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„Extraktion quantifizierbarer Information aus komplexen Systemen"

On the Optimality of the Inexact Inverse Iteration Coupled with Adaptive Finite Element Methods

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On the optimality of the inexact inverse iteration coupled with adaptive finite element methods

Andreas Zeiser*

July 20, 2010

Abstract

We study the convergence and optimality of the inverse iteration where the intermediate problems are solved only approximately. In particular we are interested in finding an approximation of an eigenvector corresponding to the smallest eigenvalue of an elliptic operator. We show that this iteration converges if the tolerances are chosen appropriately. Furthermore we identify abstract prerequisites which ensure the quasi-optimality of the method. As our main example we show the quasi-optimality of the inexact inverse iteration coupled with adaptive finite element methods (AFEM) for a class of elliptic problems.

AMS: 65N25, 65N30, 65Y20

Keywords: eigenvalue, inverse iteration, adaptive finite element method, convergence rates, complexity

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1 Introduction

The analysis of adaptive finite element methods for the solution of elliptic eigenvalue problems (EVP) is always a step behind the corresponding boundary value problem (BVP): while first convergence result for adaptive finite element methods for BVPs have been obtained in 1996 by Dörfler [8] those for EVPs have been obtained as late as 2007 by Giani and Graham [11]. The proofs on the quasi-optimality of standard adaptive methods follow more closely, i.e. 2007/2008 for BVPs [4, 15] and 2008/2009 for EVPs [3, 6]. Both problems however are closely connected, see for example the analysis in [6].

This is even more obvious in the case of the inverse iteration. Suppose we want to solve the eigenvalue problem

$$\mathsf{L}v = \lambda\, v \tag{1}$$

where L is an elliptic operator. More specific we want to find an approximation to an eigenfunction in an eigenspace corresponding to the lowest eigenvalue λ_1. For a given starting function u_0 in the course inverse iteration we have to solve

$$\mathsf{L}\tilde{u}_{k+1} = u_k \tag{2}$$

and normalize the solution $\tilde{u}_{k+1}$ to obtain the next iterate u_{k+1}. Standard convergence theory asserts that this iteration converges as long as u_0 is not orthogonal to the eigenspace.

However in general it is not possible to solve the above solution exactly - one may only find an approximation to the solution, i.e.

$$\tilde{u}_{k+1} = \mathsf{L}^{-1}u_k + \xi_k \tag{3}$$

where ξ_k is a perturbation. We will call this iteration the inexact inverse iteration. One may ask how the perturbations have to be bounded in order to ensure convergence. In the case of matrix eigenvalue problems recently the case of inexact solves has been addressed, see for example [1]. Following the argumentation of the latter work - here in the infinite dimensional setting - we show that convergence is retained if the perturbations are chosen according to the current error in the approximation.

Since the intermediate equation (2) has to be solved only approximately this opens the door for the use of adaptive finite element methods like [4, 15]. These methods not only find approximate solutions to any prescribed tolerance, but also the number of degrees of freedom is optimal in the following sense: briefly the number of degrees of freedom of the calculated solution is bound by a multiple of the minimal number of degrees of freedom needed to construct an approximation to the same tolerance.

Now the main question is the following: suppose that the intermediate equations (2) are solved by an adaptive finite element methods with quasi-optimal complexity - is the inexact inverse iteration also of quasi-optimal complexity? This question has been studied numerically in the case of partial differential equations, see [16]. The obtained results suggest that this method is also quasi-optimal, if the tolerances are chosen appropriately. However from an analytic point of view the quasi-optimality is not obvious: in general the exact solutions

$\mathsf{L}^{-1}u_k$ of the intermediate systems are not as regular as the eigenfunction. As a consequence the asymptotic convergence rate of the solution of the intermediate system is in general worse than the approximability of the eigenvector.

As our main result we will show that despite the above fact the inexact inverse iteration is indeed quasi-optimal under some rather weak assumptions on the operator. The method therefore provides an easy method for solving eigenvalue problems. This is even more important as some standard finite element packages do not provide an eigenvalue solver. Although this method might not the most efficient one compared to methods which are tailored to eigenvalue problems, e.g. [3], it might be good enough for a first approximation.

The rest of the work is organized as following. In Section 2 we analyze the inexact inverse iteration in an abstract setting of an elliptic eigenvalue problem. We identify prerequisites to ensure the quasi-optimality of this iteration. Afterwards, in Section 3, we show that the inexact inverse iteration coupled with adaptive finite element methods fit in this framework, leading to a quasi-optimal algorithm.

2 Inexact inverse iteration

In this section an abstract framework of an elliptic eigenvalue problem is introduced. For the calculation of an eigenfunction corresponding to the lowest eigenvalue we suggest the inexact inverse iteration, where the intermediate equations are only solved approximately. We will show that this iteration converges geometrically if the tolerances are chosen appropriately. The proof is very similar to the matrix case treated in [1] and references therein. However in the present case the convergence is studied with respect to the energy norm and in the infinite dimensional setting.

As the main result we identify abstract prerequisites which ensure the quasi-optimality of the inexact inverse iteration. Basically the intermediate equations have to be solved with quasi-optimal effort and the operator has to be regular enough. Here the quasi-optimality is a variant of usual complexity statements. The complexity is thereby measured by defining abstract approximation spaces based on the nonlinear best N-term approximation [7].

In summary the inexact inverse iteration is a simple but yet quasi-optimal algorithm for the solution of eigenvalue problems. In the next section we will show that this algorithm can be realized with a classical AFEM algorithm.

2.1 Problem formulation

Let $(H, (\cdot,\cdot), |\cdot|)$ be a separable Euclidean Hilbert space, and $(V, \|\cdot\|)$ a reflexive and separable Banach space such that $V \subset H$ is dense and continuously embedded in H, such that $|v| \leq \|v\|$. Denote by $(H^*, |\cdot|_*)$ and $(V^*, \|\cdot\|_*)$ the respective dual spaces of H and V. The dual pairing on V^* and V is given by $\langle\cdot,\cdot\rangle : V^* \times V \to \mathbb{R}$. The spaces $V \subset H \cong H^* \subset V^*$ form a Gelfand triple by identifying H^* and H by the Riesz representation theorem.

Let $\mathsf{L} : V \to V^*$ be a continuous and linear operator which is in addition symmetric, that is

$$\langle \mathsf{L}u, v\rangle = \langle \mathsf{L}v, u\rangle \quad \text{for all } u, v \in V$$

and strongly positive, i.e. $\langle \mathsf{L}v, v\rangle \gtrsim \|v\|^2$ for all $v \in V$. Hence the operator L induces an inner product $\langle u, v\rangle_{\mathsf{L}} = \langle \mathsf{L}u, v\rangle$ on V. The corresponding energy norm $|||\cdot||| = \langle \cdot,\cdot\rangle_{\mathsf{L}}^{1/2}$ satisfies

$$c_{\mathsf{L}}\|v\| \leq |||v||| \leq C_{\mathsf{L}}\|v\| \quad v \in V \tag{4}$$

for constants $0 < c_{\mathsf{L}} \leq C_{\mathsf{L}}$.

Now the weak eigenvalue problem can be formulated as following: find a pair $(\lambda, u) \in \mathbb{R} \times V \setminus \{0\}$ such that

$$\mathsf{L}u = \lambda\, u \quad \text{in } V^*.$$

Here and in the following we use the Riesz representation theorem with respect to the inner product on H, i.e. $u \mapsto (u, \cdot)$.

The basic example is the Poisson eigenvalue problem $-\Delta v = \lambda\, v$ with homogeneous boundary condition on a bounded domain $\Omega \subset \mathbb{R}^d$. In this case the spaces are given by $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$.

In particular we are interested in the lowest eigenvalue λ_1, while we assume that the rest of the spectrum can be bounded away by $\lambda_2 > \lambda_1$. Furthermore we assume that the eigenspace $\mathcal{E}_1$ corresponding to λ_1 is finite dimensional. Define by P_1 the H-orthogonal projector onto $\mathcal{E}_1$ and $Q_1 = I - P_1$.

We will study the following method for computing an approximation to an eigenfunction corresponding to the lowest eigenvalue.

Definition 1. Let a normalized starting vector $u_0 \in V$ be given. The *inexact inverse iteration* is defined as the sequence of vectors $(u_k)_{k\geq 0}$ generated by

$$\begin{aligned} \tilde{u}_{k+1} &= \mathsf{L}^{-1}\, u_k + \xi_k, \\ u_{k+1} &= |||\tilde{u}_{k+1}|||^{-1}\, \tilde{u}_{k+1}, \end{aligned}$$

where $(\xi_k)_{k\geq 0} \in V$ are perturbations.

A key point in the later realization of this algorithm is that the update $\tilde{u}_{k+1} = \mathsf{L}^{-1}\, u_k + \xi_k$ can be viewed as the approximate solution $\tilde{u}_{k+1}$ of the equation $\mathsf{L}u = u_k$, where the error is given by ξ_k.

2.2 Convergence analysis

For the analysis of the convergence of this algorithm we decompose a vector $v \in V$ into the parts $v = P_1 v + Q_1 v$. Since both $\mathcal{E}_1$ and its H-orthogonal complement $\mathcal{E}_1^\perp$ are invariant subspaces of L it follows that

$$|||P_1 v|||^2 + |||Q_1 v|||^2 = |||v|||^2 \tag{5}$$

for all $v \in V$. The right measure for distance of vectors to the eigenspace $\mathcal{E}_1$ is in terms of angles. For that purpose define for $v \in V$

$$t(v) = \tan(v, \mathcal{E}_1) = \frac{|||Q_1 v|||}{|||P_1 v|||} \tag{6}$$

which is the tangent of the angle between v and $\mathcal{E}_1$. For example given the tangent of an element v, the distance of v and $\mathcal{E}_1$ can be estimated by

$$\operatorname{dist}_{|||\cdot|||}(v, \mathcal{E}_1) = \inf_{u \in \mathcal{E}_1} |||v - u||| = |||Q_1 v||| \leq t(v)\, |||v|||.$$

Therefore convergence of the tangents $(t_n)_{n\geq 0}$ corresponding to a normalized sequence $(v_n)_{n\geq 0}$ implies convergence of the vectors to the space $\mathcal{E}_1$.

Now having fixed the problem and the notation the following lemma studies one step of the inexact inverse iteration.

Lemma 1. *Let $u \in V$, $|||u||| = 1$, and the estimate $\bar{t}$ be given such that $t(u) \leq \bar{t} < \infty$. Now restrict the perturbation ξ such that for a parameter $0 < \tau < \lambda_1^{-1}$*

$$|||\xi||| \leq \tau \cdot \frac{\min\{1, \bar{t}\}}{\sqrt{1+\bar{t}^2}}. \tag{7}$$

Denote by u' the next iterate of the inexact inverse iteration where the perturbation is given by ξ, see Definition 1. Then the tangent of u' satisfies

$$t(u') \leq q\, \bar{t}, \quad \text{where } q := \frac{\lambda_2^{-1} + \tau}{\lambda_1^{-1} - \tau}.$$

Proof. Let $\tilde{u} = \mathsf{L}^{-1} u + \xi$, i.e. $u' = \tilde{u}/|||u'|||$. We will derive an estimate for

$$t(u') = t(\tilde{u}) = \frac{|||Q_1 \tilde{u}|||}{|||P_1 \tilde{u}|||}.$$

The projections of $\tilde{u}$ onto $\mathcal{E}_1$ and its orthogonal complement $\mathcal{E}_1^{\perp}$ satisfy

$$P_1 \tilde{u} = \lambda_1^{-1} P_1 u + P_1 \xi, \quad Q_1 \tilde{u} = \mathsf{L}^{-1} Q_1 u + Q_1 \xi, \tag{8}$$

since $\mathcal{E}_1$ and $\mathcal{E}_1^{\perp}$ are invariant spaces with respect to L^{-1} and $P_1 u \in \mathcal{E}_1$. Let $t = t(u)$ and define by s and c the sine and cosine of the angle between u and $\mathcal{E}_1$ which can be expressed as

$$s = |||Q_1 u||| = \frac{t}{\sqrt{1+t^2}}, \quad c = |||P_1 u||| = \frac{1}{\sqrt{1+t^2}}.$$

since u is normalized. We first estimate $|||P_1 \tilde{u}|||$ from below by

$$|||P_1 \tilde{u}||| \geq \lambda_1^{-1} |||P_1 u||| - |||P_1 \xi||| \geq \lambda_1^{-1} \frac{1}{\sqrt{1+t^2}} - |||\xi|||.$$

Since $t \leq \bar{t}$ and $|||\xi||| \leq \tau/\sqrt{1+\bar{t}^2}$ we get

$$|||P_1 u'||| \geq \left(\lambda_1^{-1} - \tau\right) \frac{1}{\sqrt{1+\bar{t}^2}}.$$

Next the nominator $|||Q_1 \tilde{u}|||$ of the tangent $t(\tilde{u})$ satisfies

$$|||Q_1 \tilde{u}||| \leq |||\mathsf{L}^{-1} Q_1 u||| + |||Q_1 \xi||| \leq \lambda_2^{-1} |||Q_1 u||| + |||\xi|||$$

since the norm of L^{-1} restricted on $\mathcal{E}_1^\perp$ is bounded by λ_2^{-1}. From

$$|||\xi||| \leq \tau \frac{\bar{t}}{\sqrt{1+\bar{t}^2}}, \quad |||Q_1 u||| = \frac{t}{\sqrt{1+t^2}} \leq \frac{\bar{t}}{\sqrt{1+\bar{t}^2}}$$

it follows that

$$|||Q_1 \tilde{u}||| \leq \left(\lambda_2^{-1} + \tau\right) \frac{\bar{t}}{\sqrt{1+\bar{t}^2}}.$$

Dividing the estimates of $|||Q_1\tilde{u}|||$ and $|||P_1\tilde{u}|||$ gives the assertion □

As a consequence choosing τ small enough the tangent of the next iterate is reduced by a factor of $q < 1$. Equivalently one could also restrict the residuum for the approximate solution of $\mathsf{L}v' = v$ which leads to a result similar to [1]. However in view of the realization by adaptive finite element methods, see Section 3, we restrict ourselves to the case presented in the previous lemma.

With this result at hand we can prove the convergence of the inexact inverse iteration by induction.

Theorem 1. *Let $u_0 \in V$, $|||u_0||| = 1$, and a bound $\bar{t}_0$ be given such that $t(u_0) \leq \bar{t}_0 < \infty$. Furthermore choose $\tau > 0$ sufficiently small such that*

$$\frac{\lambda_1^{-1} + \tau}{\lambda_2^{-1} - \tau} =: q < 1.$$

If the perturbations are restricted by

$$|||\xi_k||| \leq \tau \frac{\min\{1, \bar{t}_k\}}{\sqrt{1+\bar{t}_k^2}}, \qquad \bar{t}_k = q^k\, \bar{t}_0 \tag{9}$$

then the sequence $(u_k)_{k\geq 0}$ generated by inexact inverse iteration converges geometrically to the eigenspace $\mathcal{E}_1$, i.e.

$$t(u_k) \leq q^k\, \bar{t}_0$$

for all $k \geq 0$.

2.3 Optimality

In this section the optimality of the inexact inverse iteration is studied. For that purpose an abstract approximation class based on the decay rate of the error of the nonlinear best N-term approximation [7] is introduced. This abstract framework fits for adaptive finite element methods, see Section 3, as well as for adaptive wavelet methods [5, 9], which are however not discussed here.

Let $\hat{U} \subset V$ be an initial linear space of functions and denote by $\mathbb{F}(\hat{U})$ the set of all spaces which are accessible from $\hat{U}$ through some kind of refinement. We assume that we have an algorithm

$$[u, U] = \textbf{Solve}[f, \hat{U}, \varepsilon]$$

Algorithm 1 Inexact inverse iteration.

$[u, U] = \textbf{InexactInverseIteration}[u_0, U_0, \bar{t}_0, \epsilon, \tau, q]$

Input:

u_0: initial guess, $|||u_0||| = 1$

U_0: initial space $u_0 \in U_0$

$\bar{t}_0$: upper bound for the tangent of v_0 and $\mathcal{E}_1$.

ϵ: target accuracy

parameters τ, q such that $(\lambda_2^{-1} + \tau)/(\lambda_1^{-1} - \tau) \le q < 1$.

Output:

u: approximate eigenfunction

U: final space $u \in U$

for $k = 0, 1, \ldots$ **do**

$\delta_k = \min\{\bar{t}_k, 1\}/\sqrt{1 + \bar{t}_k^2}$

$[\tilde{u}_{k+1}, U_{k+1}] = \textbf{Solve}[u_k, U_k, \tau\, \delta_k]$

$u_{k+1} = \tilde{u}_{k+1}/|||\tilde{u}_{k+1}|||$

$\bar{t}_{k+1} = q\, \bar{t}_k$

if $\bar{t}_{k+1} \le \varepsilon$ **then**

return u_{k+1}, U_{k+1}

end if

end for

which computes for all $\varepsilon > 0$ a space $U \in \mathbb{F}(\hat{U})$ and an approximation $u \in U$ such that $|||u - \mathsf{L}^{-1} f||| \le \varepsilon$ for all $f \in H$. Now this algorithm is used in the realization of the inexact inverse iteration, see Algorithm 1. The convergence of this algorithm is ensured by the results of Theorem 1.

Now in order to assess the optimality of the inexact inverse iteration we introduce the error of the best N-term approximation. For that purpose for all refinements $U \in \mathbb{F}(\hat{U})$ we denote by $\sharp U$ the effort of U, which can be as for example the number of triangles in a triangulation or the number of nonzero entries in a wavelet expansion.

Define for $v \in V$ and $N \in \mathbb{N}$ by

$$\sigma_N(v, \hat{U}) = \inf_{\substack{U \in \mathbb{F}(\hat{U}) \\ \sharp U - \sharp \hat{U} \le N}} \inf_{u \in U} |||u - v|||$$

the error of the best N-term approximation with respect to the spaces accessible from $\hat{U}$. Based on that for $s > 0$ let

$$\mathcal{A}^s_{\hat{U}} = \big\{ v \in V \,\big|\, |v|_{\mathcal{A}^s_{\hat{U}}} < \infty \big\}, \quad |v|_{\mathcal{A}^s_{\hat{U}}} = \sup_{N \ge 1} N^s \sigma_N(v, \hat{U}).$$

be the corresponding approximation spaces. The error of the best N-term approximation of elements of $\mathcal{A}^s_{\hat{U}}$ decay like N^{-s}.

Now for the inexact inverse iteration we start with an approximation u_0 from a given space U_0. Now in order to show optimality of the inexact inverse iteration we make the following assumptions. The interpretation of these assumptions is provided afterwards.

Assumption 1. (Approximability) There exists a $s > 0$ such that for the eigenspace $\mathcal{E}_1$

$$|\mathcal{E}_1|_{A^s_{U_0}} = \sup\left\{|v|_{\mathcal{A}^s_{U_0}} \,\middle|\, v \in \mathcal{E}_1,\, |||v||| = 1\right\} < \infty,$$

i.e. the eigenfunctions corresponding to the lowest eigenvalue λ_1 are approximable with a rate s.

Assumption 2. (Nestedness) For all $U \in \mathbb{F}(U_0)$ let $\mathbb{F}(U) \subset \mathbb{F}(U_0)$, i.e. all refinements of U are also refinements of U_0.

Assumption 3. (Regularity) For all $U \in \mathbb{F}(U_0)$ there exists a $s' > 0$ such that for all $v \in U$

$$|\mathsf{L}^{-1}v|_{\mathcal{A}^{s'}_{U_0}} \lesssim |||v|||,$$

i.e. solutions of the elliptic equation are in some approximation class provided that the right hand side is in V.

Assumption 4. (Optimality) Let $\hat{U} \in \mathbb{F}(U_0)$ and $f \in \hat{U}$, $|||f||| = 1$, be given. Denote by v the solution of $\mathsf{L}v = f$. Suppose that one can decompose

$$v = v_1 + v_2, \quad v_1 \in \mathcal{A}^s_{\hat{U}},\, v_2 \in \mathcal{A}^{s'}_{\hat{U}}$$

for $s, s' > 0$. Then for every $\varepsilon > 0$ the algorithm $\mathbf{Solve}[f, \hat{U}, \varepsilon]$ outputs a space $U \in \mathbb{F}(\hat{U})$ such that

$$\sharp U - \sharp\hat{U} \lesssim |v_1|^{1/s}_{\mathcal{A}^s_{\hat{U}}}\, \varepsilon^{-1/s} + |v_2|^{1/s'}_{\mathcal{A}^{s'}_{\hat{U}}}\, \varepsilon^{-1/s'}$$

where the constant is independent of $\hat{U}$ and ε.

In the course of the inexact inverse iteration we have to determine an approximation of the exact solution $w_{k+1} = \mathsf{L}^{-1}u_k$, where the starting space is given by U_k. The nestedness of Assumption 2 allows to connect the approximation spaces with respect to U_k to the approximation spaces with respect to the initial space U_0.

The intermediate solution w_k are in general not as regular as elements of the eigenspace $\mathcal{E}_1$. Therefore asymptotically the convergence rate of the intermediate solvers deteriorate from the approximability of the eigenfunctions. However the intermediate systems do only have to be solved up to a certain tolerance which is proportional to the current error. We will show that w_k can be decomposed into a part in $\mathcal{E}_1$ and a less regular part in $\mathcal{E}_1^\perp$. Now due to the regularity of Assumption 3 the second part is approximable with a rate $s' > 0$. Combining this result with the the optimality of Assumption 4 finally enables us to show quasi-optimality.

Note that the last assumption is a variant of classical optimality results (see for example [4]) in that it decomposes the approximability of the solution into two parts. Due to the non-linearity of the adaptive algorithms this result however does not follow directly from the classical one.

Theorem 2. *Let $\varepsilon > 0$. Under hypothesis of Assumptions 1-4 the Algorithm 1 outputs a space U such that*

$$\sharp U - \sharp U_0 \lesssim |\mathcal{E}_1|_{\mathcal{A}^s_{U_0}}^{1/s} \, \varepsilon^{-1/s}$$

where the constant only depends on the operator L, the initial space U_0, the initial tangent t_0, the quality of the estimate $\bar{t}_0$, the approximability s of $\mathcal{E}_1$ and the regularity s'.

Proof. The proof proceeds by induction. In the k-th step the iterate u_k is normalized, i.e. $|||u_k||| = 1$ and the corresponding tangent $t_k = t(u_k)$ satisfies $t_k \leq \bar{t}_k$, see Theorem 1. The exact solution $w_{k+1} = \mathsf{L}^{-1} u_k$ of the intermediate elliptic equation can be written as

$$w_{k+1} = \mathsf{L}^{-1} u_k = \underbrace{\mathsf{L}^{-1} P_1 u_k}_{=v_1} + \underbrace{\mathsf{L}^{-1} Q_1 u_k}_{=v_2} .$$

Now since $\mathsf{L}^{-1} P_1 u_k = \lambda_1^{-1} P_1 u_k$

$$|v_1|_{\mathcal{A}^s_{U_k}} \leq \lambda_1^{-1} |P_1 u_k|_{\mathcal{A}^s_{U_k}} \leq \lambda_1^{-1} |P_1 u_k|_{\mathcal{A}^s_{U_0}},$$

where the last inequality follows from the nestedness (Assumption 2). Due to the approximability of the eigenspace $\mathcal{E}_1$, Assumption 1, $|v_1|_{\mathcal{A}^s_{U_k}} \lesssim |\mathcal{E}_1|_{\mathcal{A}^s_{U_0}}$. Now Assumption 3 guarantees the existence of a constant $s' > 0$ such that $|v_2|_{\mathcal{A}^{s'}_{U_k}} \lesssim |||Q_1 u_k|||$. The last term can be written as

$$|||Q_1 u_k||| = \frac{t_k}{\sqrt{1+t_k^2}} \leq \frac{\bar{t}_k}{\sqrt{1+\bar{t}_k^2}}$$

since $\bar{t}_k$ is an upper bound for the exact tangent t_k of the k-th iterate, see Theorem 1. Now given the definition of δ_k in Algorithm 1 it follows that $\bar{t}_k \sim \delta_k/\tau$, where the constant depends on the estimate $\bar{t}_0$ of the initial tangent. As a consequence $|v_2|_{\mathcal{A}^{s'}_{U_k}} \lesssim \delta_k/\tau$. Therefore Assumption 4 leads to

$$\sharp U_{k+1} - \sharp U_k \lesssim |\mathcal{E}_1|_{\mathcal{A}^s_{U_0}}^{1/s} \, \delta_k^{-1/s} + \big(\delta_k/\tau\big)^{1/s'} \, \delta_k^{-1/s'} \lesssim |\mathcal{E}_1|_{\mathcal{A}^s_{U_0}}^{1/s} \, \delta_k^{-1/s}$$

since the second summand can be estimated by a constant. Finally let $K \in \mathbb{N}$ be the number of steps performed, that is the smallest integer such that $\bar{t}_0 q^K \leq \varepsilon$. Then $U = U_K$ and

$$\sharp U - \sharp U_0 = \sum_{k=0}^{K-1} \big(\sharp U_{k+1} - \sharp U_k\big) \lesssim \sum_{k=0}^{K-1} |\mathcal{E}_1|_{\mathcal{A}^s_{U_0}}^{1/s} \, \delta_k^{-1/s} \lesssim |\mathcal{E}_1|_{\mathcal{A}^s_{U_0}}^{1/s} \, \varepsilon^{-1/s},$$

where in the last inequality we used $\delta_k \sim \bar{t}_k$. □

Therefore the inexact inverse iteration coupled with a quasi-optimal solver for elliptic problems gives a quasi-optimal algorithm for the computation of an eigenvector corresponding to the smallest eigenvalue. For the actual realization of the inexact inverse iteration one has to provide information on the spectrum of L as well as an estimate $\bar{t}_0$ for the tangent of the initial vector. For the first point one may derive bounds on the eigenvalues from theoretical considerations. For the second point error estimators for eigenvalue problems may be used. Such error estimators based on the Temple-Kato inequality have been considered for example in [14, 12] and residual based estimators in [10, 3]. However these estimators rely on the fact that the function is somehow close to the actual eigenspace. Therefore as in the algorithms given in [11, 14] these restrictions on the starting vector have to be fulfilled also in our case. Of course if such an error estimator is available, it can be used during the algorithm as a stopping criteria and to improve the choice of appropriate tolerances.

3 Realization with AFEM

In this section we will analyze a realization of the inexact inverse iteration where the intermediate equations are solved by the adaptive finite element method (AFEM) described in [4]. More specific, we will show that this algorithm fits into the abstract framework of the last section.

Our main work will be to show the quasi-optimality in the sense of Assumption 4 in the case of AFEM. This prerequisite differs from usual results on quasi-optimality in such a way, that it allows to partition the solution in two parts which are in different approximation classes. Since the adaptive method is a non-linear algorithm our assumption does not follow directly from the results in [4]. However we can still rely heavily on the optimality results derived there and only slightly modify these results to fit our Assumption 4.

As a consequence we will restrict ourselves to a certain class of elliptic operators already analyzed in [4]. In the first part of this section this model problem is introduced. In order to show that the assumptions of Section 2.3 are fulfilled we have to briefly describe the adaptive finite element method of [4] as well as the involved approximation spaces, etc., which is done in Subsection 3.2. Based on the abstract results of Section 2.3 we will show the quasi-optimality of the inverse iteration coupled with the adaptive finite element method in the last part of this section.

3.1 Model problem

Let Ω be a bounded polyhedral domain in $\mathbb{R}^d$ ($d \geq 2$) that is triangulated by a conforming triangulation $\mathcal{T}_0$. Let the following eigenvalue problem be given:

$$\begin{aligned} -\mathrm{div}(A\,\nabla u) + c\,u &= \lambda\,u, && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{10}$$

Let $n \geq 1$ be given which will later specify the polynomial degree of the ansatz functions. We assume that the following properties on the coefficients hold:

1. $A : \Omega \to \mathbb{R}^d$ is piecewise polynomial of degree n over $\mathcal{T}_0$ and is symmetric positive definite such that for $0 < a_* \leq a^*$

$$a_*|\xi|^2 \leq \xi^T A(x)\,\xi \leq a^*|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d,\ x \in \Omega.$$

2. $c \in L^\infty(\Omega)$ is non-negative and piecewise polynomial of degree $\max\{n-2, 0\}$ over $\mathcal{T}_0$.

The restriction on the coefficients implies that there occurs no oscillation in the coefficients, see Section 3.2. In the abstract setting of the weak formulation, Equation (10) defines an operator L mapping $V = H_0^1(\Omega)$ into its dual. Furthermore the Hilbert space H is given by $L^2(\Omega)$. The energy norm induces by L is equivalent to the H^1-semi-norm. Generalizations to more general elliptic problems are possible.

3.2 Adaptive finite element methods

In the following we will use the adaptive finite element method described in [4] to approximately solve the boundary value problems

$$\begin{aligned} -\mathrm{div}(A\,\nabla v) + c\,u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{11}$$

corresponding to the eigenvalue problem (10) which have to be solved in the course of the inexact inverse iteration. Here the right hand side f is an element of $H_0^1(\Omega)$.

In the following we briefly describe the this method. Given an initial triangulation and a right hand side the algorithm cycles through

Solve $\rightarrow$ **Estimate** $\rightarrow$ **Mark** $\rightarrow$ **Refine**

and generates a sequence of triangulations and corresponding approximations that converges to the true solution.

Before going into detail first the refinement framework is described. For the refinement shape regular bisection is chosen, which is the newest vertex bisection in two dimensions, see also [4, Section 2.2]. All conforming refinements of some conforming discretization $\hat{\mathcal{T}}$ are denoted by $\mathbb{T}(\hat{\mathcal{T}})$. Given an initial triangulation $\mathcal{T}_0$ the algorithm therefore generates a sequence $\mathcal{T}_0, \mathcal{T}_1, \dots$ of conforming triangulations $\mathcal{T}_k \in \mathbb{T}(\mathcal{T}_0)$. By $\sharp\mathcal{T}$ we denote the number of triangles in the triangulation $\mathcal{T}$.

As ansatz spaces we choose finite elements of degree n, where the n corresponds to the restrictions in the coefficients of the eigenvalue problem (10). More precisely given a conforming triangulation $\mathcal{T}$ the ansatz space is given by

$$V_n(\mathcal{T}) = \left\{ v \in V \mid \; v|_T \in \mathbb{P}_n, \; T \in \mathcal{T} \right\}$$

where $\mathbb{P}_n$ is the set of all polynomials up to degree n.

The algorithm can be described as following. The occurring linear systems stemming from the Galerkin discretization are assumed to be solved exactly in the routine **Solve**. After that an efficient and reliable error estimator based on the residual is calculated in **Estimate**. **Mark** chooses a set of triangles which shall be refined in the next step. The routine relies on the Dörfler marking and an appropriately chosen marking parameter θ. In the last step the marked elements are refined by shape regular bisection in **Refine** and further elements are added in order to ensure conformity.

Finally if one adds an appropriate stopping criterion based on the error estimator, compare for example [15], one arrives at an algorithm

$$[u, \mathcal{T}] = \mathbf{AFEM}[f, \hat{\mathcal{T}}, \epsilon].$$

Given a right hand side $f \in L^2(\Omega)$, an initial triangulation $\hat{\mathcal{T}}$ and a target tolerance ε the algorithm computes an approximation u to the exact solution $v = \mathsf{L}^{-1} f$ such that $|||u - v||| \leq \varepsilon$, where $u \in V_n(\mathcal{T})$ and $\mathcal{T}$ is a refinement of $\hat{\mathcal{T}}$.

For the realization of the algorithm **AFEM** one may also take the algorithm described in [15]. For that purpose one has to generalize the algorithm to higher dimensions and more general problems. However following the hints in the introduction of [15] one sees that the error estimators given in [4] suffice for a generalization. Now instead of finding an approximation of the right hand side f by piecewise constants we have to allow piecewise polynomials of degree $2n - 2$. In contrast to the algorithm given in [4] one does not have to solve the intermediate linear systems exactly.

Next we introduce approximation spaces. For that purpose let $\hat{\mathcal{T}}$ be a conforming triangulation of Ω. For $v \in V$ and $N \geq 0$ define by

$$\sigma_N(v, \hat{\mathcal{T}}) = \inf_{\substack{\mathcal{T} \in \mathbb{T}(\hat{\mathcal{T}}) \\ \sharp\mathcal{T} - \sharp\hat{\mathcal{T}} \leq N}} \inf_{u \in V_n(\mathcal{T})} |||u - v|||$$

the error of the best N-term approximation of v with respect to the initial triangulation $\hat{\mathcal{T}}$. Here $\sharp\mathcal{T}$ denotes the number of triangles in a triangulation $\mathcal{T}$. For $s > 0$ define the space

$$\mathcal{A}^s_{\hat{\mathcal{T}}} = \left\{ v \in V \mid |v|_{\mathcal{A}^s_{\hat{\mathcal{T}}}} < \infty \right\}, \quad |v|_{\mathcal{A}^s_{\hat{\mathcal{T}}}} = \sup_{N \geq 1} N^s \sigma_N(v, \hat{\mathcal{T}})$$

that is the set of all functions which can be approximated at a rate N^{-s} at a conforming triangulation with N more triangles than the initial triangulation $\hat{\mathcal{T}}$.

Furthermore assume that the eigenfunctions corresponding to the lowest eigenvalue λ_1 are approximable at a given rate $s > 0$, i.e.

$$|\mathcal{E}_1|_{\mathcal{A}^s_{\mathcal{T}_0}} = \sup \left\{ |v|_{\mathcal{A}^s_{\mathcal{T}_0}} \,\middle|\, v \in \mathcal{E}_1,\ |||v||| = 1 \right\} < \infty. \tag{12}$$

3.3 Optimality of the inexact inverse iteration coupled with AFEM

In this section we show that the adaptive finite element method fits into the framework of the Subsection 2.3 and also fulfills Assumptions 1-4.

First we identify the ansatz spaces by setting $U_0 = V_n(\mathcal{T}_0)$, where $\mathcal{T}_0$ is the initial triangulation and

$$\mathbb{F}(U_0) = \left\{ V_n(\mathcal{T}) \,\middle|\, \mathcal{T} \in \mathbb{T}(\mathcal{T}_0) \right\}.$$

As a direct consequence of the nestedness of the triangulation Assumption 2 follows. Furthermore the approximation classes coincide, which directly follows from Equation (12) and hence Assumption 1 is fulfilled.

Next we study the regularity of the operator L in the sense of Assumption 3.

Theorem 3. *Let $f \in H_0^1(\Omega)$ and v be the solution of $\mathsf{L}v = f$, where the operator L and the domain Ω satisfy the assumptions of Section 3.1. Then there exits a $s' > 0$ such that*

$$|v|_{\mathcal{A}^{s'}_{\mathcal{T}_0}} \lesssim \|f\|_{H_0^1(\Omega)}.$$

The constant only depends on the initial triangulation $\mathcal{T}_0$.

Proof. The regularity theorem [13] shows that there exits a constant $\bar{s} > 0$ such that $v \in H^{1+\bar{s}}(\Omega)$. Now uniform refinement strategies suffice to show the membership of v in the approximation space $\mathcal{A}^{s'}_{\mathcal{T}_0}$ with $s' = \bar{s}/d$, see for example [2]. □

It remains to proof the optimality of the adaptive finite element method in the sense of Assumption 3. This assumption is indeed a variant of the optimality result in [4], which we will discuss in the following.

For the formulation of this result not only the solution has to be in some approximation space but also the right hand side. For that reason define for $s > 0$ the approximation class

$$\bar{\mathcal{A}}^s_{\hat{\mathcal{T}}} = \big\{ f \in H \;\big|\; |f|_{\bar{\mathcal{A}}^s_{\hat{\mathcal{T}}}} < \infty \big\}, \quad |f|_{\bar{\mathcal{A}}^s_{\hat{\mathcal{T}}}} = \sup_{N \geq 1} N^s \bar{\sigma}_N(f, \hat{\mathcal{T}})$$

where the error of the best N-term approximation is given by

$$\bar{\sigma}_N(f, \hat{\mathcal{T}}) = \inf_{\substack{\mathcal{T} \in \mathbb{T}(\hat{\mathcal{T}}) \\ \sharp\mathcal{T} - \sharp\hat{\mathcal{T}} \leq N}} \operatorname{osc}(f, \mathcal{T}),$$
$$\operatorname{osc}(f, \mathcal{T}) = \inf \Big\{ \|h_{\mathcal{T}}(f - g)\| \;\Big|\; g\big|_T \in \mathbb{P}_{2n-2},\ T \in \mathcal{T} \Big\},$$

where $h_{\mathcal{T}}$ is the local mesh width of the triangulation $\mathcal{T}$. The term $\operatorname{osc}(f, \mathcal{T})$ is the oscillation of f with respect to the triangulation $\mathcal{T}$.

The following optimality result directly follows from Theorem 5.11 and Lemma 5.3 of [4].

Theorem 4. *Let $\hat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}_0)$ be an initial triangulation and $f \in L^2(\Omega)$. Denote by v the solution of $\mathsf{L}v = f$. Suppose that v is an element of $\mathcal{A}^s_{\hat{\mathcal{T}}}$ and f an element of $\bar{\mathcal{A}}^s_{\hat{\mathcal{T}}}$ with $s > 0$. Then for appropriately chosen marking parameter the algorithm* **AFEM** *outputs a partition $\mathcal{T}$ such that*

$$\sharp\mathcal{T} - \sharp\hat{\mathcal{T}} \lesssim |v|^{1/s}_{\mathcal{A}^s_{\hat{\mathcal{T}}}} \varepsilon^{-1/s} + |f|^{1/s}_{\bar{\mathcal{A}}^s_{\hat{\mathcal{T}}}} \varepsilon^{-1/s},$$

where the constant only depends on the triangulation $\mathcal{T}_0$ and s.

In the following we prove a variant of the above theorem which will enable us to show Assumption 4. The approximability of the right hand side will be treated afterwards.

Theorem 5. *Let the assumptions of Theorem 4 be fulfilled, but now assume that the solution v can be decomposed as*

$$v = v_1 + v_2, \quad v_1 \in \mathcal{A}^s_{\hat{\mathcal{T}}},\ v_2 \in \mathcal{A}^{s'}_{\hat{\mathcal{T}}}$$

where $s, s' > 0$. Then the size of the output triangulation $\mathcal{T}$ can be bounded by

$$\sharp\mathcal{T} - \sharp\hat{\mathcal{T}} \lesssim |v_1|^{1/s}_{\mathcal{A}^s_{\hat{\mathcal{T}}}} \varepsilon^{-1/s} + |f|^{1/s}_{\bar{\mathcal{A}}^s_{\hat{\mathcal{T}}}} \varepsilon^{-1/s} + |v_2|^{1/s'}_{\mathcal{A}^{s'}_{\hat{\mathcal{T}}}} \varepsilon^{-1/s'},$$

where the constant only depends on the initial triangulation $\mathcal{T}_0$ and s.

Proof. In order to show the assertion one has to analyze the proof of optimality in [4], in particular Theorem 5.11 and Lemma 5.10. For that purpose denote by $\mathcal{T}_k$ the intermediate triangulations produced by the **AFEM** cycle and w_k the intermediate approximations. Analogously to Lemma 5.10 in [4] we can bound the number of marked elements $\mathcal{M}_k$ in the k-th step by

$$\sharp\mathcal{M}_k \lesssim |v_1|^{1/s}_{\mathcal{A}^s_{\hat{\mathcal{T}}}} \varepsilon_k^{-1/s} + |f|^{1/s}_{\bar{\mathcal{A}}^s_{\hat{\mathcal{T}}}} \varepsilon_k^{-1/s} + |v_2|^{1/s'}_{\mathcal{A}^{s'}_{\hat{\mathcal{T}}}} \varepsilon_k^{-1/s'} \tag{13}$$

in our case. Here $\varepsilon_k^2 \lesssim |||v - w_k|||^2 + \mathrm{osc}^2(f, \mathcal{T}_k)$ is an appropriate multiple of the total error. This can be shown just by following the original proof. In order to bound the size of the optimal refinement $\mathcal{T}_\varepsilon$ of $\hat{\mathcal{T}}$ leading to a total error ε_k one uses Lemma 5.3 and the overlay property, Lemma 3.7, of [4]. Now with this result at hand one can proof quasi-optimality just like in Theorem 5.11 of [4]. The proof of this result only relies on the fact that the estimate of the number of marked elements exhibits a geometrical behavior with respect to ε which is the case the inequality (13). □

Corollary 1. *Let the assumptions of Theorem 5 be fulfilled and let the right hand side f be a normalized element of $V_n(\hat{\mathcal{T}})$. Then for every $\varepsilon > 0$ the output partition satisfies*

$$\sharp\mathcal{T} - \sharp\hat{\mathcal{T}} \lesssim |v_1|^{1/s}_{\mathcal{A}^s_{\hat{\mathcal{T}}}} \varepsilon^{-1/s} + |v_2|^{1/s'}_{\mathcal{A}^{s'}_{\hat{\mathcal{T}}}} \varepsilon^{-1/s'},$$

where the constant is independent of $\hat{\mathcal{T}}$ and ε. In the case $n = 1$, i.e. linear finite elements, the result is valid for $s \leq 1/d$.

Proof. As an element of $V_n(\hat{\mathcal{T}})$ the right hand side f is piecewise polynomial of degree n on each triangle of the triangulation $\hat{\mathcal{T}}$. Therefore if $n > 1$ the oscillation $\mathrm{osc}(f, \hat{\mathcal{T}})$ is zero, hence $|f|_{\bar{\mathcal{A}}^s_{\hat{\mathcal{T}}}} = 0$ for all $s > 0$. In the case of linear finite elements, i.e. $n = 1$, uniform refinement leads to an approximation rate of $1/d$ since $f \in H^1_0$. Therefore $|f|_{\bar{\mathcal{A}}^s_{\hat{\mathcal{T}}}} \lesssim |||f||| = 1$ for $s \leq 1/d$. Finally Theorem 5 gives the assertion. □

The above result shows that Assumption 4 is fulfilled. As a consequence the inexact inverse iteration coupled with the quasi-optimal adaptive finite element method described in [4] is again quasi-optimal. Moreover note that in the course of the iteration no coarsening was needed.

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References

[1] Berns-Müller, J., Graham, I.G., Spence, A.: Inexact inverse iteration for symmetric matrices. Linear Algebra Appl. **416**(2-3), 389–413 (2006)

[2] Brenner, S.C., Scott, R.L.: The mathematical theory of finite element methods, *Texts in Applied Mathematics*, vol. 15, 3 edn. Springer, New York (2008)

[3] Carstensen, C., Gedicke, J.: An adaptive finite element eigenvalue solver of quasi-optimal computational complexity. Tech. rep., Matheon (2009)

[4] Cascon, J., Kreuzer, C., Nochetto, R.H., Siebert, K.G.: Quasi-optimal convergence rate for an adaptive finite element method. SIAM J. Numer. Anal. **46**(5), 2524–2550 (2008)

[5] Cohen, A., Dahmen, W., DeVore, R.: Adaptive wavelet methods for elliptic operator equations: Convergence rates. Math. Comput. **70**(233), 27–75 (2001)

[6] Dai, X., Xu, J., Zhou, A.: Convergence and optimal complexity of adaptive finite element eigenvalue computations. Numer. Math. **110**(3), 313–355 (2008)

[7] DeVore, R.A.: Nonlinear approximation. Acta Numerica **7**, 51–150 (1998)

[8] Dörfler, W.: A convergent adaptive algorithm for poisson's equation. SIAM J. Numer. Anal. **33**(3), 1106–1124 (1996)

[9] Gantumur, T., Harbrecht, H., Stevenson, R.: An optimal adaptive wavelet method without coarsening of the iterands. Math. Comput. **76**(258), 615–629 (2007)

[10] Garau, E.M., Morin, P., Zuppa, C.: Convergence of adaptive finite element methods for eigenvalue problems. Math. Models Methods Appl. Sci. **19**(5), 721–747 (2009)

[11] Giani, S., Graham, I.G.: A convergent adaptive method for elliptic eigenvalue problems. SIAM J. Numer. Anal. **47**(2), 1067–1091 (2009)

[12] Grubišić, L., Ovall, J.S.: On estimators for eigenvalue/eigenvector approximations. Math. Comp. **78**, 739–770 (2009)

[13] Jochmann, F.: An H^s-regularity result for the gradient of solutions to elliptic equations with mixed boundary conditions. J. Math. Anal. Appl. **238**(2), 429–450 (1999)

[14] Rohwedder, T., Schneider, R., Zeiser, A.: Perturbed preconditioned inverse iteration for operator eigenvalue problems with applications to adaptive wavelet discretization. To appear in Adv. Comp. Math. (2010)

[15] Stevenson, R.: Optimality of a standard adaptive finite element method. Found. Comput. Math. **7**(2), 245–269 (2007)

[16] Yuan, J.: An adaptive inverse iteration FEM for the inhomogeneous dielectric waveguides. J. Comput. Math. **25**(2), 169–184 (2007)

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Reports

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[57] A. Zeiser. On the Optimality of the Inexact Inverse Iteration Coupled with Adaptive Finite Element Methods. Preprint 57, DFG-SPP 1324, July 2010.