{Y}_{\ell+1}|^{j-1} \cdot (|Y_{\ell+1}| + |\widetilde{Y}_{\ell+1}|)^{j+1}\big) \\ &\leq \big(E|Y_{\ell+1} - \widetilde{Y}_{\ell+1}|^j\big)^{(j-1)/j} \cdot \big(E(|Y_{\ell+1}| + |\widetilde{Y}_{\ell+1}|)^{j(j+1)}\big)^{1/j} \\ &\leq c \cdot n^{-(j-1)}\end{aligned}$$

follows from (A3), if $j \in \{2, \ldots, q\}$.

Now, observe that $(\widehat{X}_\ell, \widetilde{X}_\ell, V_\ell)$ and $(Y_{\ell+1}, \widetilde{Y}_{\ell+1})$ are independent, and use (A2) and (A3) as well as

$$n^{-j} \cdot m^{-j/2} + n^{-(j-1)} \cdot m^{-j} \leq c \cdot m^{-1} \cdot (m^{-1} + n^{-1})^j$$

for $j \in \{2, \dots, q\}$ to derive

$$
\begin{aligned}
E|V_\ell^{q-j} \cdot \xi_\ell^j| &\le c \cdot \Big(E\big(|V_\ell|^{q-j} \cdot |\widehat{X}_\ell - \widetilde{X}_\ell|^j\big) \cdot E|1 + Y_{\ell+1}^2|^j \cdot m^{-j/2} \\
&\qquad + E(|V_\ell|^{q-j} \cdot |\widetilde{\beta}_\ell|^j) \cdot E|Y_{\ell+1} - \widetilde{Y}_{\ell+1}|^j \cdot m^{-j/2} \\
&\qquad + E\big(|V_\ell|^{q-j} \cdot |\widehat{X}_\ell - \widetilde{X}_\ell|^j \cdot (1 + |\widetilde{\beta}_\ell|^j)\big) \cdot E|Y_{\ell+1}^2 - 1|^j \cdot m^{-j} \\
&\qquad + E(|V_\ell|^{q-j} \cdot |\widetilde{\beta}_\ell|^j) \cdot (E|Y_{\ell+1}^2 - \widetilde{Y}_{\ell+1}^2|^j + n^{-j}) \cdot m^{-j} \Big) \\
&\le c \cdot \Big(E\big(|V_\ell|^{q-j} \cdot |\widehat{X}_\ell - \widetilde{X}_\ell|^j\big) \cdot m^{-j/2} + E(|V_\ell|^{q-j} \cdot |\widetilde{\beta}_\ell|^j) \cdot n^{-j} \cdot m^{-j/2} \\
&\qquad + E\big(|V_\ell|^{q-j} \cdot |\widehat{X}_\ell - \widetilde{X}_\ell|^j \cdot (1 + |\widetilde{\beta}_\ell|^j)\big) \cdot m^{-j} \\
&\qquad + E(|V_\ell|^{q-j} \cdot |\widetilde{\beta}_\ell|^j) \cdot n^{-(j-1)} \cdot m^{-j} \Big) \\
&\le c \cdot \Big(E\big(|V_\ell|^{q-j} \cdot |\widehat{X}_\ell - \widetilde{X}_\ell|^j\big) \cdot m^{-j/2} + E\big(|V_\ell|^{q-j} \cdot |\widehat{X}_\ell - \widetilde{X}_\ell|^j \cdot |\widetilde{\beta}_\ell|^j\big) \cdot m^{-j} \\
&\qquad + E(|V_\ell|^{q-j} \cdot |\widetilde{\beta}_\ell|^j) \cdot m^{-1} \cdot (m^{-1} + n^{-1})^j \Big).
\end{aligned}
$$

We obtain

$$
\begin{aligned}
&\sum_{j=2}^{q} \binom{q}{j} \cdot E(V_\ell^{q-j} \cdot \xi_\ell^j) \\
&\qquad \le c \cdot \Big(E(|V_\ell| + |\widehat{X}_\ell - \widetilde{X}_\ell|)^q \cdot m^{-1} + E(|V_\ell| + |\widehat{X}_\ell - \widetilde{X}_\ell| \cdot |\widetilde{\beta}_\ell|)^q \cdot m^{-2} \\
&\qquad\qquad + E(|V_\ell| + |\widetilde{\beta}_\ell| \cdot (n^{-1} + m^{-1}))^q \cdot m^{-1} \Big) \\
&\qquad \le c \cdot \Big(E|V_\ell|^q \cdot m^{-1} + E|\widehat{X}_\ell - \widetilde{X}_\ell|^q \cdot m^{-1} \\
&\qquad\qquad + \big(E|\widehat{X}_\ell - \widetilde{X}_\ell|^q\big)^{(q-1)/q} \cdot m^{-2} + (n^{-1} + m^{-1})^q \cdot m^{-1} \Big) \\
&\qquad \le c \cdot m^{-1} \cdot \Big(E|V_\ell|^q + E|\widehat{X}_\ell - \widetilde{X}_\ell|^q + (n^{-1} + m^{-1})^q \Big).
\end{aligned}
$$

using Lemmata 1 and 2.

Due to (29), (30), and Lemma 2, we have

$$E|\widehat{X}_\ell - \widetilde{X}_\ell|^q \le c \cdot E(V_\ell^* + \widetilde{X}_\ell^* \cdot n^{-1})^q \le c \cdot (E|V_\ell|^q + n^{-q}).$$

We conclude that

$$E|V_{\ell+1}|^q \le E|V_\ell|^q \cdot (1 + c \cdot m^{-1}) + c \cdot (n^{-1} + m^{-1})^q \cdot m^{-1},$$

such that Gronwall's inequality implies (31). □

Combine the error estimates from Lemma 1 and Lemma 3 to obtain Proposition 1.

3.4. **Proof of Theorem 1.** Recall that B_ℓ is the difference between the Brownian motion W and its linear interpolation at $t_{\ell-1}$ and t_ℓ, see (7). Thus $B_1, \ldots, B_m$ are independent Brownian bridges on the respective subintervals. For $m \in \mathbb{N}$ we consider an auxiliary approximation $\overline{X}_m$ to the solution X defined by

$$\overline{X}_m(t) = (t - t_{\ell-1}) \cdot m \cdot \widehat{X}_m(t_\ell) + (t_\ell - t) \cdot m \cdot \widehat{X}_m(t_{\ell-1}) + b(\widehat{X}_m(t_{\ell-1})) \cdot B_\ell(t)$$

for $t \in [t_{\ell-1}, t_\ell]$.

Lemma 4. *For every $r \geq 1$ we have*

$$\sup_{t\in[0,1]} E|X(t) - \overline{X}_m(t)|^r \leq c \cdot m^{-r}.$$

and

$$E\big(\sup_{t\in[0,1]} |X(t) - \overline{X}_m(t)|^r\big) \leq c \cdot m^{-r} \cdot (\ln m)^r.$$

Proof. In view of Lemma 1 it suffices to establish Lemma 4 with X replaced by the time-continuous Milstein scheme $\widehat{X}_m$.

Put

$$R(t) = 1/2 \cdot \sum_{\ell=1}^{m} (b \cdot b')(\widehat{X}_m(t_{\ell-1})) \cdot ((W(t) - W(t_{\ell-1}))^2 - (t - t_{\ell-1})) \cdot 1_{]t_{\ell-1},t_\ell]}(t).$$

Fix $t \in]t_{\ell-1}, t_\ell]$. Then

$$\begin{aligned}\overline{X}_m(t) = {} & \widehat{X}_m(t_{\ell-1}) + a(\widehat{X}_m(t_{\ell-1})) \cdot (t - t_{\ell-1}) \\ & + b(\widehat{X}_m(t_{\ell-1})) \cdot (W(t) - W(t_{\ell-1})) + (t - t_{\ell-1}) \cdot m \cdot R(t_\ell),\end{aligned}$$

and therefore,

$$\widehat{X}_m(t) - \overline{X}_m(t) = R(t) - (t - t_{\ell-1}) \cdot m \cdot R(t_\ell).$$

Use Lemma 1 to obtain

$$E|R(t)|^r \leq c \cdot E(1 + |\widehat{X}_m(t_{\ell-1})|)^r \cdot E((W(t) - W(t_{\ell-1}))^2 + m^{-1})^r \leq c \cdot m^{-r},$$

which implies the first estimate.

Furthermore, we have

$$\sup_{t\in[0,1]} |R(t)| \leq c \cdot \big(1 + \sup_{t\in[0,1]} |\widehat{X}_m(t)|\big) \cdot \big(\max_{\ell=1,\ldots,m} \sup_{t\in[t_{\ell-1},t_\ell]} |W(t) - W(t_{\ell-1})|^2 + m^{-1}\big)$$

and the well known estimate

$$E\big(\sup_{t\in[0,1]} |W(t) - W(t_{\ell-1})|^{2r}\big) \leq c \cdot (\ln m/m)^r. \tag{32}$$

Hence, by Lemma 1

$$E\big(\sup_{t\in[0,1]} |R(t)|^r\big) \leq c \cdot (\ln m/m)^r,$$

which yields the second estimate. □

Recall that $S_k(B)$ is a k-quantization of a Brownian bridge B on $[0,1]$. In the sequel we consider fixed parameters $p, q \geq 1$ with $q \geq \min\{r \in 2\mathbb{N} : r \geq p\}$. Moreover, we put

$$\delta_k = e^{(p)}(B, S_k(B), L_p)$$

for $k \in \mathbb{N}$, and for $m, n, N \in \mathbb{N}$ we define

$$A_{m,n,N} = m^{-(p+2)/2} \cdot \sum_{\ell=1}^{m} \big| b(\widetilde{X}^{(q)}_{m,n}(t_{\ell-1})) \big|^p \cdot \delta^p_{K_{\ell,N}}$$

with random variables $K_{\ell,N}$ given by (9).

Lemma 5.

$$e^{(p)}(X, \widetilde{X}^{(p,q)}_{m,n,N}, L_p) \leq c \cdot (m^{-1} + n^{-1}) + (E A_{m,n,N})^{1/p}$$

Proof. In view of Lemma 4 it suffices to establish Lemma 5 with X replaced by the approximation $\overline{X}_m$. By definition,

$$\overline{X}_m - \widetilde{X}^{(p,q)}_{m,n,N} = U_1 + U_2,$$

where

$$\begin{aligned} U_1(t) = {} & (t - t_{\ell-1}) \cdot m \cdot \big(\widehat{X}_m(t_\ell) - \widetilde{X}^{(q)}_{m,n}(t_\ell)\big) \\ & + (t_\ell - t) \cdot m \cdot \big(\widehat{X}_m(t_{\ell-1}) - \widetilde{X}^{(q)}_{m,n}(t_{\ell-1})\big) \\ & + \big(b(\widehat{X}_m(t_{\ell-1})) - b(\widetilde{X}^{(q)}_{m,n}(t_{\ell-1}))\big) \cdot B_\ell(t), \end{aligned} \tag{33}$$

and

$$U_2(t) = b(\widetilde{X}^{(q)}_{m,n}(t_{\ell-1})) \cdot \big(B_\ell(t) - \widetilde{B}_{\ell,K_{\ell,N}}(t)\big)$$

for $t \in [t_{\ell-1}, t_\ell]$.

Observe that $(\widehat{X}_m(t_0), \widetilde{X}^{(q)}_{m,n}(t_0), \ldots, \widehat{X}_m(t_m), \widetilde{X}^{(q)}_{m,n}(t_m))$ and $(B_1, \ldots, B_m)$ are are independent, and use Proposition 1 to obtain

$$E\|U_1\|^p_{C[0,1]} \leq c \cdot E\big(\max_{\ell=1,\ldots,m} |\widehat{X}_m(t_\ell) - \widetilde{X}^{(q)}_{m,n}(t_\ell)|^p\big) \leq c \cdot (m^{-1} + n^{-1})^p. \tag{34}$$

For the analysis of the process U_2 we use the fact that

$$E\|B_\ell - \widetilde{B}_{\ell,k}\|^p_{L_p[t_{\ell-1},t_\ell]} = m^{-(p/2+1)} \cdot \delta^p_k \tag{35}$$

holds for every $k \in \mathbb{N}$, which is a straightforward consequence of the definition (8) of $\widetilde{B}_{\ell,k}$ and the scaling property of a Brownian bridge. Put

$$V = (\widetilde{X}^{(q)}_{m,n}(t_0), \ldots, \widetilde{X}^{(q)}_{m,n}(t_{m-1})), \tag{36}$$

and note that $k_{\ell,N}$ is measurable w.r.t. $\sigma(V)$. Observe that V and B_ℓ are independent, and use (35) to derive

$$E\big(\|B_\ell - \widetilde{B}_{\ell,K_{\ell,N}}\|^p_{L_p[t_{\ell-1},t_\ell]} \big| V\big) = m^{-(p/2+1)} \cdot \delta^p_{K_{\ell,N}}.$$

Hence

$$E\big(\|U_2\|^p_{L_p[0,1]}\big| V\big) = \sum_{\ell=1}^{m} |b(\widetilde{X}^{(q)}_{m,n}(t_{\ell-1}))|^p \cdot E\big(\|B_\ell - \widetilde{B}_{\ell,K_{\ell,N}}\|^p_{L_p[t_{\ell-1},t_\ell]}\big| V\big) = A_{m,n,N},$$

which completes the proof. □

Consider a sequence of integers $m_N \in \mathbb{N}$ that satisfies (11). We analyze the asymptotic behavior of the corresponding sequence of random variables $A_{m_N,m_N,N}$. Recall that $C^{(p)}$ is given by (12) and that $\kappa^{(p)} = \limsup_{k\to\infty}(\ln k)^{1/2} \cdot \delta_k$ by definition.

Lemma 6.

$$\limsup_{N\to\infty} \Big((\ln N)^{1/2} \cdot \big(EA_{m_N,m_N,N}\big)^{1/p}\Big) \le \kappa^{(p)} \cdot C^{(p)}.$$

Proof. Put

$$\kappa_N = \max_{\ell=1,\dots,m_N} (\ln k_{\ell,N})^{1/2} \cdot \delta_{k_{\ell,N}}$$

as well as

$$\widetilde{D}_{r,N} = \left(\frac{1}{m_N} \cdot \sum_{\ell=1}^{m_N} \big|b(\widetilde{X}^{(q)}_{m_N,m_N}(t_{\ell-1}))\big|^r\right)^{1/r}$$

for $r \ge 1$, and define $D_{r,N}$ as $\widetilde{D}_{r,N}$ with $\widetilde{X}^{(q)}_{m_N,m_N}(t_{\ell-1})$ replaced by $X(t_{\ell-1})$. Moreover, let

$$p^* = 2p/(p+2).$$

Assume $N \ge 2$. By definition of $K_{\ell,N}$ we have

$$\begin{aligned} &\big|b(\widetilde{X}^{(q)}_{m_N,m_N}(t_{\ell-1}))\big|^p \cdot (\ln K_{\ell,N})^{-p/2} \\ &\qquad\le m_N^{p/2} \cdot \big|b(\widetilde{X}^{(q)}_{m_N,m_N}(t_{\ell-1}))\big|^{p^*} \cdot \widetilde{D}^{p\cdot p^*/2}_{p^*,N} \cdot (\ln N)^{-p/2}, \end{aligned}$$

which implies

$$A_{m_N,m_N,N} \le \Big(\kappa_N \cdot \widetilde{D}_{p^*,N}\Big)^p \cdot (\ln N)^{-p/2}.$$

Note that $\ln(K_{\ell,N}) \ge (\ln N)/(m_N \cdot \ln m_N)$. Hence

$$\lim_{N\to\infty} \min_{\ell=1,\dots,m_N} K_{\ell,N} = \infty,$$

due to (11), and consequently, $\limsup_{N\to\infty} \kappa_N/\kappa^{(p)} \le 1$. Furthermore,

$$\lim_{N\to\infty} D_{p^*,N} = \|b(X)\|_{L_{p^*}([0,1])},$$

since the process $(b(X(t)))_{t\in[0,1]}$ has continuous trajectories. Thus

$$\limsup_{N\to\infty} E\left(\kappa_N \cdot D_{p^*,N}\right)^p \le E\left(\limsup_{N\to\infty} \left(\kappa_N \cdot D_{p^*,N}\right)^p\right) = \left(\kappa^{(p)} \cdot C^{(p)}\right)^p.$$

It remains to show that

$$\lim_{N\to\infty} \big|(E\big(\kappa_N \cdot \widetilde{D}_{p^*,N}\big)^p)^\alpha - (E\big(\kappa_N \cdot D_{p^*,N}\big)^p)^\alpha\big| = 0$$

for some $\alpha > 0$. Clearly, we may assume $\kappa^{(p)} < \infty$, which implies

$$\sup_{N\in\mathbb{N}} \kappa_N \leq c. \tag{37}$$

First, we consider the case $p \geq 2$. Then $1 \leq p^* < p$, and therefore

$$\begin{aligned} |\widetilde{D}_{p^*,N} - D_{p^*N}| &\leq \left(\frac{1}{m_N} \cdot \sum_{\ell=1}^{m_N} \left| b(\widetilde{X}^{(q)}_{m_N,m_N}(t_{\ell-1})) - b(X(t_{\ell-1})) \right|^{p^*} \right)^{1/p^*} \\ &\leq c \cdot \max_{\ell=1,\dots,m_N} \left| \widetilde{X}^{(q)}_{m_N,m_N}(t_{\ell-1}) - X(t_{\ell-1}) \right|. \end{aligned} \tag{38}$$

Observe (37) and use Proposition 1 to conclude that

$$\left| (E(\kappa_N \cdot \widetilde{D}_{p^*,N})^p)^{1/p} - (E(\kappa_N \cdot D_{p^*,N})^p)^{1/p} \right| \leq c \cdot \left(E|\widetilde{D}_{p^*,N} - D_{p^*,N}|^p \right)^{1/p} \leq c \cdot m_N^{-1}.$$

Next, we assume $1 \leq p < 2$. Then $p^* \leq 1$, and we have

$$\begin{aligned} |\widetilde{D}^{p^*}_{p^*,N} - D^{p^*}_{p^*,N}| &\leq \frac{1}{m_N} \cdot \sum_{\ell=1}^{m_N} \left| b(\widetilde{X}^{(q)}_{m_N,m_N}(t_{\ell-1})) - b(X(t_{\ell-1})) \right|^{p^*} \\ &\leq c \cdot \max_{\ell=1,\dots,m_N} \left| \widetilde{X}^{(q)}_{m_N,m_N}(t_{\ell-1}) - X(t_{\ell-1}) \right|^{p^*}. \end{aligned}$$

Consequently, by (37) and Proposition 1,

$$\begin{aligned} &\left| (E(\kappa_N^p \cdot \widetilde{D}^p_{p^*,N}))^{p^*/p} - (E(\kappa_N^p \cdot D^p_{p^*,N}))^{p^*/p} \right| \\ &\qquad \leq c \cdot \left(E\left| \widetilde{D}^{p^*}_{p^*,N} - D^{p^*}_{p^*,N} \right|^{p/p^*} \right)^{p^*/p} \leq c \cdot m_N^{-p^*}, \end{aligned}$$

which finishes the proof. □

Combine Lemma 5 with Lemma 6 and observe (11) to obtain the error estimate for the quantization $\widetilde{X}^{(p,q)}_N$ of X in Theorem 1. Moreover, the bound for the size of the range of $\widetilde{X}^{(p,q)}_N$ is a straightforward consequence of (10) and (11).

3.5. **Proof of Theorem 2.** We fix $s, q \geq 1$ with $q \geq \min\{r \in 2\mathbb{N} : r \geq s\}$. For $N \in \mathbb{N}$ we put

$$\delta_N = e^{(s)}(W, S_N(W), C).$$

First, we analyze the quantization $\widetilde{B}^{(\gamma,q)}_{m,n,N}$ of a weighted combination $B^{(\gamma)}$ of the Brownian bridges $B_1, \dots, B_m$, see (13) and (16).

Lemma 7. *For every $\gamma \in \mathbb{R}^m \setminus \{0\}$ that satisfies* (14),

$$\left(E\|B^{(\gamma)} - \widetilde{B}^{(\gamma,q)}_{m,n,N}\|^s_{C[0,1]} \right)^{1/s} \leq \delta_N + c \cdot n^{-1}.$$

Proof. Due to (15) we have

$$\|B^{(\gamma)} - \widetilde{B}^{(\gamma,q)}_{m,n,N}\|_{C[0,1]} \le \|W^{(\gamma)} - S_N(W^{(\gamma)})\|_{C[0,1]} + \max_{\ell=1,\dots,m} |W^{(\gamma)}(s_\ell) - \widetilde{W}^{(\gamma,q)}_{m,n}(s_\ell)|.$$

Put

$$U_\ell = W^{(\gamma)}(s_\ell) - \widetilde{W}^{(\gamma,q)}_{m,n}(s_\ell) = m^{-1/2} \cdot \sum_{j=1}^{\ell} \gamma_j \cdot \big(Z_j - \widetilde{Z}^{(q)}_{j,n}\big)$$

and note that the sequence $(U_\ell)_{\ell=0,\dots,m}$ is a centered martingale, due to property (A1). Hence

$$E\big(\max_{\ell=1,\dots,m} |U_\ell|^q\big) \le c \cdot E|U_m|^q.$$

Use property (A3) and observe (14) to derive

$$\begin{aligned}
E|U_{\ell+1}|^q &= E\left(\sum_{j=0}^{q} \binom{q}{j} U_\ell^{q-j} \cdot \big(m^{-1/2} \cdot \gamma_{\ell+1} \cdot (Z_j - \widetilde{Z}^{(q)}_{j,n})\big)^j\right) \\
&= E|U_\ell|^q + \sum_{j=2}^{q} \binom{q}{j} E(U_\ell^{q-j}) \cdot (\gamma_{\ell+1}/m^{1/2})^j \cdot E(Z_j - \widetilde{Z}^{(q)}_{j,n})^j \\
&\le E|U_\ell|^q + c \cdot \gamma_{\ell+1}^2/m \cdot \sum_{j=2}^{q} \binom{q}{j} E|U_\ell|^{q-j} \cdot n^{-j} \\
&\le E|U_\ell|^q + c \cdot \gamma_{\ell+1}^2/m \cdot E(|U_\ell| + n^{-1})^q \\
&\le E|U_\ell|^q \cdot (1 + c \cdot \gamma_{\ell+1}^2/m) + c \cdot \gamma_{\ell+1}^2/m \cdot n^{-q}.
\end{aligned}$$

Hence

$$E|U_m|^q \le c \cdot n^{-q},$$

due to Gronwall's inequality, which completes the proof. □

Put

$$A_{m,n} = \left(\frac{1}{m} \sum_{j=0}^{m-1} b_m^2(\widetilde{X}^{(q)}_{m,n}(t_\ell))\right)^{1/2}$$

for $m, n \in \mathbb{N}$.

Lemma 8.

$$e^{(s)}(X, \widetilde{X}^{(q)}_{m,n,N}, C) \le c \cdot (m^{-1} \cdot \ln m + n^{-1}) + (EA^s_{m,n})^{1/s} \cdot (\delta_N + c \cdot n^{-1}).$$

Proof. In view of Lemma 4 it suffices to establish Lemma 8 with X replaced by $\overline{X}_m$. By definition,

$$\overline{X}_m - \widetilde{X}^{(q)}_{m,n,N} = U_1 + U_2 + U_3$$

with U_1 given by (33),

$$U_2(t) = \big(b(\widetilde{X}^{(q)}_{m,n}(t_{\ell-1})) - b_m(\widetilde{X}^{(q)}_{m,n}(t_{\ell-1}))\big) \cdot B_\ell(t)$$

for $t \in [t_{\ell-1}, t_\ell]$, and

$$U_3 = A_{m,n} \cdot \big(B^{(\Gamma)} - \widetilde{B}^{(\Gamma,q)}_{m,n,N}\big).$$

Clearly $|b(x) - b_m(x)| \le m^{-1}$, and therefore

$$E\|U_2\|^s_{C[0,1]} \le c \cdot m^{-s} \cdot (\ln m/m)^{s/2} \tag{39}$$

follows from (32). Recall the definition (36) of V, and note that $(B_1, Z_1, \ldots, B_m, Z_m)$ and V are independent, while Γ and $A_{m,n}$ are measurable w.r.t. $\sigma(V)$. Hence

$$E\big(\|U_3\|^s_{C[0,1]} \big| V\big) \le A^s_{m,n} \cdot (\delta_N + c \cdot n^{-1})^s \tag{40}$$

follows from Lemma 7. Combine (34), (39), and (40) to complete the proof. □

Consider a sequence of integers $m_N \in \mathbb{N}$ that satisfies (18) and recall the definition of $\widetilde{D}_{r,N}$ and $D_{r,N}$ in the proof of Lemma 6. Use (38) and Proposition 1 to establish

$$\big(EA^s_{m_N,m_N}\big)^{1/s} \le c \cdot m_N^{-1} + \big(ED^s_{2,N}\big)^{1/s}$$

so that

$$\limsup_{N\to\infty} \big(EA^s_{m_N,m_N}\big)^{1/s} \le C^{(\infty,s)}.$$

Now, apply Lemma 8 and observe (18) to obtain the error estimate for the quantization $\widetilde{X}^{(q)}_N$ of X in Theorem 2. The bound for the size of the range of $\widetilde{X}^{(q)}_N$ is a straightforward consequence of (17) and (18).

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