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„Extraktion quantifizierbarer Information aus komplexen Systemen"

Sparse Recovery and Kronecker Products

S. Jokar

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Sparse Recovery and Kronecker Products

Sadegh Jokar

(Invited Paper)

***Abstract*—In this note will consider sufficient conditions for sparse recovery such as Spark, coherence, restricted isometry property (RIP) and null space property (NSP). Then we will discuss the solution of underdetermined linear equations when the matrix is the Kronecker product of matrices. Specially we will explain how NSP behave in the case where the matrix is the Kronecker product of matrices.**

***Index Terms*—Spark, coherence, null space property, restricted isometry property, compressed sensing, Kronecker product, sparse solution of linear systems.**

I. Introduction

In this short paper we will consider the computation of sparse solutions of underdetermined linear systems

$$Ax = b,$$

where $A \in \mathbb{R}^{m,n}$, with $m \leq n$ is given as a Kronecker product, i.e.

$$A = A_1 \otimes A_2 \otimes \ldots \otimes A_N, \quad A_i \in \mathbb{R}^{m_i,n_i}, i = 1, \ldots, N. \quad (1)$$

Since the solution is typically non-unique it is an important topic in many applications, in particular in sparse signal recovery, see e.g. [1], [3], [4], [5], [6], [9], [10], [19] to find the sparsest solution,

$$\min \|x\|_0, \ \ s.t. \ Ax = b, \quad (2)$$

where $\|x\|_0$ denotes the number of nonzero entries of a vector x, see Section II.

In general, the problem of finding the sparsest solution is known to be NP-hard [21]. However, in the context of compressed sensing, conditions have been derived on the size of the *support of* x, i.e. the number of nonzero elements of x, that allow one to compute the sparsest solution using ℓ_1-minimization via the so called *basis pursuit algorithm* [3], [5], [7], [8], [10], [11], [12], i.e, by computing

$$\min \|x\|_1, \ \ s.t. \ Ax = b, \quad (3)$$

where $\|x\|_1 = \sum_i |x_i|$.

Sufficient conditions for this approach to work are that some properties of the matrix A called *spark* [10], [22], *coherence* [7], [12], or the *restricted isometry property (RIP)* [2], [3], [4] or the *null space property* [8] are studied. We will introduce these properties in Section II.

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For general matrices it is possible to determine the coherence, while analyzing the spark, the restricted isometry property or null space property is difficult. If, however, the matrix A has the form (1) then we show in Section III that these properties can be derived from the corresponding properties of the factors.

II. Preliminary Observations

For $m, n \in \mathbb{N}$, where $\mathbb{N} = \{1, 2, \ldots\}$, we denote by $\mathbb{R}^{m,n}$ the set of real $m \times n$ matrices, by I_n the $n \times n$ identity matrix, and by $\langle \cdot, \cdot \rangle$ the Euclidean inner product in $\mathbb{R}^n$. For $1 \leq p \leq \infty$, the ℓ_p-norm of $x \in \mathbb{R}^n$ is defined by

$$\|x\|_p := \Big(\sum_{j=1}^{n} |x_j|^p \Big)^{\frac{1}{p}},$$

with the special case

$$\|x\|_\infty := \max_{j \in \{1,\ldots,n\}} |x_j|,$$

if $p = \infty$. Finally, for $x \in \mathbb{R}^n$, we introduce the notation

$$\|x\|_0 := \# \operatorname{supp}(x),$$

where $\operatorname{supp}(x) := \{j \in \{1, \ldots, n\} : x_j \neq 0\}$ is the *support* of x. We use the term *k-sparse* for all vectors x such that $\|x\|_0 \leq k$.

Definition II.1. *[17], [20] The* Kronecker product *of* $A = [a_{i,j}] \in \mathbb{R}^{p,q}$ *and* $B = [b_{i,j}] \in \mathbb{R}^{r,s}$ *is denoted by* $A \otimes B$ *and is defined to be the block matrix*

$$A \otimes B := \begin{bmatrix} a_{1,1}B & \cdots & a_{1,q}B \\ \vdots & \ddots & \vdots \\ a_{p,1}B & \cdots & a_{p,q}B \end{bmatrix} \in \mathbb{R}^{pr,qs}.$$

As our first special property we introduce the spark of a matrix.

Definition II.2. *[10], [22] Let* $A = [a_1, \ldots, a_n] \in \mathbb{R}^{m,n}$, $2 \leq m \leq n$ *have columns* a_i *that are normalized so that* $\|a_i\|_2 = 1, \ i = 1, \ldots, n.$ *The* spark *of* A, *denoted as* $\operatorname{spark}(A)$ *is defined as the cardinality of the smallest subset of linearly dependent columns of* A.

The quantity $\operatorname{spark}(A)$ can be used to derive sufficient conditions for the existence of sparse solutions.

Lemma II.3. *[10], [16] Consider the linear system* $Ax = b$ *with* $A \in \mathbb{R}^{m,n}$, $m \leq n$. *A sufficient condition for the linear system* $Ax = b$ *to have a unique k-sparse solution* x *is that* $k \leq \operatorname{spark}(A)/2$.

The second property that we consider is the coherence.

Definition II.4. *[12] Let $A = [a_1, \dots, a_n] \in \mathbb{R}^{m,n}$, $m \le n$ have columns a_i that are normalized so that $\|a_i\|_2 = 1$, $i = 1, \dots, n$. Then the* coherence *$\mathcal{M}(A)$ is defined by*

$$\mathcal{M}(A) := \max_{i \neq j} |\langle a_i, a_j \rangle|.$$

Note that, since the columns of A are normalized, by the triangle inequality we always have $\mathcal{M}(A) \le 1$. On the other hand, if A has orthonormal columns, then $\mathcal{M}(A) = 0$.

The following lemma relates the sparsest solution as defined in (2) and the ℓ_1-solution as defined in (3) of the linear equation $Ax = b$ in terms of the coherence of a matrix A.

Lemma II.5. *[10], [15], [14] Suppose that $A \in \mathbb{R}^{m,n}$, $m \le n$ has columns a_i that are normalized so that $\|a_i\|_2 = 1$, $i = 1, \dots, n$. If there exists a solution x for a given b of the equation $Ax = b$ satisfying*

$$\|x\|_0 < \frac{1 + \frac{1}{\mathcal{M}(A)}}{2},$$

then the ℓ_1-norm minimal solution in (3) coincides with the ℓ_0-minimal solution in (2).

The third quantity that is important in the context of sparse recovery and compressed sensing is the *restricted isometry property*.

Definition II.6. *[2], [3], [4], [5] Let $A = [a_1, \dots, a_n] \in \mathbb{R}^{m,n}$, $m \le n$ have columns a_i that are normalized so that $\|a_i\|_2 = 1$, $i = 1, \dots, n$. The* k-restricted isometry constant *of A is the smallest number δ_k such that*

$$(1 - \delta_k)\|x\|_2^2 \le \|Ax\|_2^2 \le (1 + \delta_k)\|x\|_2^2 \tag{4}$$

for all $x \in \mathbb{R}^n$ with $\|x\|_0 \le k$.

The following lemma gives the relation between the sparsest solution (as defined in (2)) of a linear system $Ax = b$ and the ℓ_1-solution as defined in (3) in terms of the k-restricted isometry constant.

Lemma II.7. *[2] Let $A = [a_1, \dots, a_n] \in \mathbb{R}^{m,n}$, $m \le n$ have columns a_i that are normalized so that $\|a_i\|_2 = 1$, $i = 1, \dots, n$.*

Suppose that

$$\delta_{2k} < \sqrt{2} - 1.$$

Then for all k-sparse solution vectors x of $Ax = b$ the solution of (3) is equal to the solution of (2).

For $A \in \mathbb{R}^{m,n}$ with $m < n$, a vector of the form $b = Ax$ *represents (encodes)* the vector x in terms of the columns of A. To extract the information that b holds about x, we may use a *decoder* Δ which is a (not necessary linear) mapping. Then $y = \Delta(b) = \Delta(Ax)$ is our approximation to x from the information given in b.

Let $\Sigma_k = \{z \in \mathbb{R}^n : \|z\|_0 \le k\}$ denote the vectors of support less than or equal to k. In the following we use the classical ℓ_l-norm.

We introduce the distance

$$\sigma_k(x)_p := \min_{z \in \Sigma_k} \|x - z\|_p. \tag{5}$$

Definition II.8. *In going further, we say that Φ has the null space property of order k with constant C_k if*

$$\|\eta\|_1 \le C_k \sigma_k(\eta)_1$$

holds for all $\eta \in \mathcal{N}$.

Theorem II.9. *[8] Let $a = \ell/k, b = \ell'/k$ with $\ell, \ell' \ge k$ integers. If Φ satisfies the RIP of order $(a+b)k$ with $\delta = \delta_{(a+b)k} < 1$, then Φ satisfies the null space property in ℓ_1 of order ak with constant*

$$C = 1 + \frac{\sqrt{a}(1+\delta)}{\sqrt{b}(1-\delta)}. \tag{6}$$

From this Theorem one could get the following result.

Theorem II.10. *[8] Let $A \in \mathbb{R}^{m,n}$ satisfy (4) in the form*

$$(1 - \delta_{3k})\|x\|_2^2 \le \|Ax\|_2^2 \le (1 + \delta_{3k})\|x\|_2^2$$

with

$$\delta_{3k} \le \delta < \frac{(\sqrt{2}-1)^2}{3}.$$

Define a decoder Δ for A via

$$\Delta(b) = \mathrm{argmin}_{b=Ax}\|x\|_1. \tag{7}$$

Then

$$\|x - \Delta(Ax)\|_1 \le C\sigma_k(x)_1,$$

where

$$C = \frac{2\sqrt{2} + 2 - (2\sqrt{2} - 2)\delta}{\sqrt{2} - 1 - (\sqrt{2}+1)\delta}.$$

Theorem II.10 shows that the ℓ_1-norm solution can be as good as best k-term approximation.

In the following we will explain a way to calculate and estimate the NSP constant C_k.

Theorem II.11. *Suppose that $A \in \mathbb{R}^{m,n}$ is normalized ($\|a_i\|_2 = 1$) where $m < n$. If $k < \mathrm{spark}(A)$, then the constant C_k in NSP is:*

$$C_k = 1 + \max_{\substack{S \subset \{1, \cdots n\} \\ \#S = k}} \|A_S^\dagger A_{S^c}\|_1. \tag{8}$$

where $A_S = [a_i]_{i \in S}$ and $A_S^\dagger = (A_S^{\mathrm{T}} A_S)^{-1} A_S^T$ is the psedoinverse of A_S. In the special case where $k = 1$, we have:

$$C_1 = 1 + \mathcal{M}(A). \tag{9}$$

Proof: Since $k < \mathrm{spark}(A)$, this problem is well defined. Let assume that $S \subset \{1, \dots, n\}$ with $\#S = k$. Then from $A\eta = 0$, we get:

$$A_S \eta_S + A_{S^c}\eta_{S^c} = [A_S \ A_{S^c}] \begin{bmatrix} \eta_S \\ \eta_{S^c} \end{bmatrix} = 0,$$

and therefore $\eta_S = -A_S^\dagger A_{S^c}\eta_{S^c}$. By taking the ℓ_1-norm in both direction we have:

$$\|\eta_S\|_1 = \|A_S^\dagger A_{S^c}\eta_{S^c}\|_1 \le \|A_S^\dagger A_{S^c}\|_1 \|\eta_{S^c}\|_1.$$

By adding $\|\eta_{S^c}\|_1$ in both direction we get:

$$\|\eta\|_1 \leq (1 + \|A_S^\dagger A_{S^c}\|_1)\|\eta_{S^c}\|_1.$$

By taking all subsets S of cardinality k we have (8) for all subset S with $\#S \leq k$ and $\eta \in N(A)$.

For the case where $k = 1$, we have:

$$\max_{\substack{S \subset \{1,\cdots n\} \\ \#S=1}} \|A_S^\dagger A_{S^c}\|_1 = \max_{1\leq i\leq n} \|[a_i]^{\mathrm{T}}[a_j]_{j\neq i}\|_1.$$

Therefore

$$C_1 = 1 + \max_{i\neq j} |\langle a_i, a_j\rangle|.$$

Remark II.12. *Note that if $k \geq \operatorname{spark}(A)$ then there exist $\eta \neq 0$ with $\|\eta\|_0 = k$ such that $A\eta = 0$. But in null space property we must have*

$$0 < \|\eta\|_1 \leq C_k \sigma_k(\eta)_1 = 0$$

which is impossible. Therefore null space property of order k does not make sense.

After introducing the concepts of spark, coherence, k-restricted isometry property and null space property, in the next section we analyze these concepts for Kronecker product of matrices.

III. Sparse representation and Kronecker Products

In this section we study sparse solutions for linear system $Ax = b$, where the matrix A is given as a Kronecker product as in (1).

In [18], we charatrezied $\operatorname{spark}(A \otimes B)$ in terms of $\operatorname{spark}(A)$ and $\operatorname{spark}(B)$. Note that if A, B have normalized columns then $A \otimes B$ has normalized columns as well.

Theorem III.1. *[18] Let $A = [a_1, \ldots, a_q] \in \mathbb{R}^{p,q}$ and $B = [b_1, \ldots, b_s] \in \mathbb{R}^{r,s}$ be rank-deficient matrices with normalized columns, i.e., $\|a_i\|_2 = 1$, $i = 1, \ldots, q$, $\|b_i\|_2 = 1$, $i = 1, \ldots, s$. Then*

$$\operatorname{spark}(A \otimes B) = \operatorname{spark}(B \otimes A) = \min\{\operatorname{spark}(A), \operatorname{spark}(B)\}. \quad (10)$$

If A is an invertible matrix and B is rank-deficient matrix, then

$$\operatorname{spark}(A \otimes B) = \operatorname{spark}(B). \quad (11)$$

If both A and B are square and invertible then

$$\operatorname{spark}(A \otimes B) = (\operatorname{spark}(A) - 1)(\operatorname{spark}(B) - 1) + 1 = qs + 1.$$

Corollary III.2. *[18] Consider rank-deficient matrices $\{A_i\}_{i=1}^N$ with normalized columns. Then*

$$\operatorname{spark}(A_1 \otimes \ldots \otimes A_N) = \min_{1\leq i\leq N} \{\operatorname{spark}(A_i)\}.$$

We immediately have the following corollary of III.2.

Corollary III.3. *[18] Consider a linear system $(A_1 \otimes \ldots \otimes A_N)x = b$ with rank-deficient matrices $A_i \in \mathbb{R}^{p_i,q_i}$ that have normalized columns. A sufficient condition for this linear system to have a unique k-sparse solution x is that*

$$k \leq \frac{\min_{1\leq i\leq N} \{\operatorname{spark}(A_i)\}}{2}.$$

Similar to the analysis of $\operatorname{spark}(A \otimes B)$, it has been shown in [18], an estimate of $\mathcal{M}(\otimes_{i=1}^N A_i)$ in terms of each $\mathcal{M}(A_i)$'s.

Theorem III.4. *[18] Consider matrices $\{A_i\}_{i=1}^N$ with normalized columns and let $A = A_1 \otimes \ldots \otimes A_N$. Then,*

$$\mathcal{M}(A) = \max_{1\leq i\leq n} \mathcal{M}(A_i).$$

Theorem III.4 shows that if one of the matrices A_i has a large coherence, then it will dominate the coherence of A, regardless of all the other factors in the Kronecker product.

We immediately have the following corollary of Theorem III.4.

Corollary III.5. *Consider a linear system $(A_1 \otimes \ldots \otimes A_N)x = b$ with rank-deficient matrices $A_i \in \mathbb{R}^{p_i,q_i}$ that have normalized columns. A sufficient condition for this linear system to have a unique k-sparse solution x is that*

$$k \leq \frac{1 + \frac{1}{\min_{1\leq i\leq N}\{\mathcal{M}(A_i)\}}}{2}.$$

One could also have similiar results which relates the k-restricted isometry constant of $\delta_k^{A\otimes B}$ to those of δ_k^A and δ_k^B.

Theorem III.6. *[18] Let $A \in \mathbb{R}^{p,q}$ and $B \in \mathbb{R}^{r,s}$ have normalized columns. Then*

$$\delta_k^{A\otimes B} = \delta_k^{B\otimes A} \geq \max\{\delta_k^A, \delta_k^B\}. \quad (12)$$

Remark III.7. *In [13], they gave also an upper bound for the RIP where*

$$\delta_k^{A_1\otimes\cdots\otimes A_N} \leq \prod_{i=1}^N (1 + \delta_k^{A_i}) - 1.$$

We have the following obvious corollary.

Corollary III.8. *Suppose that matrices A_i for $i = 1, \ldots, N$ have normalized columns. Then*

$$\max_{1\leq i\leq N} \{\delta_k^{A_i}\} \leq \delta_k^{A_1\otimes\ldots\otimes A_N} \leq \prod_{i=1}^N (1 + \delta_k^{A_i}) - 1.$$

According to Lemma II.7, if the restricted isometry constant δ_{2k} is small enough ($\delta_{2k} < \sqrt{2} - 1$), then one can recover all k-sparse solutions using ℓ_1-minimization. On the other hand, Corollary III.8 implies that if the k-restricted isometry constant δ_k of A is small (for example less than $1/2$), then A can not be written as a Kronecker product of matrices A_i with smaller sizes.

We will show in the following that one could also relates the k-null space constant of $C_k^{A\otimes B}$ to those of C_k^A and C_k^B.

Theorem III.9. *Let $A \in \mathbb{R}^{p,q}$ and $B \in \mathbb{R}^{r,s}$ have normalized columns. Then*

$$C_k^{A\otimes B} \geq \max\{C_k^A, C_k^B\}. \tag{13}$$

Proof: Let $A \in \mathbb{R}^{p,q}$ and $B \in \mathbb{R}^{r,s}$. Then, we have:

$$C_k^{A\otimes B} = 1 + \max_{\substack{S\subset\{1,\cdots n\}\\ \#S=k}} \|(A\otimes B)_S^\dagger (A\otimes B)_{S^c}\|_1$$

We choose S for example, such that

$$(A\otimes B)_S = \begin{bmatrix} a_{1,1}b_1 & \cdots & a_{1,1}b_k \\ \vdots & \ddots & \vdots \\ a_{p,1}b_1 & \cdots & a_{p,1}b_k \end{bmatrix}.$$

Then there exists a permutation matrix P such that

$$C_k^{A\otimes B} \geq \quad 1 + \|(\langle b_\ell, b_w\rangle)^{-1}_{\ell,w=1,\cdots,k}(A\otimes B)_S^T(A\otimes B)_{S^c}\|_1$$

where

$$(A\otimes B)_{S^c} = \begin{bmatrix} a_{1,1}b_{k+1} & \cdots & a_{1,1}b_s & \\ \vdots & \ddots & \vdots & D \\ a_{p,1}b_{k+1} & \cdots & a_{p,1}b_s & \end{bmatrix} P$$

and matrix D is the rest of $(A\otimes B)_{S^c}$. For simplicity let assume that $E_k = (\langle b_\ell, b_w\rangle)_{\ell,w=1,\cdots,k}$, $E_{s-k} = (\langle b_\ell, b_w\rangle)^{k,s}_{\ell,w=1,k+1}$. and $F = (A\otimes B)_S^T D$. Then we have:

$$(A\otimes B)_S^T(A\otimes B)_{S^c} = [E_{s-k}F]P$$

Thus:

$$\begin{aligned} C_k^{A\otimes B} &\geq \quad 1 + \|[E_k^{-1}E_{s-k} \quad E_k^{-1}F]\|_1 \\ &\geq \quad 1 + \|[E_k^{-1}E_{s-k}]\|_1 \end{aligned}$$

Therefore we have $C_k^{A\otimes B} \geq C_k^B$. Using the fact that there exist permutation matrices Π_1, Π_2 such that $\Pi_1(B\otimes A)\Pi_2 = A\otimes B$, we conclude that $C_k^{A\otimes B} \geq C_k^A$.

Using Theorem II.9 and II.10, if the NSP constant $C_k^{A\otimes B}$ is small enough, then one can recover all k-sparse solutions using ℓ_1-minimization. On the other hand, Theorem III.6 implies that if one k-th NSP constant C_k of A or B is not small, then $A\otimes B$ can not have a good NSP constant which means that one could not hope to recover sparse signals of high order.

IV. Conclusion

We have analyzed the recently introduced concepts of the spark, the coherence, the k-restricted isometry property and specially NSP of matrix in Kronecker product form to that of the Kronecker factors.

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