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„Extraktion quantifizierbarer Information aus komplexen Systemen”

Best m-Term Approximation and Sobolev-Besov Spaces of Dominating Mixed Smoothness - the Case of Compact Embeddings

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Best m-Term Approximation and Sobolev-Besov Spaces of Dominating Mixed Smoothness – the Case of Compact Embeddings

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Abstract

We shall investigate the asymptotic behaviour of the widths of best m-term approximation with respect to tensor products of Sobolev as well as Besov spaces in case of compact embeddings. Our approach leads to final assertions. In addition we compare best m-term approximation with optimal linear approximation and entropy numbers.

1 Introduction

Let $\Phi := (\psi_j)_j$ denote a tensor product wavelet system satisfying some additional smoothness, integrability, and moment conditions, see Subsection 3.2.2 for an exact definition. We consider best m-term approximation with respect to Φ, i.e., we investigate the quantity

$$\sigma_m(f,\Phi)_X := \inf \left\{ \| f - \sum_{j\in\Lambda} c_j \, \psi_j \, \|_X : \quad |\Lambda| \le m \,, \quad c_j \in \mathbb{C} \,, \, j \in \Lambda \right\} , \qquad m \in \mathbb{N}_0 \,.$$

Associated widths are defined as follows. Let X and Y be quasi-Banach spaces such that $Y \hookrightarrow X$. Then we define

$$\sigma_m(Y,X,\Phi) := \sup \left\{ \sigma_m(f,\Phi)_X : \quad \| \, f \, |Y\| \le 1 \right\} , \qquad m \in \mathbb{N}_0 \,. \qquad (1)$$

Here we shall choose $X = L_{p_1}([0,1]^d)$. Concerning Y we shall consider two different cases, namely tensor products of Sobolev spaces of fractional order, denoted by $S^{r_0}_{p_0}H([0,1]^d)$, and tensor products of Besov spaces, denoted by $S^{r_0}_{p_0}B([0,1]^d)$. Both spaces are special cases of Besov-Lizorkin-Triebel spaces of dominating mixed smoothness. Throughout the paper we assume

$$r_0 > \max\left(0, \frac{1}{p_0} - \frac{1}{p_1}\right), \tag{2}$$

which guarantees the compactness of the embedding $Y \hookrightarrow X$ with X and Y as above. Whenever $1 < p_1 < \infty$ holds, then our approach leads to a final characterization of the asymptotic behaviour of $\sigma_m(Y, X, \Phi)$. This paper is a direct continuation of [25] where we have investigated the limiting situation $t = 1/p_0 - 1/p_1$, $p_0 < p_1$. A part of our proofs for the case (2) can be directly reduced to the proofs given in [25].

Concerning the wavelet system Φ we wish to remark the following. As mentioned above, an exact definition will be given in (20). When we deal with the widths $\sigma_m(Y, X, \Phi)$ it is always assumed that Y and X allow a characterization by means of the same system Φ, see Propositions 3, 4 for sufficient conditions.

The paper is organized as follows. In Section 2 we state and comment on our main results. Here we also compare best m-term approximation with approximation numbers (optimal linear approximation) and entropy numbers. The next section is devoted to some basics in the theory of Besov and Lizorkin-Triebel spaces of dominating mixed smoothness. Section 4 has some preparatory character. On the one hand we recall some basics of approximation spaces with respect to sequence spaces and on the other hand we derive some assertions on real interpolation of sequence spaces related to function spaces of dominating mixed smoothness. All proofs will be collected in Sections 5 and 6. In Section 5 we study the behaviour of best m-term approximation with respect to sequence spaces associated to spaces of dominating mixed smoothness. In Section 6 we transfer the results from sequence spaces to function spaces of dominating mixed smoothness. This will be done by means of a discrete wavelet transform, see Subsection 3.2.2. Here we allow a greater generality for X and Y as in Section 2. In particular, we shall prove in Cor. 6 below

$$\sigma_m\left(S^{r_0}_{p_0,q_0}B(\Omega), S^{r_1}_{p_1,q_1}B(\Omega), \Phi_\Omega\right) \asymp m^{-r_0+r_1} (\log m)^{(d-1)\left(r_0 - r_1 - \frac{1}{q_0} + \frac{1}{q_1}\right)_+}, \quad m \geq 2,$$

if

$$r_0 - r_1 > \max\left(0, \frac{1}{p_0} - \frac{1}{p_1}\right).$$

The latter condition is the necessary and sufficient for the compactness of the embedding $S^{r_0}_{p_0,q_0}B(\Omega) \hookrightarrow S^{r_1}_{p_1,q_1}B(\Omega)$. Here Ω is an open, nontrivial and bounded subset of $\mathbb{R}^d$, $d \geq 2$.

Notation

As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the integers and $\mathbb{R}$ the real numbers. For a real number a we put $a_+ := \max(a, 0)$. If $j \in \mathbb{N}_0^d$, i.e., if $j = (j_1, \ldots, j_d)$,

$j_\ell \in \mathbb{N}_0$, $\ell = 1, \ldots, d$, then we put

$$|j|_1 := j_1 + \ldots + j_d \,.$$

If X and Y are two quasi-Banach spaces, then the symbol $Y \hookrightarrow X$ indicates that the embedding is continuous. As usual, the symbol c denotes positive constants which depend only on the fixed parameters r, p, q and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. Sometimes we will use the symbols "$\lesssim$" and "$\gtrsim$" instead of "$\le$" and "$\ge$", respectively. The meaning of $A \lesssim B$ is given by: there exists a constant $c > 0$ such that $A \le c\,B$. Similarly $\gtrsim$ is defined. The symbol $A \asymp B$ will be used as an abbreviation of $A \lesssim B \lesssim A$. For a discrete set ∇ the symbol $|\nabla|$ denotes the cardinality of this set. Finally, the symbol $\mathcal{I}$ will be used for identity operators.

Tensor products of Besov and Sobolev spaces are investigated in [45], [43] and [44]. General information about Besov and Lizorkin-Triebel spaces of dominating mixed smoothness can be found, e.g., in [1, 3, 4, 5, 41, 40, 52] ($S^r_{p,q}B(\mathbb{R}^d)$, $S^r_{p,q}F(\mathbb{R}^d)$). We will not give definitions here. However, the wavelet characterizations, recalled in Subsection 3.2.2, can be taken as definitions as well. The reader, who is interested in more elementary descriptions of these spaces, e.g., by means of differences, is referred to [1, 41] and [51].

Agreement: We shall deal with function and sequence spaces depending on three parameters r, p, q. If there is no additional restriction given, then it is assumed that $r \in \mathbb{R}$ and $0 < p, q \le \infty$. Furthermore, Ω always denotes an open, nontrivial and bounded subset of $\mathbb{R}^d$. Moreover, if we consider $S^r_{p,q}F(\Omega)$, $S^r_{p,q}F(\mathbb{R}^d)$, $s^r_{p,q}f(\Omega)$, or $s^r_{p,q}f$ then it is always assumed that $p < \infty$.

2 The asymptotic behaviour of the widths of best m-term approximation

Our main interest lies in the asymptotic behaviour of the widths $\sigma_m(Y, X, \Phi)$ for different choices of the spaces X and Y. Tensor products of Sobolev spaces of fractional order are denoted by $S^r_pH(I^d)$ and I is the interval $[0, 1]$. Tensor products of Besov spaces are denoted by $S^r_pB(I^d)$. Of course, $d \in \mathbb{N}$, $d \ge 2$. Definitions are given in Section 3.

2.1 Widths of best m-term approximation on the cube

Since $Y \hookrightarrow X$ is a necessary condition, we recall the needed embedding assertions. Let $1 < p_1 < \infty$. Then we have

$$S^{r_0}_{p_0}Y(I^d) \hookrightarrow L_{p_1}(I^d)\,, \qquad Y \in \{B, H\}\,,$$

if (2) holds. The embedding is always compact, see [52]. Furthermore, if $r_0 = 1/p_0 - 1/p_1 > 0$, then either the spaces are incomparable or the embedding is not compact. The latter case we have dealt with in [25].

Theorem 1. *Let $1 < p_0, p_1 < \infty$ and r_0 as in (2). Let Φ be a wavelet system which satisfies the conditions in Prop. 3 with respect to the spaces $S^{r_0}_{p_0,2}F(\mathbb{R}^d)$ and $S^{0}_{p_1,2}F(\mathbb{R}^d)$. Then we have*

$$\sigma_m\Big(S^{r_0}_{p_0}H([0,1]^d), L_{p_1}([0,1]^d), \Phi\Big) \asymp m^{-r_0}\,(\log m)^{(d-1)r_0} \tag{3}$$

for all $m \geq 2$.

Remark 1. (i) The asymptotic behaviour of $\sigma_m\Big(S^{r_0}_{p_0}H([0,1]^d), L_{p_1}([0,1]^d), \Phi\Big)$ is independent of p_0 and p_1 (up to constants). Let us fix p_1. Then the scale $S^{r_0}_{p_0}H([0,1]^d)$ is monotone in p_0, i.e., we have

$$S^{r_0}_{p_0}H([0,1]^d) \hookrightarrow S^{r_0}_{p}H([0,1]^d) \qquad \text{if} \qquad p_0 > p\,.$$

Looking for the largest space (by keeping the rate $m^{-r_0}\,(\log m)^{(d-1)r_0}$ fixed) we have to choose p_0 as small as possible. Here (2) comes into play. This motivates to study the case $r_0 = 1/p_0 - 1/p_1 > 0$. At least if $1 < p_0 \leq 2 \leq p_1 < \infty$, $p_0 < p_1$, then

$$\sigma_m\Big(S^{r_0}_{p_0}H([0,1]^d), L_{p_1}([0,1]^d), \Phi\Big) \asymp m^{-r_0}\,(\log m)^{(d-1)r_0}\,, \qquad r_0 := \frac{1}{p_0} - \frac{1}{p_1}\,, \tag{4}$$

holds for all $m \geq 2$, see [25].
(ii) Thm. 1 is the nonperiodic analog of a result of Temlyakov [48, Thm. 3.3]. However, in [48] (3) is proved under an additional restriction on r_0 which we could avoid here. The main reason for this progress is given by the fact that we have been able to deal with the limiting situation $r_0 = 1/p_0 - 1/p_1 > 0$. By combining a "standard estimate" for high smoothness, see Cor. 2 below, with our knowledge about the behaviour of σ_m in the limiting situation we closed this gap with tools from interpolation theory.
(iii) Oswald [32] has studied best m-term approximation with respect to the tensor product Haar system in case of Hilbert spaces, i.e. $p_0 = p_1 = 2$. More precisely, he considered the behaviour of

$$\sigma_m\Big(S^{r_0}_{2}H([0,1]^2), H^{r_1}_{2}([0,1]^2), \Phi\Big)\,, \qquad -\frac{1}{2} < r_0 \leq 1\,, \quad -1 < r_1 < \frac{1}{2}$$

(with Φ being the tensor product Haar system). Here $H_2^{r_1}([0,1]^2) = B_{2,2}^{r_1}([0,1]^2)$ denotes the isotropic Besov space on $[0,1]^2$. Since $S_2^{r_0}H([0,1]^2$ can be characterized by the tensor product Haar system if, and only if $-1/2 < r_0 < 1/2$ only the cases $r_1 = 0 < r_0 < 1/2$ corresponds to the situation in Thm. 1.

Theorem 2. *Let $0 < p_0 < \infty$, $1 < p_1 < \infty$ and r_0 as in (2). Let Φ be a wavelet system which satisfies the conditions in Prop. 3 with respect to the spaces $S^{r_0}_{p_0,p_0}F(\mathbb{R}^d)$ and $S^0_{p_1,2}F(\mathbb{R}^d)$. Then we have*

$$\sigma_m\Big(S^{r_0}_{p_0}B([0,1]^d), L_{p_1}([0,1]^d), \Phi\Big) \asymp m^{-r_0}\,(\log m)^{(d-1)(r_0-\frac{1}{p_0}+\frac{1}{2})_+} \qquad (5)$$

for all $m \geq 2$.

Remark 2. (i) In contrast to the situation in Thm. 1 the asymptotic behaviour of σ_m depends on p_0. If p_0 is increasing, then the right-hand side in (5) becomes smaller. However, note in this connection, that the spaces $S^{r_0}_{p_0}B([0,1]^d)$ are incomparable for different p_0, i.e.

$$S^{r_0}_{p_0}B([0,1]^d) \setminus S^{r_0}_{p}B([0,1]^d) \neq \emptyset \qquad \text{and} \qquad S^{r_0}_{p}B([0,1]^d) \setminus S^{r_0}_{p_0}B([0,1]^d) \neq \emptyset$$

if $p_0 \neq p$.
(ii) Under certain additional restrictions the periodic counterpart of (5) has been proved by Dinh Dung [18, 19, 20]. He applied a similar discretization argument to reformulate the approximation problem as one for sequence spaces. Afterwards, he explicitly constructed continuous mappings which yield almost optimal approximants. On the one hand the cases treated in his work complement those ones in our explicit constructions, on the other hand his arguments are restricted to Banach spaces. By employing monotonicity arguments as in Rem. 1(i) it becomes clear that p_0 small, e.g., $p_0 < 1$, is of some importance.
(iii) Temlyakov [47], Romanyuk [37] and Bazarkhanov [6] have studied best m-term approximation in a slightly different context. Under certain restrictions on the parameters they determined the asymptotic behaviour of the quantities

$$\sigma_m\Big(S^{r_0}_{p_0,q_0}B(\mathbb{T}^d), L_{p_1}(\mathbb{T}^d), \mathcal{D}\Big)$$

where $\mathbb{T}^d$ is the d-dimensional torus and the dictionary $\mathcal{D}$ is given by the trigonometric system. As it is known from the isotropic situation, see [17], best m-term approximation with respect to this dictionary is not always of the same quality as in case of appropriate wavelet systems Φ.

2.2 Approximation and entropy numbers – a comparison with best m-term approximation

We would like to answer the question under which circumstances best m-term approximation performs essentially better than the optimal linear approximation. In addition we compare the widths σ_m with entropy numbers.

2.2.1 Approximation numbers of embeddings

Denote by $\mathcal{L}(Y,X)$ the set of all bounded linear operators $L : Y \to X$. As usual,

$$\|L\|_{\mathcal{L}(Y,X)} := \| \, L \, | Y \to X\| := \sup_{\|f\|_Y \le 1} \| \, Lf\|_X \, .$$

Let $Y \hookrightarrow X$. Then we define the n-th approximation number of the embedding operator $\mathcal{I}$ as

$$a_n(\mathcal{I}, Y, X) := \inf \Big\{ \| \, \mathcal{I} - L_n \, | \mathcal{L}(Y,X)\| : \quad \operatorname{rank} L_n < n \Big\} \, .$$

We have to subdivide our consideration into the cases $p_0 \le p_1$ and $p_0 > p_1$.

Proposition 1. *Let* $\max(1,p_0) < p_1 < \infty$ *and let* r_0 *be as in (2).*
(i) *Let* $1 < p_0 < \infty$*. Then there exists a positive number* ε *s.t.*

$$a_m(\mathcal{I}, S^{r_0}_{p_0}H([0,1]^d), L_{p_1}([0,1]^d)) \gtrsim m^{-r_0+\varepsilon} \, , \qquad m \in \mathbb{N} \, ,$$

holds.
(ii) *There exists a positive number* ε *s.t.*

$$a_m(\mathcal{I}, S^{r_0}_{p_0}B([0,1]^d), L_{p_1}([0,1]^d)) \gtrsim m^{-r_0+\varepsilon} \, , \qquad m \in \mathbb{N} \, ,$$

holds.

Remark 3. (i) Under the conditions of Prop. 1 we obviously have

$$\lim_{m\to\infty} \frac{\sigma_m(S^{r_0}_{p_0}Y([0,1]^d), L_{p_1}([0,1]^d))}{a_m(\mathcal{I}, S^{r_0}_{p_0}B([0,1]^d), L_{p_1}([0,1]^d))} = 0 \, , \qquad Y \in \{H, B\}.$$

(ii) The result is an easy consequence of the well-known behaviour of $a_m(\mathcal{I}, B^{r_0}_{p_0,p_0}([0,1]^d), L_{p_1}([0,1]^d))$, see [53] and the references given there. From this one can derive immediately more information on ε. For all details we refer to [26].

Proposition 2. *Let* $r_0 > 0$ *and* $1 < p_1 \le p_0 < \infty$*.*
(i) *Then*

$$a_m(\mathcal{I}, S^{r_0}_{p_0}H([0,1]^d), L_{p_1}([0,1]^d)) \asymp m^{-r_0} \, (\log m)^{(d-1)r_0} \, , \qquad m \in \mathbb{N} \, , \qquad (6)$$

holds.

(ii) *Let* $2 \le p_0 < \infty$. *Then*

$$a_m(\mathcal{I}, S^{r_0}_{p_0}B([0,1]^d), L_{p_1}([0,1]^d)) \asymp m^{-r_0} (\log m)^{(d-1)(r_0-\frac{1}{p_0}+\frac{1}{2})}, \qquad m \in \mathbb{N}, \qquad (7)$$

holds.

Remark 4. (i) This time, it means, in all cases listed in Prop. 2, we have

$$\frac{\sigma_m(S^{r_0}_{p_0}Y([0,1]^d), L_{p_1}([0,1]^d))}{a_m(\mathcal{I}, S^{r_0}_{p_0}B([0,1]^d), L_{p_1}([0,1]^d))} \asymp 1, \qquad Y \in \{H, B\}.$$

(ii) Furthermore, under the given restrictions in Prop. 2, an optimal linear approximation (of rank m) is obtained by approximating from an adapted hyperbolic cross. In fact, one can choose

$$P_n f := \sum_{|j|\le n} \sum_{k \in \nabla_j} \langle f, \psi_{j,k} \rangle \, \psi_{j,k}$$

where the connection between n and m is given by $m = \operatorname{rank} P_n \asymp n^{d-1}\, 2^n$. For details we refer to [26].

(iii) We did not find a convenient reference for approximation numbers in the nonperiodic context. In the periodic situation much more is known about the behaviour of the approximation numbers, but also here also without having a complete knowledge. We refer to the papers of Galeev [23] and Romanyuk [35, 36, 38] and the references given there. Of course, one can transfer some results from the periodic case to the nonperiodic one. This is, however, connected with some technical difficulties. So, it seems to be easier to give a self-contained treatment of the nonperiodic case which will be done in [26].

2.2.2 Entropy numbers

Another well-known method to measure the degree of compactness are entropy numbers. It seems that entropy numbers are closer to the widths of best m-term approximation than all s-numbers, see, e.g., [11] or [53], to which the approximation numbers belong to. For the definition and properties of these quantities we refer to [11] and [21].

Following [52] it holds

$$\begin{aligned} e_m\Big(\mathcal{I}, S^{r_0}_{p_0}Y([0,1]^d), L_{p_1}([0,1]^d)\Big) &\asymp m^{-r_0} (\log m)^{(d-1)(r_0-\frac{1}{p_0}+\frac{1}{2})_+} \\ &\asymp \sigma_m\Big(S^{r_0}_{p_0}Y([0,1]^d), L_{p_1}([0,1]^d), \Phi\Big), \qquad m \ge 2, \end{aligned}$$

$Y \in \{H, B\}$, if $r_0 > 1/p_0 - 1/2$. If $1/p_0 - 1/p_1 < r_0 \le 1/p_0 - 1/2$, then there are gaps between the estimates from below and from above. We also refer in this connection to Belinsky [7], Ding Dung [19] and Temlyakov [46] for corresponding estimates in the periodic situation, see [52, Sect. 4.6] for a detailed comparison.

Remark 5. If $r_0 > 1/p_0 - 1/2$, then the widths of best m-term approximation behave asymptotically like the entropy numbers of the corresponding embedding operator. It has been proved by Temlyakov [47] that, under certain extra conditions, a lower bound for entropy numbers represents a lower bound for the widths of best m-term approximation. However, entropy numbers have the additional nice property that $\lim_{m\to\infty} e_m(\mathcal{I}) = 0$ if, and only if $\mathcal{I}$ is compact. A corresponding result with e_m replaced by σ_m is not true. As mentioned above, if we consider the limiting case $r_0 = 1/p_0 - 1/p_1 > 0$, then either the embedding $S^{r_0}_{p_0}Y([0,1]^d) \to L_{p_1}([0,1]^d)$ does not exist or it is not compact. In the latter case we always have $\lim_{m\to\infty} \sigma_m\big(S^{r_0}_{p_0}Y([0,1]^d), L_{p_1}([0,1]^d)\big) = 0$, see [25].

3 Besov-Lizorkin-Triebel spaces of dominating mixed smoothness

Besov-Lizorkin-Triebel spaces of dominating mixed smoothness are widely investigated, mainly on $\mathbb{R}^d$ or on the d-dimensional torus. We refer to [1], [41], [40], [3, 4, 5] and [52]. The most prominent examples are the Sobolev spaces of dominating mixed smoothness.

3.1 Sobolev spaces of dominating mixed smoothness

Let $1 < p < \infty$ and $r \in \mathbb{N}$. Let $I \subset \mathbb{R}$ be a finite or infinite interval. Then $S^r_pW(I^d)$ is the collection of all functions in $L_p(I^d)$ s.t.

$$\| f\, |S^r_pW(I^d)\| := \sum_{\alpha\le(r,\dots,r)} \| D^\alpha f|L_p(I^d)\| < \infty\,.$$

Here $D^\alpha f$ denotes the distributional derivative of order α of f. The derivative $D^\alpha f$ of the highest order is the mixed one, given by $\alpha = (r,\dots,r)$. This explains the name *Sobolev space of dominating mixed smoothness.*

The connection to tensor product spaces is as follows, see [43]. Let α_p denote the p-nuclear tensor norm, see e.g. [27]. Then

$$S^r_pW(I) = W^r_p(I) \otimes_{\alpha_p} W^r_p(I) \tag{8}$$

and

$$S^r_pW(I^{d+1}) = S^r_pW(I^d) \otimes_{\alpha_p} W^r_p(I) = W^r_p(I) \otimes_{\alpha_p} S^r_pW(I^d)\,. \tag{9}$$

Remark 6. Let us mention that many times $S^r_2W(I^d)$ is also denoted as $H^r_{mix}(I^d)$.

For fractional order $r > 0$ of smoothness one defines $H^r_p(I) := F^r_{p,2}(I)$ and $S^r_pH(I^d) := S^r_{p,2}F(I^d)$, where $F^r_{p,2}(I)$ and $S^r_{p,2}F(I^d)$ are Lizorkin-Triebel spaces. We also use the convention

$$S^r_pH(I^d) = S^r_pW(I^d)\,, \qquad r \in \mathbb{N}_0\,.$$

Then (8) and (9) remain true in the context of fractional order $r > 0$, i.e.

$$S^r_pH(I^2) = H^r_p(I) \otimes_{\alpha_p} H^r_p(I)$$

and

$$S^r_pH(I^{d+1}) = S^r_pH(I^d) \otimes_{\alpha_p} H^r_p(I) = H^r_p(I) \otimes_{\alpha_p} S^r_pH(I^d)\,,$$

see [43].

3.2 Tensor product wavelet systems and Besov-Lizorkin-Triebel spaces of dominating mixed smoothness

As mentioned in the introduction we will not give definitions of the above classes. However, the characterizations by means of tensor product wavelet systems may be taken as a definition. First we introduce sequence spaces related to the characterization of Besov and Triebel-Lizorkin spaces of dominating mixed smoothness in terms of wavelet coefficients.

3.2.1 Sequence spaces

Let $\mathcal{X}$ denote the characteristic function of the interval $[0,1]$. Then we put

$$\mathcal{X}_{j,k}(x) := \mathcal{X}(2^{j_1}x_1 - k_1) \cdots \mathcal{X}(2^{j_d}x_d - k_d)\,, \qquad x = (x_1, \ldots, x_d) \in \mathbb{R}^d\,,$$

where $j = (j_1, \ldots, j_d) \in \mathbb{N}_0^d$ and $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$. In other words, $\mathcal{X}_{j,k}$ is the characteristic function of the cube

$$Q_{j,k} = \big(2^{-j_1}([0,1] + k_1)\big) \times \cdots \times \big(2^{-j_d}([0,1] + k_d)\big)\,.$$

Definition 1. *Let $d > 1$, $0 < q \le \infty$ and $r \in \mathbb{R}$. Let $\nabla = (\nabla_j)_{j \in \mathbb{N}_0^d}$ be a sequence of nontrivial subsets of $\mathbb{Z}^d$.*

(i) *Let $0 < p \le \infty$. Then $s^r_{p,q}b(\nabla)$ consists of all sequences $a = (a_{j,k})_{j,k}$ such that*

$$\big\|\, a \,|s^r_{p,q}b(\nabla)\big\| := \Bigg(\sum_{j \in \mathbb{N}_0^d} 2^{|j|_1\,(r + \frac{1}{2} - \frac{1}{p})q} \Big(\sum_{k \in \nabla_j} |a_{j,k}|^p\Big)^{\frac{q}{p}}\Bigg)^{\frac{1}{q}} < \infty\,. \qquad (10)$$

(ii) *Let $0 < p < \infty$. Then $s^r_{p,q}f(\nabla)$ consists of all sequences $a = (a_{j,k})_{j,k}$ such that*

$$\big\|\, a \,|s^r_{p,q}f(\nabla)\big\| := \Bigg\| \Big(\sum_{j \in \mathbb{N}_0^d} \sum_{k \in \nabla_j} 2^{|j|_1\,(r + \frac{1}{2})q}\, |a_{j,k}|^q\, \mathcal{X}_{j,k}(\,\cdot\,)\Big)^{\frac{1}{q}} \Big| L_p(\mathbb{R}^d)\Bigg\| < \infty\,. \qquad (11)$$

Remark 7.

(i) Many times we shall use the notation $s^r_{p,q}x(\nabla)$, $x \in \{b, f\}$. If, then we always assume $p < \infty$ if $x = f$.

(ii) Two special cases of sequences ∇ are of particular importance. The first one is simply $\nabla_j = \mathbb{Z}^d$ for all j, and we will denote the corresponding spaces by $s^r_{p,q}b$ and $s^r_{p,q}f$, respectively. The second one will be related to function spaces on bounded open (nontrivial) subsets of $\mathbb{R}^d$ and will be discussed below, see Def. 2.

(iii) Obviously we have $s^r_{p,p}b(\nabla) = s^r_{p,p}f(\nabla)$.

3.2.2 Spaces of dominating mixed smoothness

For a definition of spaces of dominating mixed smoothness in Fourier-analytic terms we refer to the monographs [1] and [41]. Characterizations in terms of atoms and wavelets have been given in papers by Bazarkhanov [3, 4, 5] and Vybiral [52]. Here we are going to recall a few results from [52].

Let ϕ denote an univariate scaling function associated with the wavelet ψ such that $\phi, \psi \in C^s(\mathbb{R})$ are compactly supported, the wavelet satisfies the moment condition

$$\int_{-\infty}^{\infty} t^\ell\, \psi(t)\, dt = 0\,, \quad 0 \le \ell < s\,, \tag{12}$$

and the collection of functions, consisting of

$$\begin{aligned} \psi_{0,k}(t) &:= \phi(t-k)\,, \qquad k \in \mathbb{Z}\,, \\ \psi_{j,k}(t) &:= 2^{(j-1)/2}\,\psi(2^{j-1}t - k)\,, \qquad k \in \mathbb{Z}, \quad j \in \mathbb{N}\,, \end{aligned}$$

forms an orthonormal basis for the space $L_2(\mathbb{R})$. Here $s \in \mathbb{N}$ will be chosen later on. Next we need the corresponding tensor product systems. We put

$$\psi_{j,k}(x) := \psi_{j_1,k_1}(x_1) \cdot \ldots \cdot \psi_{j_d,k_d}(x_d)\,, \qquad x = (x_1, \ldots, x_d)\,.$$

Then the tensor product wavelet system Φ is the collection

$$\psi_{j,k}\,, \qquad j \in \mathbb{N}_0^d\,, \quad k \in \mathbb{Z}^d\,. \tag{13}$$

For the following propositions we refer to [52]. To begin with we deal with Lizorkin-Triebel spaces.

Proposition 3. *Let $r \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \le \infty$. If φ, ψ are satisfying the above conditions for sufficiently large $s = s(r, p, q)$ then we have the following: For every $f \in S^r_{p,q}F(\mathbb{R}^d)$, we have*

$$f = \sum_{j \in \mathbb{N}_0^d} \sum_{k \in \mathbb{Z}^d} \langle f,\, \psi_{j,k}\rangle\, \psi_{j,k}\,, \tag{14}$$

convergence in $\mathcal{S}'(\mathbb{R}^d)$ *(and in* $S^r_{p,q}F(\mathbb{R}^d)$ *if* $q<\infty$*), and*

$$\| f\,|S^r_{p,q}F(\mathbb{R}^d)\| \asymp \| (\langle f,\, \psi_{j,k}\rangle)_{j,k}\,|s^r_{p,q}f\| \,. \tag{15}$$

The counterpart in case of Besov spaces reads as follows.

Proposition 4. *Let* $r\in\mathbb{R}$, $0<p\le\infty$ *and* $0<q\le\infty$. *If* φ,ψ *are satisfying the above conditions for sufficiently large* $s=s(r,p,q)$ *then we have the following: For every* $f\in S^r_{p,q}B(\mathbb{R}^d)$*, the identity (14) holds with convergence in* $\mathcal{S}'(\mathbb{R}^d)$ *(and in* $S^r_{p,q}B(\mathbb{R}^d)$ *if* $\max(p,q)<\infty$*), and*

$$\| f\,|S^r_{p,q}B(\mathbb{R}^d)\| \asymp \| (\langle f,\, \psi_{j,k}\rangle)_{j,k}\,|s^r_{p,q}b\| \,. \tag{16}$$

Remark 8. Here we are not interested in optimal conditions with respect to Φ. Propositions 3 and 4 will allow us to translate the major part of our calculations from function spaces to sequence spaces.

3.2.3 Spaces on domains

Let $\Omega\subset\mathbb{R}^d$ be a bounded and open set. Then we define the spaces $S^r_{p,q}F(\Omega)$ and $S^r_{p,q}B(\Omega)$ by restrictions. More exactly, we put

$$\begin{aligned} S^r_{p,q}X(\Omega) &:= \Big\{ f\in D'(\Omega):\quad f=g_{|\Omega}\quad \text{for some } g\in S^r_{p,q}X(\mathbb{R}^d)\Big\} \\ \| f\,|S^r_{p,q}X(\Omega)\| &:= \inf \| g\,|S^r_{p,q}X(\mathbb{R}^d)\| \,, \end{aligned}$$

where the infimum is taken over all $g\in S^r_{p,q}X(\mathbb{R}^d)$ such that $f=g_{|\Omega}$. Here $X\in\{F,B\}$. For our purpose it is enough to observe the following. The univariate scaling function ϕ and the associated wavelet ψ are compactly supported, say

$$\Big(\operatorname{supp}\phi\,\cup\,\operatorname{supp}\psi\Big)\subset[-N,N]$$

for some $N>0$. For given $f\in S^r_{p,q}X(\Omega)$ let $\mathcal{E}f$ denote an extension of f such that

$$\| \mathcal{E}f\,|S^r_{p,q}X(\mathbb{R}^d)\| \le 2\,\| f\,|S^r_{p,q}X(\Omega)\| \le 2\,\| \mathcal{E}f\,|S^r_{p,q}X(\mathbb{R}^d)\| \,.$$

Then

$$\mathcal{E}f=\sum_{j\in\mathbb{N}_0^d}\sum_{k\in\mathbb{Z}^d}\langle \mathcal{E}f,\,\psi_{j,k}\rangle\,\psi_{j,k}\,.$$

Hence, also

$$\mathcal{E}^*f:=\sum_{j\in\mathbb{N}_0^d}\ \sum_{\operatorname{supp}\psi_{j,k}\cap\Omega\neq\emptyset}\langle \mathcal{E}f,\,\psi_{j,k}\rangle\,\psi_{j,k} \tag{17}$$

is an extension of f such that

$$\| \mathcal{E}^*f\,|S^r_{p,q}X(\mathbb{R}^d)\| \asymp \| f\,|S^r_{p,q}X(\Omega)\| \,. \tag{18}$$

Moreover, we have

$$\operatorname{supp} \mathcal{E}^* f \subset \Gamma := \{x \in \mathbb{R}^d : \ \operatorname{dist}(x, \Omega) < 2N\} . \tag{19}$$

For spaces on bounded domains we define Φ to be the collection of all functions $\psi_{j,k}$ such that

$$\Omega \cap \operatorname{supp} \psi_{j,k} \neq \emptyset , \qquad j \in \mathbb{N}_0^d, \ k \in \mathbb{Z}^d . \tag{20}$$

In case we want to distinguish between a wavelet system on $\mathbb{R}^d$ and on Ω we shall use the notation Φ_Ω for the latter one.
Now we are ready to define related sequence spaces.

Definition 2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open (nontrivial) set. Let r, p, q as in Def. 1. Let $(\psi_{j,k})_{j,k}$ satisfy the restrictions in Prop. 3 (f-case) or in Prop. 4 (b-case). Then we define $\nabla = \nabla(\Omega)$ according to (20) by*

$$\nabla_j = \Big\{ k \in \mathbb{Z}^d : \quad \operatorname{supp} \ \psi_{j,k} \cap \Omega \neq \emptyset \Big\} . \tag{21}$$

The corresponding sequence spaces will be denoted by $s^r_{p,q}b(\Omega) := s^r_{p,q}b(\nabla(\Omega))$ and $s^r_{p,q}f(\Omega) := s^r_{p,q}f(\nabla(\Omega))$ respectively.

Remark 9. We want to mention the following fact, which we will make constant use of without always explicitly mentioning it. There exist positive constants $C_1 = C_1(\Omega)$ and $C_2 = C_2(\Omega)$ and an appropriate integer $J = J(\Omega)$, such that

$$C_1 \leq 2^{-|j|_1} \, |\nabla_j(\Omega)| \leq C_2, \qquad |j|_1 \geq J. \tag{22}$$

For simplicity we will always assume $J = 0$.

Lemma 1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open (nontrivial) set. Let $0 < q \leq \infty$.*
(i) *Let $0 < p_0 < p_1 \leq \infty$. Then*

$$s^r_{p_1,q}b(\Omega) \hookrightarrow s^r_{p_0,q}b(\Omega) .$$

(ii) *Let $0 < p_0 < p_1 < \infty$. Then*

$$s^r_{p_1,q}f(\Omega) \hookrightarrow s^r_{p_0,q}f(\Omega) .$$

Proof. *Step 1.* Proof of (ii). Let φ and ψ be the generators of the wavelet system used in Def. 2. We put

$$A := \sup_{x,y \in \operatorname{supp} \varphi} |x - y| \qquad \text{and} \qquad B := \sup_{x,y \in \operatorname{supp} \psi} |x - y| .$$

Let $C := \max(A, B)$. Then, with

$$\Omega^* := \{x \in \mathbb{R}^d : \quad \operatorname{dist}(x, \Omega) \leq C\}$$

we obtain

$$\begin{aligned}
\big\| \, a \, \big| s^r_{p_0,q} f(\Omega) \big\| &= \Bigg\| \Big(\sum_{j\in\mathbb{N}_0^d} \sum_{k\in\nabla_j(\Omega)} 2^{|j|_1 \, (r+\frac{1}{2})q} \, |a_{j,k}|^q \, \mathcal{X}_{j,k}(\,\cdot\,) \Big)^{\frac{1}{q}} \, \Big| L_{p_0}(\Omega^*) \Bigg\| \\
&\le |\Omega^*|^{\frac{1}{p_0}-\frac{1}{p_1}} \Bigg\| \Big(\sum_{j\in\mathbb{N}_0^d} \sum_{k\in\nabla_j(\Omega)} 2^{|j|_1 \, (r+\frac{1}{2})q} \, |a_{j,k}|^q \, \mathcal{X}_{j,k}(\,\cdot\,) \Big)^{\frac{1}{q}} \, \Big| L_{p_1}(\Omega^*) \Bigg\| \\
&= |\Omega^*|^{\frac{1}{p_0}-\frac{1}{p_1}} \, \big\| \, a \, \big| s^r_{p_1,q} f(\Omega) \big\| \, ,
\end{aligned}$$

using Hölder's inequality.

Step 2. Proof of (ii). We use (22). It follows

$$\begin{aligned}
\big\| \, a \, \big| s^r_{p_0,q} b(\Omega) \big\| &= \Bigg(\sum_{j\in\mathbb{N}_0^d} 2^{|j|_1 \, (r+\frac{1}{2}-\frac{1}{p_0})q} \Big(\sum_{k\in\nabla_j(\Omega)} |a_{j,k}|^{p_0} \Big)^{\frac{q}{p_0}} \Bigg)^{\frac{1}{q}} \\
&\lesssim |\Omega^*|^{\frac{1}{p_0}-\frac{1}{p_1}} \, \big\| \, a \, \big| s^r_{p_1,q} b(\Omega) \big\| \, ,
\end{aligned}$$

also by using Hölder's inequality. ■

For later use we also recall the Sobolev-type embedding.

Lemma 2. *Let $\Omega \subset \mathbb{R}^d$ be an open (nontrivial) set. Suppose*

$$r_0 - r_1 > \max\Big(0, \frac{1}{p_0} - \frac{1}{p_1}\Big)$$

Then, with $x, y \in \{b, f\}$, we have the continuous embedding

$$s^{r_0}_{p_0,q_0} x(\Omega) \hookrightarrow s^{r_1}_{p_1,q_1} y(\Omega) \, .$$

Proof. The corresponding result for function spaces is proved in [52]. It can be shifted to the sequence spaces by means of Prop. 3, 4. ■

Remark 10. By γ_p we denote the projective tensor p-norm, see [43]. We define

$$\delta_p := \begin{cases} \alpha_p & \text{if} \quad 1 < p < \infty \, , \\ \gamma_p & \text{if} \quad 0 < p \le 1 \, . \end{cases}$$

Then

$$S^r_{p,p} B(I^2) = B^r_{p,p}(I) \otimes_{\delta_p} B^r_{p,p}(I) \tag{23}$$

and

$$S^r_{p,p} B(I^{d+1}) = S^r_{p,p} B(I^d) \otimes_{\delta_p} B^r_{p,p}(I) = B^r_{p,p}(I) \otimes_{\delta_p} S^r_{p,p} B(I^d) \, , \tag{24}$$

see [43] and [44].

4 Sequence spaces and approximation spaces

In this section, we introduce several of our main tools. There is an abstract theory of approximation spaces with a deep interplay with interpolation theory. This interplay we are going to use for our sequence spaces $s^r_{p,q}b$ and $s^r_{p,q}f$.

4.1 Approximation spaces

Let $\mathcal{D}$ be a subset of the quasi-Banach space X. Then we define

$$\sigma_m\big(a, X, \mathcal{D}\big) := \inf \left\{ \Big\| a - \sum_{j\in\Lambda} c_j\, \psi_j \Big\|_X \ : \ |\Lambda| \le m\,,\ c_j \in \mathbb{C}\,,\ \psi_j \in \mathcal{D}\,,\ j \in \Lambda \right\}\,,$$

Obviously $\sigma_0(a, X, \mathcal{D}) = \|\, a\,\|_X$. We are interested in approximation spaces relative to σ_m. We do not need these spaces in the generality as used in the recent papers of Luther and Almira [29, 30]. For us the following version will be sufficient. Let $\mathcal{A}^{s,t}_q(X, \mathcal{D})$ be the collection of all elements $a \in X$, such that

$$\begin{aligned} &\big\|\, a \,\big|\mathcal{A}^{s,t}_q(X,\mathcal{D})\big\| \\ := &\begin{cases} \Big(\sum\limits_{m=0}^{\infty} \Big[(m+1)^s\big(1+\log(m+1)\big)^t\, \sigma_m\big(a,X,\mathcal{D}\big)\Big]^q \dfrac{1}{m+1}\Big)^{1/q} & \text{if} \quad 0<q<\infty\,, \\ \sup\limits_{m=0,1,\ldots} \ (m+1)^s\big(1+\log(m+1)\big)^t\, \sigma_m\big(a,X,\mathcal{D}\big) & \text{if} \quad q=\infty\,, \end{cases} \end{aligned}$$

where $s > 0$ and $t \in \mathbb{R}$. Later on we shall need some assertions on real interpolation of these scales. For the basics in real interpolation we refer to [9, 49].

Proposition 5. *Let X be a quasi-Banach space and $\mathcal{D}$ a subset of X. Let $0 < \Theta < 1$.*
(i) *Let $0 < u, q \le \infty$, $s > 0$ and $t \in \mathbb{R}$. Then it holds*

$$\big(X, \mathcal{A}^{s,t}_u(X,\mathcal{D})\big)_{\Theta,q} = \mathcal{A}^{\Theta s,\Theta t}_q(X,\mathcal{D})\,. \tag{25}$$

(ii) *Let $0 < u, u_0, u_1 \le \infty$, $s_0, s_1 > 0$, $s_0 \ne s_1$, and $t_0, t_1 \in \mathbb{R}$. Then, with $s := (1-\Theta)\, s_0 + \Theta\, s_1$ and $t := (1-\Theta)\, t_0 + \Theta\, t_1$, it holds*

$$\big(\mathcal{A}^{s_0,t_0}_{u_0}(X,\mathcal{D}), \mathcal{A}^{s_1,t_1}_{u_1}(X,\mathcal{D})\big)_{\Theta,u} = \mathcal{A}^{s,t}_u(X,\mathcal{D})\,. \tag{26}$$

Remark 11. The classical counterparts of these assertions, i.e., the case $t = 0$, are well-known, see e.g. [33, 10] and [16, 15]. The above generalizations (25) and (26) can be found in [29].

At this point we only want to mention two further facts, which are almost trivial but nonetheless important for our later considerations. Let $s > 0$, $t \in \mathbb{R}$, and $0 < u_0 \le u_1 \le \infty$. Then we have the embedding

$$\mathcal{A}^{s,t}_{u_0}(X,\mathcal{D}) \hookrightarrow \mathcal{A}^{s,t}_{u_1}(X,\mathcal{D})$$

(use the monotonicity of σ_m with respect to m and switch to dyadic subsequences). Moreover, if X and Y are two quasi-Banach spaces with $\mathcal{D} \subset X$ and $X \hookrightarrow Y$, then it holds

$$\mathcal{A}^{s,t}_u(X,\mathcal{D}) \hookrightarrow \mathcal{A}^{s,t}_u(Y,\mathcal{D})$$

for all $s > 0$, $t \in \mathbb{R}$, and $0 < u \le \infty$.

For the convenience of the reader we summarize some further assertions in the following lemma.

Lemma 3. *Let X and Y by quasi-Banach spaces s.t. $Y \hookrightarrow X$. Further, let $\mathcal{D} \subset X$. Then the following assertions are equivalent:*

(i) *The Jackson-type inequality*

$$\sigma_m(f,X,\mathcal{D}) \le c\, m^{-s}\big(\log m\big)^{-t}\, \big\|f\big|Y\big\|$$

holds for some $s > 0$, $t \in \mathbb{R}$, and some constant $c > 0$, which is independent of $f \in Y$ and $m \ge 2$.

(ii) *The m-term width satisfies*

$$\sigma_m\big(Y,X,\mathcal{D}\big) \le c\, m^{-s}\big(\log m\big)^{-t}$$

for some $s > 0$, $t \in \mathbb{R}$, and some constant $c > 0$, which is independent of $m \ge 2$.

(iii) *We have the continuous embedding $Y \hookrightarrow \mathcal{A}^{s,t}_\infty\big(X,\mathcal{D}\big)$.*

Later on we shall need another interesting property of these approximation spaces. As one can easily check, we have $\mathcal{D} \subset \mathcal{A}^{s,t}_u(X,\mathcal{D})$ for all parameters s, t and u. Hence, it makes sense also to consider the approximation spaces $\mathcal{A}^{s_1,t_1}_v\big(\mathcal{A}^{s_2,t_2}_u(X,\mathcal{D}),\mathcal{D}\big)$.

Proposition 6. *Let $0 < u, v \le \infty$, $s_1, s_2 > 0$, and $t_1, t_2 \in \mathbb{R}$. Let X be a quasi-Banach space and $\mathcal{D}$ a fixed subset of X. Then we have*

$$\mathcal{A}^{s_1,t_1}_u\big(\mathcal{A}^{s_2,t_2}_v(X,\mathcal{D}),\mathcal{D}\big) = \mathcal{A}^{s_1+s_2,t_1+t_2}_u(X,\mathcal{D}) \tag{27}$$

in the sense of equivalent quasi-norms.

Remark 12. This proposition, known as the *Reiteration theorem*, can be found in its classical version, i.e, $t_1 = t_2 = 0$, in [34]. The general assertion (27) is due to Luther [29].

4.2 Sequence spaces

We need some general assertions on best m-term approximation on the level of sequence spaces. In this connection we concentrate on best m-term approximation with respect to the canonical orthonormal basis of $\ell_2(I)$, where I is a fixed infinite, but countable, index set. We put

$$\mathcal{B} := \{e^j \ : \ j \in I\}\,, \quad e^j := (e^j_k)_k\,, \quad e^j_k := \delta_{j,k}\,,\ j,k \in I.$$

By $\ell_{p,u}(I)$ we denote the Lorentz sequence spaces. Here $\ell_{p,u}(I)$ is the collection of all sequences $a = (a_j)_{j\in I}$, such that

$$\big\| \, a \,\big|\ell_{p,u}(I)\big\| := \Big\| \big(n^{\frac{1}{p}-\frac{1}{u}} a^*_n\big)_{n\in\mathbb{N}} \Big| \ell_u(\mathbb{N})\Big\| < \infty, \qquad 0 < p, u \le \infty\,,$$

where $a^* = (a^*_n)_n$ denotes the non-increasing rearrangement of a. Our point of departure is the following nice result of Pietsch [34, Ex. 1].

Proposition 7. *Let $0 < p_1, u \le \infty$. Let I be a fixed index set. Then $a \in \ell_{p_1}(I)$ belongs to the approximation space $\mathcal{A}^s_u(\ell_{p_1}(I), \mathcal{B})$, if and only if $a \in \ell_{p_0,u}(I)$, where $1/p_0 := s + 1/p_1$. Furthermore,*

$$\big\| \, a \,\big|\mathcal{A}^s_u\big(\ell_{p_1}(I), \mathcal{B}\big)\big\| \asymp \big\| \, a \,\big|\ell_{p_0,u}(I)\big\|\,, \tag{28}$$

where the constants of equivalence do not depend on I.

Based on Prop. 7 it is easy to derive the following, see [25] for details.

Corollary 1. *Let ∇ be as in Definition 1. Let $0 < p_0 < p_1$. Then we have*

$$\mathcal{A}^{\frac{1}{p_0}-\frac{1}{p_1}}_{p_0}\big(s^0_{p_1,p_1}b(\nabla), \mathcal{B}\big) = s^{\frac{1}{p_0}-\frac{1}{p_1}}_{p_0,p_0}b(\nabla)\,,$$

in the sense of equivalent quasi-norms.

4.3 Gagliardo-Nirenberg-type inequalities

So far, we collected results which are parallel to the isotropic case, see [24]. However, there are essential differences between the isotropic and the dominating mixed case. The first example in this direction is given by the following Gagliardo-Nirenberg-type inequalities. Its influence on approximation-theoretical aspects, mainly via interpolation, will be discussed in the subsequent sections.

Here we will use the notation $\ell^r_q(\mathbb{N}^d_0)$ for the quasi-Banach space of all sequences $a = (a_j)_{j\in\mathbb{N}^d_0}$, such that $\big(2^{|j|_1 r} a_j\big)_{j\in\mathbb{N}^d_0} \in \ell_q(\mathbb{N}^d_0)$.

Proposition 8. *Let $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \le \infty$, $r_0, r_1 \in \mathbb{R}$ and $0 < \Theta < 1$. We put $\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$ and $r = (1-\Theta)r_0 + \Theta r_1$. Then the following assertions are equivalent:*

(i) *It holds*

$$\frac{1}{q} \le \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}\,. \tag{29}$$

(ii) *There is some positive constant* c_1*, such that*

$$\left\|a\middle|s^r_{p,q}f\right\| \le c_1 \left\|a\middle|s^{r_0}_{p_0,q_0}f\right\|^{1-\Theta}\left\|a\middle|s^{r_1}_{p_1,q_1}f\right\|^{\Theta} \tag{30}$$

holds for all $a \in s^{r_0}_{p_0,q_0}f \cap s^{r_1}_{p_1,q_1}f$.

(iii) *There is some positive constant* c_2*, such that*

$$\left\|a\middle|s^r_{p,q}b\right\| \le c_2 \left\|a\middle|s^{r_0}_{p_0,q_0}b\right\|^{1-\Theta}\left\|a\middle|s^{r_1}_{p_1,q_1}b\right\|^{\Theta} \tag{31}$$

holds for all $a \in s^{r_0}_{p_0,q_0}b \cap s^{r_1}_{p_1,q_1}b$.

(iv) *There is some positive constant* c_3*, such that*

$$\left\|\lambda\middle|\ell^r_q(\mathbb{N}^d_0)\right\| \le c_3 \left\|\lambda\middle|\ell^{r_0}_{q_0}(\mathbb{N}^d_0)\right\|^{1-\Theta}\left\|\lambda\middle|\ell^{r_1}_{q_1}(\mathbb{N}^d_0)\right\|^{\Theta} \tag{32}$$

holds for all $\lambda \in \ell^{r_0}_{q_0}(\mathbb{N}^d_0) \cap \ell^{r_1}_{q_1}(\mathbb{N}^d_0)$.

Remark 13. *A corresponding result holds for the spaces* $s^r_{p,q}b(\Omega)$ *and* $s^r_{p,q}f(\Omega)$.

Proof. We show two chains of implications, at first (i)$\Longrightarrow$(ii)$\Longrightarrow$(iv)$\Longrightarrow$(i) and then (i)$\Longrightarrow$(iii)$\Longrightarrow$(iv).

Step 1. (i)$\Longrightarrow$(ii) follows with $c_1 = 1$ by using the monotonicity of the ℓ_q-spaces and applying Hölder's inequality twice. Now let $b = (b_j)_{j\in\mathbb{N}^d_0}$ be an arbitrary sequence of complex numbers. Then define a by

$$a_{j,k} = \begin{cases} 2^{-|j|_1/2}b_j, & j \in \mathbb{N}^d_0\,,\ Q_{j,k} \subset [0,1]^d\,, \\ 0, & \text{else.} \end{cases}$$

A simple calculation shows

$$\left\|a\middle|s^r_{p,q}f\right\| = \left\|a\middle|s^r_{p,q}f\right\| = \left\|b\middle|\ell^r_q(\mathbb{N}^d_0)\right\|.$$

Hence (ii)$\Longrightarrow$(iv) and at the same time (iii)$\Longrightarrow$(iv). Finally, consider sequences a^n, defined by

$$(a^n)_{j,k} = \begin{cases} 1, & j \in \mathbb{N}^d_0, j_1 + \cdots + j_d = n\,, \\ 0, & \text{else.} \end{cases}$$

It is known, that

$$S(n,d) := \left|\left\{j \in \mathbb{N}^d_0 : j_1 + \cdots + j_d = n\right\}\right| = \binom{n+d-1}{n} \asymp n^{d-1}\,.$$

This implies $\left\|a^n\middle|\ell^r_q(\mathbb{N}^d_0)\right\| = 2^{rn}S(n,d)^{1/q}$. From (32) we conclude

$$S(n,d)^{1/q} \le c_3 S(n,d)^{\frac{1-\Theta}{q_0}} S(n,d)^{\frac{\Theta}{q_1}}\,,$$

and thus (29) follows.

Step 2. (i)$\Longrightarrow$(iii) with $c_2 = 1$ is again a matter of Hölder's inequality. The implication (iii)$\Longrightarrow$(iv) has been proved in Step 1. ■

Remark 14. *If we assume* (29)*, then the inequalities* (30)–(32) *hold with* $c_1 = c_2 = c_3 = 1$*, and this remains valid for the spaces* $s^r_{p,q}b(\nabla)$ *and* $s^r_{p,q}f(\nabla)$ *for arbitrary* ∇.

4.4 Interpolation of sequence spaces

In the sequel, we shall need interpolation results for the spaces $s^r_{p,q}b$ and $s^r_{p,q}f$ and their respective variants. Although more precise statements would be possible in an analogous way as for the isotropic function spaces (see [49, 50]), we are mainly interested in embeddings into $(\,.\,,.)_{\Theta,\infty}$-interpolation spaces. There the Gagliardo-Nirenberg-type inequalities (Prop. 8) come into play.

First, we have a look at the dual spaces of $s^r_{p,q}b(\nabla)$ and $s^r_{p,q}f(\nabla)$.

Lemma 4. *Let $1 \le p, q < \infty$ and $r \in \mathbb{R}$. Then*

$$\big(s^r_{p,q}x(\nabla)\big)' = s^{-r}_{p',q'}x(\nabla)\,, \qquad x \in \{b, f\}\,,$$

where $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$ ($p>1$ if $x=f$). More precisely, there is a canonical isomorphism $T : s^{-r}_{p',q'}x(\nabla) \longrightarrow \big(s^r_{p,q}x(\nabla)\big)'$, which admits the representation

$$\big(Ta\big)(b) = \sum_{j\in\mathbb{N}_0^d}\sum_{k\in\nabla_j} a_{j,k}b_{j,k} = \int_{\mathbb{R}^d}\sum_{j\in\mathbb{N}_0^d}\sum_{k\in\nabla_j} 2^{|j|}a_{j,k}b_{j,k}\mathcal{X}_{j,k}(x)dx \tag{33}$$

for $a \in s^{-r}_{p',q'}x(\nabla)$ and $b \in s^r_{p,q}x(\nabla)$.

Proof. A direct proof for the Besov-type sequence spaces is based on the assertion

$$\big(\ell_q(I, \ell_p(J))\big)' = \ell_{q'}(I, \ell_{p'}(J))\,.$$

Here I denotes an arbitrary nonempty, countable index set, $J = (J_i)_{i\in I}$ be a family of nonempty, countable index sets , and the corresponding norm is given by

$$\big\|a\big|\ell_q\big(I, \ell_p(J)\big)\big\| := \left(\sum_{i\in I}\left(\sum_{j\in J_i}|a_{i,j}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}.$$

This is to be understood via the same isomorphism T as in (33), which is even an isometry in this case. The proof of this mapping being an isometry follows along the lines of the proof of the well-known result $\big(\ell_p(I)\big)' = \ell_{p'}(I)$. For the f-spaces one generalizes the proof of Frazier and Jawerth in [22], given there in the isotropic situation. One needs only one modification. Instead of using the classical Hardy-Littlewood maximal operator one has to use an iterated maximal operator, see [2] for the related maximal inequality. ■

Next we recall the following assertion in interpolation theory, see, e.g., [8, Proposition 5.2.10].

Lemma 5. *Let $\{X_0, X_1\}$ be an interpolation couple of Banach spaces, and let X be an intermediate space. Furthermore, let $0 < \Theta < 1$. Then the embedding*

$$\big(X_0, X_1\big)_{\Theta,1} \hookrightarrow X \hookrightarrow X_0 + X_1$$

holds if, and only if, for some constant $c>0$ the estimate

$$\big\|f|X\big\| \le c\,\big\|f|X_0\big\|^{1-\Theta}\big\|f|X_1\big\|^{\Theta} \tag{34}$$

is fulfilled for all $f \in X_0 \cap X_1$.

Now we are able to present our interpolation results.

Theorem 3. *Let $0 < p_0 < p_1 \le \infty$, $0 < q, q_0, q_1 \le \infty$ and $r_0, r_1 \in \mathbb{R}$. Moreover, let $0 < \Theta < 1$,*

$$\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}\,, \qquad r := (1-\Theta)r_0 + \Theta r_1\,, \qquad \text{and} \qquad \frac{1}{q} \ge \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}\,. \tag{35}$$

Then we have the embedding

$$s^{r}_{p,q}x(\nabla) \hookrightarrow \Big(s^{r_0}_{p_0,q_0}x(\nabla), s^{r_1}_{p_1,q_1}x(\nabla)\Big)_{\Theta,\infty}\,, \qquad x \in \{b,f\}\,. \tag{36}$$

Thereby $p_0, p_1 < \infty$ if $x = f$.

Proof. A similar interpolation assertion in the isotropic situation has been proved in [24]. Since that argumentation carries over without essential changes, we shall be brief. By the monotonicity of the ℓ_q-norms, it is sufficient to treat the case of equality in (35).
Step 1. Suppose in addition $1 < p, p_0, p_1, q, q_0, q_1 \le \infty$. Then we have $1 \le p', p_0', p_1', q', q_0', q_1' < \infty$ for the respective conjugated indices. By (35) and Prop. 8 we obtain

$$\big\|a|s^{-r}_{p',q'}x(\nabla)\big\| \le c\,\big\|a|s^{-r_0}_{p_0',q_0'}x(\nabla)\big\|^{1-\Theta}\big\|a|s^{-r_1}_{p_1',q_1'}x(\nabla)\big\|^{\Theta}\,.$$

It is known that

$$[s^{-r_0}_{p_0',q_0'}f(\nabla), s^{-r_1}_{p_1',q_1'}f(\nabla)]_\theta = s^{-r}_{p',q'}f(\nabla) \qquad \text{and} \qquad [s^{-r_0}_{p_0',q_0'}b(\nabla), s^{-r_1}_{p_1',q_1'}b(\nabla)]_\theta = s^{-r}_{p',q'}b(\nabla)\,,$$

where $[\,\cdot\,,\,\cdot\,]_\theta$ refers to the complex method of interpolation, see [52, Thm. 4.6]. This implies that $s^{-r}_{p',q'}f(\nabla)$ is an intermediate space with respect to the pair $(s^{-r_0}_{p_0',q_0'}f(\nabla), s^{-r_1}_{p_1',q_1'}f(\nabla))$ and that $s^{-r}_{p',q'}b(\nabla)$ is an intermediate space with respect to the pair $(s^{-r_0}_{p_0',q_0'}b(\nabla), s^{-r_1}_{p_1',q_1'}b(\nabla))$. Lemmas 5 and 4 and well-known duality assertions for interpolation spaces (see e.g. [49, Theorem 1.11.2]; the required density properties are valid due to the density of the finite sequences) yield the embedding (36).
Step 2. We remove the restriction $1 < p, p_0, p_1, q, q_0, q_1 \le \infty$. Let $\varepsilon > 0$. For a given sequence $a := (a_{\bar{j},\bar{k}})$ we define the sequences $|a|$ and $|a|^\varepsilon$ by

$$|a| := \big(|a_{\bar{j},\bar{k}}|\big)_{\bar{j},\bar{k}} \qquad \text{and} \qquad |a|^\varepsilon := \big(|a_{\bar{j},\bar{k}}|^\varepsilon\big)_{\bar{j},\bar{k}}\,,$$

respectively. Let $K(t,a;X,Y)$ denote the K-functional of a with respect to the interpolation couple (X,Y). Using the lattice property of our sequence spaces $s^{u}_{v,w}x(\nabla)$, $x \in \{b,f\}$ we can prove

$$K\big(t,a;s^{r_0}_{p_0,q_0}x(\nabla), s^{r_1}_{p_1,q_1}x(\nabla)\big) = K\big(t,|a|;s^{r_0}_{p_0,q_0}x(\nabla), s^{r_1}_{p_1,q_1}x(\nabla)\big)\,, \qquad t > 0\,,$$

and

$$K\big(|a|^\varepsilon, t; s^{r_0\varepsilon}_{p_0/\varepsilon,q_0/\varepsilon}x(\nabla), s^{r_1\varepsilon}_{p_1/\varepsilon,q_1/\varepsilon}x(\nabla)\big) \asymp K\big(t^{1/\varepsilon}, |a|; s^{r_0}_{p_0,q_0}x(\nabla), s^{r_1}_{p_1,q_1}x(\nabla)\big)^\varepsilon, \qquad t > 0,$$

if $\varepsilon < \min(1, p, p_0, p_1, q, q_0, q_1)$. Based on these formulas one can argue as in [24]. ■

5 Widths of best m-term approximation for embeddings of $s^t_{p,q}x(\Omega)$-spaces

Within this section we deal with the behaviour of the widths of best m-term approximation with respect to pairs $(s^t_{p_0,q_0}x(\Omega), s^0_{p_1,q_1}y(\Omega))$, $x, y \in \{b, f\}$. A few more notations are needed. For $\mu \in \mathbb{N}$ we define

$$\begin{aligned} M\big(\mu, d\big) &:= \big\{j \in \mathbb{N}^d_0 : \ |j|_1 = \mu\big\}, \\ \nabla_\mu(\Omega) &:= \Big\{(j, k) \in \mathbb{N}^d_0 \times \mathbb{Z}^d : \ j \in M\big(\mu, d\big), \ k \in \nabla_j(\Omega)\Big\}, \\ S\big(\mu, d\big) &:= |M\big(\mu, d\big)| \qquad \text{and} \qquad D_\mu := |\nabla_\mu|. \end{aligned}$$

Obviously, $S\big(\mu, d\big) \asymp \mu^{d-1}$ and from (22) we conclude $D_\mu \asymp S\big(\mu, d\big)2^\mu \asymp \mu^{d-1}2^\mu$. Furthermore,

$$\mathcal{B} := \Big\{e^{j,k} : \quad j \in \mathbb{N}^d_0, \ k \in \nabla_j(\Omega)\Big\}$$

denotes the canonical basis with respect to $\nabla(\Omega)$.

5.1 Estimates from below

We recall a result from [25].

Proposition 9. *We suppose $s^t_{p_0,q_0}x(\Omega) \hookrightarrow s^0_{p_1,q_1}y(\Omega)$. With $x, y \in \{b, f\}$ it follows*

$$\sigma_m\Big(s^t_{p_0,q_0}x(\Omega), s^0_{p_1,q_1}y(\Omega), \mathcal{B}\Big) \gtrsim m^{-t}\big(\log m\big)^{(d-1)(t-\frac{1}{q_0}+\frac{1}{q_1})_+}, \qquad m \in \mathbb{N}. \tag{37}$$

5.2 Upper bounds and explicit constructions

The aim of this subsection consists in deriving upper bounds for σ_m with respect to pairs $(s^t_{p_0,q_0}x(\Omega), s^0_{p_0,q_0}y(\Omega))$. We always work with an explicit construction of the approximant.

Proposition 10. *Let*

$$\gamma_0 := \min(p_0, q_0) \ < \ \delta_1 := \max(p_1, q_1) \qquad \text{and} \qquad t > \frac{1}{\gamma_0} - \frac{1}{\delta_1}. \tag{38}$$

Let $n \in \mathbb{N}$ and $m := n^{d-1}\, 2^n$. For $x, y \in \{b, f\}$ there exists an approximation $S_m a$ of $a \in s^t_{p_0,q_0}x(\Omega)$, $\big\|a\big|s^0_{p_0,q_0}x(\Omega)\big\| \le 1$, of the form

$$S_m a := \sum_{\mu=0}^{\infty} \sum_{(j,k)\in\Lambda_\mu} a_{j,k}\, e^{j,k}\,,$$

(for the definition of Λ_μ see formula (45) below) consisting of at most $c_0 m$ terms, such that

$$\big\|a - S_m a\big|s^0_{p_1,q_1}y(\Omega)\big\| \le c_1\, m^{-t}\big(\log m\big)^{(d-1)\left(t-\frac{1}{q_0}+\frac{1}{q_1}\right)}\,, \tag{39}$$

where c_0 and c_1 are positive constants independent of a and m.

Proof. Since Ω is fixed we drop it in notation (at least partly).

Step 1. First we deal with the b-b case.

Step 1.1. Preparations. Any sequence $a \in s^r_{p,q}b(\Omega)$ may be decomposed into an infinite series of "building blocks". For this purpose, we define for $\mu \in \mathbb{N}$ restriction operators R_μ by

$$\big(R_\mu a\big)_{j,k} = \begin{cases} a_{j,k} & \text{if} \quad j \in M(\mu,d)\,, \quad k \in \nabla_j\,, \\ 0, & \text{else}\,. \end{cases}$$

Then we have componentwise $a = \sum_{\mu=0}^{\infty} R_\mu a$. Moreover, we find for $0 < q < \infty$

$$\big\|a\big|s^r_{p,q}b(\Omega)\big\|^q = \sum_{\mu=0}^{\infty} \big\|\, R_\mu a\,\big|s^r_{p,q}b(\Omega)\big\|^q\,, \tag{40}$$

and similarly for $q = \infty$

$$\big\|a\big|s^r_{p,\infty}b(\Omega)\big\| = \sup_{\mu\in\mathbb{N}_0} \big\|\, R_\mu a\,\big|s^r_{p,\infty}b(\Omega)\big\|\,. \tag{41}$$

In particular, we have

$$\big\|\, R_\mu a\,\big|s^r_{p,q}b(\Omega)\big\| \le \big\|a\big|s^r_{p,q}b(\Omega)\big\| \tag{42}$$

for all $\mu \in \mathbb{N}_0$. Furthermore, if $u \le p$, then

$$\big\|\, R_\mu a\,\big|s^r_{u,q}b(\Omega)\big\| \lesssim \big\|\, R_\mu a\,\big|s^r_{p,q}b(\Omega)\big\|\,, \tag{43}$$

see lemma 1. If $v \le q$ we find

$$\big\|\, R_\mu a\,\big|s^r_{p,v}b(\Omega)\big\| \le S(\mu,d)^{\frac{1}{v}-\frac{1}{q}} \big\|\, R_\mu a\,\big|s^r_{p,q}b(\Omega)\big\|\,. \tag{44}$$

Step 1.2. Let $a \in s^t_{p_0,q_0}b(\Omega)$ with $\big\|a\big|s^t_{p_0,q_0}b(\Omega)\big\| \le 1$. Furthermore we put

$$\Lambda_\mu := \Big\{(j,k) \in \nabla_\mu :\ |a_{j,k}| \ge \varepsilon_\mu\Big\}\,, \tag{45}$$

and

$$\varepsilon_\mu := \begin{cases} 0 & \text{if} \quad \mu \le n\,, \\ 2^{\mu\alpha}\, 2^{n\beta}\, S\big(n,d\big)^{\eta} & \text{if} \quad \mu > n\,, \end{cases} \qquad \mu \in \mathbb{N}_0. \tag{46}$$

The parameters α, β and η will be chosen later on. Clearly, Λ_μ depends in a nonlinear way on a. By assumption $\gamma_0 < \infty$. In case $\mu > n$ we find

$$\begin{aligned}
|\Lambda_\mu| &= \sum_{(j,k)\in\Lambda_\mu} 1 \le \sum_{(j,k)\in\Lambda_\mu} \frac{|a_{j,k}|^{\gamma_0}}{\varepsilon_\mu^{\gamma_0}} \\
&\le \varepsilon_\mu^{-\gamma_0} \sum_{j\in M(\mu,d)} 2^{-\mu(t+\frac{1}{2}-\frac{1}{\gamma_0})\gamma_0} 2^{|j|_1(t+\frac{1}{2}-\frac{1}{\gamma_0})\gamma_0} \sum_{k\in\nabla_j} \big|a_{j,k}\big|^{\gamma_0} \\
&= \varepsilon_\mu^{-\gamma_0} 2^{-\mu(t+\frac{1}{2}-\frac{1}{\gamma_0})\gamma_0} \big\| \, R_\mu a \, \big| s^t_{\gamma_0,\gamma_0} b(\Omega) \big\|^{\gamma_0} \\
&\lesssim \varepsilon_\mu^{-\gamma_0} 2^{-\mu(t+\frac{1}{2}-\frac{1}{\gamma_0})\gamma_0} S\big(\mu,d\big)^{\gamma_0(\frac{1}{\gamma_0}-\frac{1}{q_0})} \big\| \, R_\mu a \, \big| s^t_{p_0,q_0} b(\Omega) \big\|^{\gamma_0} \\
&\le \varepsilon_\mu^{-\gamma_0} 2^{-\mu(t+\frac{1}{2}-\frac{1}{\gamma_0})\gamma_0} S\big(\mu,d\big)^{\gamma_0(\frac{1}{\gamma_0}-\frac{1}{q_0})} \big\| a \big| s^t_{p_0,q_0} b(\Omega) \big\|^{\gamma_0} ,
\end{aligned}$$

where we used (42)–(44). Summing up and inserting (46) we obtain

$$\begin{aligned}
&\sum_{\mu=0}^{\infty} |\Lambda_\mu| = \sum_{\mu=0}^{n} |\nabla_\mu| + \sum_{\mu=n+1}^{\infty} |\Lambda_\mu| \\
&\lesssim \sum_{\mu=0}^{n} D_\mu + \sum_{\mu=n+1}^{\infty} \varepsilon_\mu^{-\gamma_0} \, 2^{-\mu(t+\frac{1}{2}-\frac{1}{\gamma_0})\gamma_0} \, S\big(\mu,d\big)^{\gamma_0(\frac{1}{\gamma_0}-\frac{1}{q_0})} \\
&\lesssim \sum_{\mu=0}^{n} S(\mu,d) \, 2^\mu + \sum_{\mu=n+1}^{\infty} 2^{-\mu\alpha\gamma_0} \, 2^{-n\beta\gamma_0} \, S\big(n,d\big)^{-\eta\gamma_0} \, 2^{-\mu(t+\frac{1}{2}-\frac{1}{\gamma_0})\gamma_0} \, S\big(\mu,d\big)^{\gamma_0(\frac{1}{\gamma_0}-\frac{1}{q_0})} .
\end{aligned}$$

We need a further parameter. Define

$$\vartheta := \frac{1}{2} \, \frac{t - \frac{1}{\gamma_0} + \frac{1}{\delta_1}}{1 - \gamma_0/\delta_1} \, .$$

By assumption $\vartheta > 0$, see (38). Now we choose

$$\begin{aligned}
\alpha &:= -t - \tfrac{1}{2} + \tfrac{1}{\gamma_0} + \vartheta \, , \\
\beta &:= -\tfrac{1}{\gamma_0} - \vartheta \qquad (\iff -\beta\gamma_0 = 1 + \vartheta\gamma_0) \, , \\
\eta &:= -\tfrac{1}{q_0} \qquad (\iff -\eta\gamma_0 = 1 - \gamma_0\big(\tfrac{1}{\gamma_0} - \tfrac{1}{q_0}\big)) \, .
\end{aligned}$$

Since $S(\mu,d) \asymp \mu^{d-1}$ we obtain

$$\sum_{\mu=0}^{\infty} |\Lambda_\mu| \lesssim n^{d-1} \, 2^n + 2^{n(1+\vartheta\gamma_0)} \, n^{-(d-1)\eta\gamma_0} \sum_{\mu=n+1}^{\infty} 2^{-\mu\vartheta\gamma_0} \, \mu^{(d-1)(1+\eta\gamma_0)} \lesssim n^{d-1} \, 2^n \, .$$

Hence, S_m is a $c_0 m$-term approximation of a. We did not use the special value of ϑ here, only its positivity. However, the concrete value will be used in the next step.

Step 1.3. Here we estimate $\big\| \, a - S_m a \, \big| s^0_{p_1,q_1} b(\Omega) \big\|$. Let

$$T_\mu := \sum_{(j,k)\in\nabla_\mu\setminus\Lambda_\mu} a_{j,k} \, e^{j,k} = R_\mu(a - S_m a) \, .$$

As in Step 2 we obtain

$$\begin{aligned}\big\|T_\mu\big|s^0_{p_1,q_1}b(\Omega)\big\| &\lesssim S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{1}{\delta_1}}\big\|T_\mu\big|s^0_{\delta_1,\delta_1}b(\Omega)\big\| \\ &= S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{1}{\delta_1}}2^{\mu(\frac{1}{2}-\frac{1}{\delta_1})}\Bigg(\sum_{(j,k)\in\nabla_\mu\setminus\Lambda_\mu}|a_{j,k}|^{\delta_1}\Bigg)^{\frac{1}{\delta_1}}.\end{aligned}$$

Next we use the definition of Λ_μ and find in case $\delta_1<\infty$

$$|a_{j,k}|^{\delta_1}=|a_{j,k}|^{\delta_1-\gamma_0}\,|a_{j,k}|^{\gamma_0}\le\varepsilon_\mu^{\delta_1-\gamma_0}\,|a_{j,k}|^{\gamma_0}$$

if $(j,k)\in\nabla_\mu\setminus\Lambda_\mu$. In case $\delta_1=\infty$ we simply use $|a_{j,k}|\le\varepsilon_\mu$. This implies

$$\begin{aligned}&\big\|T_\mu\big|s^0_{p_1,q_1}b(\Omega)\big\| \\ &\lesssim S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{1}{\delta_1}}2^{\mu(\frac{1}{2}-\frac{1}{\delta_1})}\varepsilon_\mu^{1-\frac{\gamma_0}{\delta_1}}\Bigg(\sum_{(j,k)\in\nabla_\mu\setminus\Lambda_\mu}2^{-\mu(t+\frac{1}{2}-\frac{1}{\gamma_0})\gamma_0}2^{|j|_1(t+\frac{1}{2}-\frac{1}{\gamma_0})\gamma_0}|a_{j,k}|^{\gamma_0}\Bigg)^{\frac{1}{\delta_1}} \\ &\le S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{1}{\delta_1}}2^{\mu(\frac{1}{2}-\frac{1}{\delta_1})}2^{-\mu(t+\frac{1}{2}-\frac{1}{\gamma_0})\frac{\gamma_0}{\delta_1}}\varepsilon_\mu^{1-\frac{\gamma_0}{\delta_1}}\Bigg(\sum_{(j,k)\in\nabla_\mu}2^{|j|_1(t+\frac{1}{2}-\frac{1}{\gamma_0})\gamma_0}|a_{j,k}|^{\gamma_0}\Bigg)^{\frac{1}{\gamma_0}\cdot\frac{\gamma_0}{\delta_1}} \\ &= S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{1}{\delta_1}}2^{-\mu(t+\frac{1}{2}-\frac{\delta_1}{2\gamma_0})\frac{\gamma_0}{\delta_1}}\varepsilon_\mu^{1-\frac{\gamma_0}{\delta_1}}\big\|\,R_\mu a\,\big|s^t_{\gamma_0,\gamma_0}b(\Omega)\big\|^{\frac{\gamma_0}{\delta_1}} \\ &\lesssim S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{1}{\delta_1}}2^{-\mu(t+\frac{1}{2}-\frac{\delta_1}{2\gamma_0})\frac{\gamma_0}{\delta_1}}\varepsilon_\mu^{1-\frac{\gamma_0}{\delta_1}}S\big(\mu,d\big)^{(\frac{1}{\gamma_0}-\frac{1}{q_0})\frac{\gamma_0}{\delta_1}}\big\|\,R_\mu a\,\big|s^t_{p_0,q_0}b(\Omega)\big\|^{\frac{\gamma_0}{\delta_1}} \\ &\le S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{\gamma_0}{q_0\delta_1}}2^{-\mu(t+\frac{1}{2}-\frac{\delta_1}{2\gamma_0})\frac{\gamma_0}{\delta_1}}\varepsilon_\mu^{1-\frac{\gamma_0}{\delta_1}}.\end{aligned}$$

Temporarily we assume $q_1<\infty$. With $T_\mu:=0$ if $\mu\le n$ and by means of the definition of ε_μ we conclude

$$\begin{aligned}\big\|a-S_m a\big|s^0_{p_1,q_1}b(\Omega)\big\|^{q_1} &= \Bigg\|\sum_{\mu=0}^\infty T_\mu\Big|s^0_{p_1,q_1}b(\Omega)\Bigg\|^{q_1} = \sum_{\mu=n+1}^\infty\big\|T_\mu\big|s^0_{p_1,q_1}b(\Omega)\big\|^{q_1} \\ &\lesssim \sum_{\mu=n+1}^\infty\Big(S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{\gamma_0}{q_0\delta_1}}2^{-\mu(t+\frac{1}{2}-\frac{\delta_1}{2\gamma_0})\frac{\gamma_0}{\delta_1}}\varepsilon_\mu^{1-\frac{\gamma_0}{\delta_1}}\Big)^{q_1} \\ &= 2^{n\beta(1-\frac{\gamma_0}{\delta_1})q_1}S\big(n,d\big)^{\eta(1-\frac{\gamma_0}{\delta_1})q_1}\sum_{\mu=n+1}^\infty\Big(S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{\gamma_0}{q_0\delta_1}}2^{-\mu(t+\frac{1}{2}-\frac{\delta_1}{2\gamma_0})\frac{\gamma_0}{\delta_1}}2^{\mu\alpha(1-\frac{\gamma_0}{\delta_1})}\Big)^{q_1} \\ &= 2^{-n(\frac{1}{\gamma_0}+\vartheta)(1-\frac{\gamma_0}{\delta_1})q_1}\,S\big(n,d\big)^{-(1-\frac{\gamma_0}{\delta_1})\frac{q_1}{q_0}} \\ &\qquad\times\sum_{\mu=n+1}^\infty\Big(S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{\gamma_0}{q_0\delta_1}}2^{-\mu(t+\frac{1}{2}-\frac{\delta_1}{2\gamma_0})\frac{\gamma_0}{\delta_1}}\,2^{\mu(-t-\frac{1}{2}+\frac{1}{\gamma_0}+\vartheta)(1-\frac{\gamma_0}{\delta_1})}\Big)^{q_1} \\ &= 2^{-n(\frac{1}{\gamma_0}+\vartheta)(1-\frac{\gamma_0}{\delta_1})q_1}\,S\big(n,d\big)^{-(1-\frac{\gamma_0}{\delta_1})\frac{q_1}{q_0}} \\ &\qquad\times\sum_{\mu=n+1}^\infty\Big(S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{\gamma_0}{q_0\delta_1}}2^{-\mu(t-\frac{1}{\gamma_0}+\frac{1}{\delta_1})}2^{\mu\vartheta(1-\frac{\gamma_0}{\delta_1})}\Big)^{q_1}.\end{aligned}$$

Since $t > \frac{1}{\gamma_0} - \frac{1}{\delta_1}$ the sum on the right-hand side is finite and we get

$$
\begin{aligned}
&\big\|a - S_m a\big|s^0_{p_1,q_1}b(\Omega)\big\|^{q_1} \\
&\qquad \lesssim 2^{-n(\frac{1}{\gamma_0}+\vartheta)(1-\frac{\gamma_0}{\delta_1})q_1}\, S\big(n,d\big)^{-(1-\frac{\gamma_0}{\delta_1})\frac{q_1}{q_0}} \left(S\big(n,d\big)^{\frac{1}{q_1}-\frac{\gamma_0}{q_0\delta_1}}\, 2^{-\frac{1}{2}n(t-\frac{1}{\gamma_0}+\frac{1}{\delta_1})}\right)^{q_1} \\
&\qquad = 2^{-n(\frac{1}{\gamma_0}+\vartheta)(1-\frac{\gamma_0}{\delta_1})q_1}\, S\big(n,d\big)^{1-\frac{q_1}{q_0}} 2^{-n\vartheta(1-\frac{\gamma_0}{\delta_1})q_1} \\
&\qquad = 2^{-n(\frac{1}{\gamma_0}-\frac{1}{\delta_1})q_1}\, S\big(n,d\big)^{1-\frac{q_1}{q_0}}\, 2^{-n(t-\frac{1}{\gamma_0}+\frac{1}{\delta_1})q_1} = 2^{-ntq_1}\, n^{(d-1)(1-\frac{q_1}{q_0})}\,.
\end{aligned}
$$

In case $q_1 = \infty$ some obvious modifications have to be made. We omit details.

Step 2. Now we investigate the f-f case.

Step 2.1. Instead of (40)-(44) we shall use the following set of inequalities. Obviously.

$$
\big\|\, R_\mu a\, \big|s^r_{p,q}f(\Omega)\big\| \le \big\|a\big|s^r_{p,q}f(\Omega)\big\|\,. \tag{47}
$$

Using Minkowski's inequality one can prove

$$
\big\|a\big|s^r_{p,q}f(\Omega)\big\|^u \le \sum_{\mu=0}^{\infty}\big\|\, R_\mu a\big|s^r_{p,q}f(\Omega)\big\|^u\,, \tag{48}
$$

where $u = \min(p,q)$. Moreover, the analogues of (43) and (44) are

$$
\big\|\, R_\mu a\, \big|s^r_{v,q}f(\Omega)\big\| \lesssim \big\|\, R_\mu a\, \big|s^r_{p,q}f(\Omega)\big\|\,, \qquad v \le p\,, \tag{49}
$$

see Lemma 1, and

$$
\big\|\, R_\mu a\, \big|s^r_{p,w}f(\Omega)\big\| \le S(\mu,d)^{\frac{1}{w}-\frac{1}{q}}\big\|\, R_\mu a\, \big|s^r_{p,q}f(\Omega)\big\|\,, \qquad w \le q\,. \tag{50}
$$

The latter inequality follows from the observation that for a fixed x the sum $\sum_{k\in\nabla_j} a_{j,k}\,\mathcal{X}_{j,k}(x)$ consists of exactly one summand. Hence, the cardinality of the summands in $\sum_{|j|=\mu}\sum_{k\in\nabla_j} a_{j,k}\,\mathcal{X}_{j,k}(x)$ is uniformly bounded by $S(\mu,d)$. Hölder's inequality now yields (50).

Step 2.2. Let $a \in s^t_{p_0,q_0}f(\Omega)$ with $\big\|a\big|s^t_{p_0,q_0}f(\Omega)\big\| \le 1$. With ε_μ and Λ_μ as in step 1.2 and with $\mu > n$ we derive

$$
\begin{aligned}
|\Lambda_\mu| &\le \varepsilon_\mu^{-\gamma_0} 2^{-\mu(t+\frac{1}{2}-\frac{1}{\gamma_0})\gamma_0}\big\|\, R_\mu a\, \big|s^t_{\gamma_0,\gamma_0}b(\Omega)\big\|^{\gamma_0} \\
&\lesssim \varepsilon_\mu^{-\gamma_0} 2^{-\mu(t+\frac{1}{2}-\frac{1}{\gamma_0})\gamma_0} S\big(\mu,d\big)^{\gamma_0(\frac{1}{\gamma_0}-\frac{1}{q_0})}\big\|\, R_\mu a\, \big|s^t_{p_0,q_0}f(\Omega)\big\|^{\gamma_0} \\
&\le \varepsilon_\mu^{-\gamma_0} 2^{-\mu(t+\frac{1}{2}-\frac{1}{\gamma_0})\gamma_0} S\big(\mu,d\big)^{\gamma_0(\frac{1}{\gamma_0}-\frac{1}{q_0})}\big\|a\big|s^t_{p_0,q_0}f(\Omega)\big\|^{\gamma_0}\,.
\end{aligned}
$$

As in step 1 this implies that $S_m a$ is a linear combination of at most $c_0 m$ elements of $\mathcal{B}$. Concerning T_μ we obtain

$$
\begin{aligned}
\big\|T_\mu\big|s^0_{p_1,q_1}f(\Omega)\big\| &\lesssim S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{1}{\delta_1}}\big\|T_\mu\big|s^0_{\delta_1,\delta_1}f(\Omega)\big\| = S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{1}{\delta_1}}\big\|T_\mu\big|s^0_{\delta_1,\delta_1}b(\Omega)\big\| \\
&\le S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{1}{\delta_1}} 2^{-\mu(t+\frac{1}{2}-\frac{\delta_1}{2\gamma_0})\frac{\gamma_0}{\delta_1}}\varepsilon_\mu^{1-\frac{\gamma_0}{\delta_1}}\big\|\, R_\mu a\, \big|s^t_{\gamma_0,\gamma_0}f(\Omega)\big\|^{\frac{\gamma_0}{\delta_1}} \\
&\lesssim S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{1}{\delta_1}} 2^{-\mu(t+\frac{1}{2}-\frac{\delta_1}{2\gamma_0})\frac{\gamma_0}{\delta_1}}\varepsilon_\mu^{1-\frac{\gamma_0}{\delta_1}} S\big(\mu,d\big)^{(\frac{1}{\gamma_0}-\frac{1}{q_0})\frac{\gamma_0}{\delta_1}}\big\|\, R_\mu a\, \big|s^t_{p_0,q_0}f(\Omega)\big\|^{\frac{\gamma_0}{\delta_1}} \\
&\le S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{\gamma_0}{q_0\delta_1}} 2^{-\mu(t+\frac{1}{2}-\frac{\delta_1}{2\gamma_0})\frac{\gamma_0}{\delta_1}}\varepsilon_\mu^{1-\frac{\gamma_0}{\delta_1}}\,.
\end{aligned}
$$

Finally, the estimate of $\|a - S_m a|s^0_{p_1,q_1} f(\Omega)\|$ is similar to that given for $\|a - S_m a|s^0_{p_1,q_1} b(\Omega)\|$. Observe, that the difference between (48) and (40) does not cause problems, since for all $u \in (0,\infty]$ we have

$$\Big(\sum_{\mu=n+1}^{\infty} \mu^{\tau}\, 2^{-\mu\lambda u}\Big)^{1/u} \asymp n^{\tau}\, 2^{-n\lambda}\,, \qquad \lambda > 0\,, \quad \tau \in \mathbb{R}\,.$$

Step 3. It remains to deal with the b-f and f-b case. Here the claim follows from a combination of the estimates given in Steps 1 and 2. ■

Remark 15. (i) The mapping $a \to S_m a$ is nonlinear and not continuous. However, by using soft-thresholding in (46) one could also obtain a continuous version with similar properties.
(ii) Observe, that $S_m a$ is explicitly known, see Substep 1.2 of the above proof, in particular (45) and (46).
(iii) The proof given above uses a combination of ideas from [13] and [52], see also [12].

Next we deal with the case $\gamma_0 \ge \delta_1$.

Proposition 11. *Let $\Omega \subset \mathbb{R}^d$ be open, bounded and nontrivial. We suppose*

$$\gamma_0 = \min(p_0,q_0) \ge \delta_1 = \max(p_1,q_1) \qquad \text{and} \qquad t > 0\,. \tag{51}$$

Let $x, y \in \{b, f\}$. Then the linear operator

$$S_n a := \sum_{\mu=0}^{n} \sum_{|j|_1=\mu} \sum_{k\in\nabla_j} a_{j,k}\, e^{j,k} \tag{52}$$

satisfies

$$\big\| I - S_n \,|s^t_{p_0,q_0} x(\Omega) \to s^0_{p_1,q_1} y(\Omega)\big\| \lesssim 2^{-nt}\, n^{(d-1)\left(-\frac{1}{q_0}+\frac{1}{q_1}\right)}\,, \qquad n \in \mathbb{N}\,. \tag{53}$$

Proof. We shall use the notation from the proof of the previous proposition. By Hölder's inequality and the inequalities (43), (44), (47)-(50) we find

$$\begin{aligned}
\big\| R_\mu a \,|s^0_{p_1,q_1} y(\Omega)\big\| &\lesssim S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{1}{\delta_1}} \big\| R_\mu a \,|s^0_{\delta_1,\delta_1} y(\Omega)\big\| = S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{1}{\delta_1}}\, 2^{\mu(\frac12-\frac{1}{\delta_1})} \big\| R_\mu a \,|\ell^{D_\mu}_{\delta_1}\big\| \\
&\le S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{1}{\delta_1}}\, 2^{\mu(\frac12-\frac{1}{\delta_1})}\, D_\mu^{\frac{1}{\delta_1}-\frac{1}{\gamma_0}} \big\| R_\mu a \,|\ell^{D_\mu}_{\gamma_0}\big\| \\
&\lesssim S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{1}{\delta_1}}\, 2^{\mu(\frac12-\frac{1}{\delta_1})}\, D_\mu^{\frac{1}{\delta_1}-\frac{1}{\gamma_0}}\, 2^{-\mu(t+\frac12-\frac{1}{\gamma_0})} \big\| R_\mu a \,|s^t_{\gamma_0,\gamma_0} x(\Omega)\big\| \\
&\lesssim S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{1}{\delta_1}}\, D_\mu^{\frac{1}{\delta_1}-\frac{1}{\gamma_0}}\, 2^{-\mu(t-\frac{1}{\gamma_0}+\frac{1}{\delta_1})}\, S\big(\mu,d\big)^{\frac{1}{\gamma_0}-\frac{1}{q_0}} \big\| R_\mu a \,|s^t_{p_0,q_0} x(\Omega)\big\|\,.
\end{aligned}$$

Because of $D_\mu \asymp S(\mu,d)\, 2^\mu$ this implies

$$\big\| R_\mu a \,|s^0_{p_1,q_1} y(\Omega)\big\| \lesssim S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{1}{q_0}}\, 2^{-\mu t} \big\|a|s^t_{p_0,q_0} x(\Omega)\big\|\,.$$

Summing up with respect to μ we find in case $y = b$

$$\begin{aligned}\big\|a - S_n a\big|s^0_{p_1,q_1}b(\Omega)\big\|^{q_1} &= \Big\|\sum_{\mu=\lambda+1}^{\infty} R_\mu a\,\Big|s^0_{p_1,q_1}b(\Omega)\Big\|^{q_1} = \sum_{\mu=\lambda+1}^{\infty} \|\, R_\mu a\,|s^0_{p_1,q_1}b(\Omega)\big\|^{q_1}\\ &\lesssim \sum_{\mu=n+1}^{\infty}\Big(S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{1}{q_0}}\, 2^{-\mu t}\Big)^{q_1}\,\big\|a\big|s^t_{p_0,q_0}x(\Omega)\big\|^{q_1}\\ &\lesssim 2^{-\lambda t q_1}\, S\big(n,d\big)^{(\frac{1}{q_1}-\frac{1}{q_0})q_1}\,\big\|a\big|s^t_{p_0,q_0}x(\Omega)\big\|^{q_1}\,.\end{aligned}$$

If $y = f$, then we employ (48) and obtain

$$\begin{aligned}\big\|a - S_n a\big|s^0_{p_1,q_1}f(\Omega)\big\|^{u} &\le \sum_{\mu=\lambda+1}^{\infty} \|\, R_\mu a\,|s^0_{p_1,q_1}f(\Omega)\big\|^{u}\\ &\lesssim \sum_{\mu=n+1}^{\infty}\Big(S\big(\mu,d\big)^{\frac{1}{q_1}-\frac{1}{q_0}}\, 2^{-\mu t}\Big)^{u}\,\big\|a\big|s^t_{p_0,q_0}x(\Omega)\big\|^{u}\\ &\lesssim 2^{-ntu}\, S\big(n,d\big)^{(\frac{1}{q_1}-\frac{1}{q_0})u}\,\big\|a\big|s^t_{p_0,q_0}x(\Omega)\big\|^{u}\,.\end{aligned}$$

Since $S(n,d) \asymp n^{d-1}$ this proves the claim. ∎

Remark 16. The operator S_n defined in (52) is a projection with respect to the hyperbolic cross

$$H(n,d) := \bigcup_{\mu=0}^{n} \nabla_\mu\,.$$

Observe

$$\operatorname{rank} S_n \asymp |H(n,d)| \asymp 2^n\, n^{d-1}\,, \qquad n \in \mathbb{N}\,.$$

It is well-known, that in certain situations these operators are realizing the approximation numbers of related embedding operators (up to constants). For the periodic setting we refer to Galeev [23], Romanyuk [35, 36, 38] and [42]. For functions on $\mathbb{R}^d$ hyperbolic cross approximation with respect to tensor product wavelet systems was considered in DeVore, Konyagin, Temlyakov [14]. They concentrated on the approximation of functions from $S^r_pW(\mathbb{R}^d)$ in the norm of $L_p(\mathbb{R}^d)$ and associated approximation spaces.

Now it's time for a first summary. We combine Prop. 10, Prop. 11 and Prop. 9. This leads to the following.

Corollary 2. *Let $\Omega \subset \mathbb{R}^d$ be open, bounded and nontrivial. We suppose*

$$t > \max\Big(0\,,\ \frac{1}{\min(p_0,q_0)} - \frac{1}{\max(p_1,q_1)}\Big)\,. \tag{54}$$

Let $x, y \in \{b, f\}$. Then we have

$$\sigma_m\Big(s^t_{p_0,q_0}x(\Omega), s^0_{p_1,q_1}y(\Omega), \mathcal{B}\Big) \asymp m^{-t}\big(\log m\big)^{(d-1)\big(t-\frac{1}{q_0}+\frac{1}{q_1}\big)}\,, \qquad m \ge 2\,,$$

or, with other words,

$$s^t_{p_0,q_0}x(\Omega) \hookrightarrow \mathcal{A}_\infty^{t,-(d-1)\left(t-\frac{1}{q_0}+\frac{1}{q_1}\right)}\left(s^0_{p_1,q_1}y(\Omega),\mathcal{B}\right).$$

Proof. Combining Prop. 10 and Prop. 11 we obtain the estimates from above in case of $m = c\,2^n\,n^{d-1}$, $n \in \mathbb{N}$, where $c > 0$ is independent of n. The claim for the remaining natural numbers m follows immediately by the monotonicity properties of σ_m with respect to m. ■

5.3 Estimates from above – limiting cases and consequences

We recall some more results obtained in [25]. The first one deals with Lizorkin-Triebel sequence spaces.

Proposition 12. *Let $p_0 < p_1$, $q_1 \le p_1$ and $p_0 \le q_0 \le \infty$. Let $t := \frac{1}{p_0} - \frac{1}{p_1}$. Then we have*

$$\sigma_m\Big(s^t_{p_0,q_0}f(\Omega), s^0_{p_1,q_1}f(\Omega),\mathcal{B}\Big) \asymp m^{-\frac{1}{p_0}+\frac{1}{p_1}}\left(\log m\right)^{(d-1)(\frac{1}{p_0}-\frac{1}{p_1}-\frac{1}{q_0}+\frac{1}{q_1})}, \qquad (55)$$

if $m \ge 2$. Moreover, if $q_0 \le p_0 < p_1 \le q_1$ then we find for $m \in \mathbb{N}$

$$\sigma_m\Big(s^t_{p_0,q_0}f(\Omega), s^0_{p_1,q_1}f(\Omega),\mathcal{B}\Big) \asymp m^{-\frac{1}{p_0}+\frac{1}{p_1}}. \qquad (56)$$

By using the techniques from Section 4 we can supplement this as follows.

Corollary 3. *Let $0 < p_0 < p_1 < \infty$, $0 < q_0, q_1 \le \infty$, and $t := \frac{1}{p_0} - \frac{1}{p_1}$.*
(i) *Let $\frac{1}{p_0} - \frac{1}{p_1} > \frac{1}{q_0} - \frac{1}{q_1}$ and $q_1 \le p_1$. Then it holds*

$$\sigma_m\Big(s^t_{p_0,q_0}f(\Omega)\,,s^0_{p_1,q_1}f(\Omega)\,,\mathcal{B}\Big) \lesssim m^{-\frac{1}{p_0}+\frac{1}{p_1}}\left(\log m\right)^{(d-1)\left(\frac{1}{p_0}-\frac{1}{p_1}-\frac{1}{q_0}+\frac{1}{q_1}\right)}$$

for every natural number $m \ge 2$.
(ii) *Let $\frac{1}{p_0} - \frac{1}{p_1} < \frac{1}{q_0} - \frac{1}{q_1}$ and $p_1 \le q_1$. Then it holds*

$$\sigma_m\Big(s^t_{p_0,q_0}f(\Omega)\,,s^0_{p_1,q_1}f(\Omega)\,,\mathcal{B}\Big) \lesssim m^{-\frac{1}{p_0}+\frac{1}{p_1}}, \quad m \in \mathbb{N}.$$

Proof. *Step 1.* Proof of (i). *Substep 1.1.* We assume $q_0 \le q_1$. This implies $v \le q_0$. Hence, for every $0 < \Theta < 1$ we find parameters $0 < u < p_0 < p_1$ and $0 < v \le q_0$, such that

$$\frac{1}{p_0} := \frac{1-\Theta}{p_1} + \frac{\Theta}{u} \qquad \text{and} \qquad \frac{1}{q_0} := \frac{1-\Theta}{q_1} + \frac{\Theta}{v}. \qquad (57)$$

To apply formula (55) with respect to the pair $\big(s^{\frac{1}{u}-\frac{1}{p_1}}_{u,v}f(\Omega),\, s^0_{p_1,q_1}f(\Omega)\big)$, it is necessary to have $u \le v$. This is equivalent to

$$\frac{\Theta}{v} \le \frac{\Theta}{u} \quad \Longleftrightarrow \quad \Theta\Big(\frac{1}{q_1} - \frac{1}{p_1}\Big) \le \frac{1}{p_0} - \frac{1}{p_1} - \frac{1}{q_0} + \frac{1}{q_1}, \qquad (58)$$

and hence, satisfied, if Θ is chosen sufficiently small (or equivalently by choosing u and v sufficiently small). We find

$$s^{\frac{1}{u}-\frac{1}{p_1}}_{u,v} f(\Omega) \hookrightarrow \mathcal{A}^{\frac{1}{u}-\frac{1}{p_1}, -(d-1)(\frac{1}{u}-\frac{1}{p_1}-\frac{1}{v}+\frac{1}{q_1})}_{\infty} \big(s^0_{p_1,q_1} f(\Omega)\,, \mathcal{B}\big)\,.$$

Next we employ Thm. 3. In combination with Prop. 5 this yields

$$\begin{aligned} s^{\frac{1}{p_0}-\frac{1}{p_1}}_{p_0,q_0} f(\Omega) &\hookrightarrow \Big(s^0_{p_1,q_1} f(\Omega)\,, s^{\frac{1}{u}-\frac{1}{p_1}}_{u,v} f(\Omega)\Big)_{\Theta,\infty} \\ &\hookrightarrow \Big(s^0_{p_1,q_1} f(\Omega)\,, \mathcal{A}^{\frac{1}{u}-\frac{1}{p_1}, -(d-1)(\frac{1}{u}-\frac{1}{p_1}-\frac{1}{v}+\frac{1}{q_1})}_{\infty} \big(s^0_{p_1,q_1} f(\Omega)\,, \mathcal{B}\big)\Big)_{\Theta,\infty} \\ &= \mathcal{A}^{\frac{1}{p_0}-\frac{1}{p_1}, -(d-1)(\frac{1}{p_0}-\frac{1}{p_1}-\frac{1}{q_0}+\frac{1}{q_1})}_{\infty} \big(s^0_{p_1,q_1} f(\Omega)\,, \mathcal{B}\big)\,. \end{aligned}$$

This proves (i) under the extra condition $q_0 \le q_1$.

Substep 1.2. Let $q_1 < q_0$. Again we use (57). This implies $q_0 < v \le \infty$ and moreover, a lower bound for Θ, namely $\Theta \ge 1 - q_1/q_0$. Again we want to apply formula (55) with respect to the pair $(s^{\frac{1}{u}-\frac{1}{p_1}}_{u,v} f(\Omega), s^0_{p_1,q_1} f(\Omega))$. Hence, we need to have $u \le v$ and this is equivalent to

$$\Theta \le \frac{\frac{1}{p_0} - \frac{1}{p_1} - \frac{1}{q_0} + \frac{1}{q_1}}{\frac{1}{q_1} - \frac{1}{p_1}}\,,$$

see (58). In case $p_1 = q_1$ the condition (58) is automatically satisfied for every Θ. Because of

$$1 - \frac{q_1}{q_0} < \frac{\frac{1}{p_0} - \frac{1}{p_1} - \frac{1}{q_0} + \frac{1}{q_1}}{\frac{1}{q_1} - \frac{1}{p_1}} \qquad \Longleftrightarrow \qquad \frac{p_0}{p_1} < \frac{q_0}{q_1}$$

there exists a $\Theta \in (0,1)$ which satisfies all our requirements. Now we continue as in Substep 1.1. This completes the proof of (i).

Step 2. Proof of (ii). We argue as in Step 1. This time we always have $q_0 < q_1$. Again we assume (57). An application of (56) with respect to the pair $\big(s^{\frac{1}{u}-\frac{1}{p_1}}_{u,v} f(\Omega),\ s^0_{p_1,q_1} f(\Omega)\big)$ requires $v \le u < p_1 \le q_1$. The inequality $v \le u$ is guaranteed if

$$\frac{\Theta}{u} \le \frac{\Theta}{v} \iff \frac{1}{p_0} - \frac{1-\Theta}{p_1} \le \frac{1}{q_0} - \frac{1-\Theta}{q_1} \iff \Theta\Big(\frac{1}{p_1} - \frac{1}{q_1}\Big) \le \frac{1}{q_0} - \frac{1}{q_1} - \frac{1}{p_0} + \frac{1}{p_1}\,.$$

Thus, the assumptions of (56) are satisfied by choosing Θ sufficiently small (which is possible by choosing u and v small). We obtain in this way

$$s^{\frac{1}{u}-\frac{1}{p_1}}_{u,v} f(\Omega) \hookrightarrow \mathcal{A}^{\frac{1}{u}-\frac{1}{p_1}}_{\infty} \big(s^0_{p_1,q_1} f(\Omega)\,, \mathcal{B}\big)\,.$$

Now we conclude from Thm. 3 and Prop. 5

$$\begin{aligned} s^{\frac{1}{p_0}-\frac{1}{p_1}}_{p_0,q_0} f(\Omega) &\hookrightarrow \Big(s^0_{p_1,q_1} f(\Omega)\,, s^{\frac{1}{u}-\frac{1}{p_1}}_{u,v} f(\Omega)\Big)_{\Theta,\infty} \\ &\hookrightarrow \Big(s^0_{p_1,q_1} f(\Omega)\,, \mathcal{A}^{\frac{1}{u}-\frac{1}{p_1}}_{\infty} \big(s^0_{p_1,q_1} f(\Omega)\,, \mathcal{B}\big)\Big)_{\Theta,\infty} \\ &= \mathcal{A}^{\frac{1}{p_0}-\frac{1}{p_1}}_{\infty} \big(s^0_{p_1,q_1} f(\Omega)\,, \mathcal{B}\big)\,. \end{aligned}$$

This proves (ii). ■

As a consequence of Lemma 1, Cor. 3 and Prop. 9 we derive the following. Let

$$1/p_* := t + 1/p_1 \, . \tag{59}$$

Corollary 4. *We suppose*

$$t = \frac{1}{p_*} - \frac{1}{p_1} > \max\Big(0, \frac{1}{p_0} - \frac{1}{p_1}\Big) \, .$$

(i) *Let* $t > \frac{1}{q_0} - \frac{1}{q_1}$ *and* $q_1 \le p_1$*. Then it holds*

$$\sigma_m\Big(s^t_{p_0,q_0}f(\Omega)\, , s^0_{p_1,q_1}f(\Omega)\, , \mathcal{B}\Big) \lesssim m^{-t}\big(\log m\big)^{(d-1)\big(t-\frac{1}{q_0}+\frac{1}{q_1}\big)}$$

for every natural number $m \ge 2$*.*
(ii) *Let* $t < \frac{1}{q_0} - \frac{1}{q_1}$ *and* $p_1 \le q_1$*. Then we have*

$$\sigma_m\Big(s^t_{p_0,q_0}f(\Omega)\, , s^0_{p_1,q_1}f(\Omega)\, , \mathcal{B}\Big) \lesssim m^{-\frac{1}{p_0}+\frac{1}{p_1}} \, , \qquad m \in \mathbb{N} \, .$$

Proof. *Step 1.* Proof of (i). By assumption $p_* < p_0$. This implies the continuous embedding $s^t_{p_0,q_0}f(\Omega) \hookrightarrow s^t_{p_*,q_0}f(\Omega)$, see Lemma 1. Now we apply Cor. 3, see (55), and obtain

$$\begin{aligned}\sigma_m\Big(s^t_{p_*,q_0}f(\Omega), s^0_{p_1,q_1}f(\Omega), \mathcal{B}\Big) &\asymp m^{-\frac{1}{p_*}+\frac{1}{p_1}}\,\big(\log m\big)^{(d-1)(\frac{1}{p_*}-\frac{1}{p_1}-\frac{1}{q_0}+\frac{1}{q_1})} \\ &\asymp m^{-t}\,\big(\log m\big)^{(d-1)(t-\frac{1}{q_0}+\frac{1}{q_1})} \, .\end{aligned}$$

This yields the estimate from above. The estimate from below is a consequence of Prop. 9.
Step 2. Proof of (ii). We argue as in Step 1, but replacing (55) by (56). The proof is complete. ■

Next we turn to the Besov case. Also in [25] the following result can be found.

Proposition 13. *Let* $p_0 < p_1$ *and* $q_0 \le q_1$*. Let*

$$t := \frac{1}{p_0} - \frac{1}{p_1} \qquad \text{and} \qquad r := \min\Big(\frac{1}{p_0} - \frac{1}{p_1}, \frac{1}{q_0} - \frac{1}{q_1}\Big) \, .$$

Then

$$\sigma_m\Big(s^t_{p_0,q_0}b(\Omega), s^0_{p_1,q_1}b(\Omega), \mathcal{B}\Big) \asymp m^{-r} \, , \qquad m \in \mathbb{N} \, .$$

Only in case $\frac{1}{p_0} - \frac{1}{p_1} \le \frac{1}{q_0} - \frac{1}{q_1}$ it makes sense to apply it for our problem.

Corollary 5. *We assume*

$$\Big(\frac{1}{p_0} - \frac{1}{p_1}\Big)_+ < t \le \frac{1}{q_0} - \frac{1}{q_1} \, .$$

Then

$$\sigma_m\Big(s^t_{p_0,q_0}b(\Omega), s^0_{p_1,q_1}b(\Omega), \mathcal{B}\Big) \asymp m^{-t} \, , \qquad m \in \mathbb{N} \, .$$

Proof. Again we shall use the continuous embedding $s^t_{p_0,q_0}b(\Omega) \hookrightarrow s^t_{p_*,q_0}b(\Omega)$, see Lemma 1. Then we apply Prop. 13 with respect to the pair $(s^t_{p_*,q_0}b(\Omega), s^0_{p_1,q_1}b(\Omega))$ and obtain the estimate from above. The estimate from below is a consequence of Prop. 9. ■

5.4 The widths of best m-term approximation in the case of two Lizorkin-Triebel sequence spaces

Our strategy is as follows. If t is large enough, then we employ Cor. 2. A few other cases are covered by Cor. 4. To close the gap, at least partly, we use interpolation theory.

Theorem 4. *Let either*

$$t > \max\Big(0, \frac{1}{p_0} - \frac{1}{p_1}, \frac{1}{q_0} - \frac{1}{q_1}\Big)\,, \tag{60}$$

or

$$\max\Big(0, \frac{1}{p_0} - \frac{1}{p_1}\Big) < t < \frac{1}{q_0} - \frac{1}{q_1} \qquad \textit{and} \qquad p_1 \le q_1\,. \tag{61}$$

Then we have

$$\sigma_m\Big(s^t_{p_0,q_0}f(\Omega), s^0_{p_1,q_1}f(\Omega), \mathcal{B}\Big) \asymp m^{-t}\big(\log m\big)^{(d-1)\left(t-\frac{1}{q_0}+\frac{1}{q_1}\right)_+}, \quad m \ge 2\,.$$

Proof. The estimates from below can be found in Proposition 9. The proof of the estimates from above will be split into two parts.

Step 1. Let the parameters be as in (60). The case $q_1 < p_1$ is covered by Cor. 4. Hence, it is enough to deal with $q_1 \ge p_1$.

Substep 1.1. Some technical preparations. We choose $0 < u < \infty$, $0 < v \le \infty$, $r > 0$, and $0 < \Theta < 1$, such that

$$t = \Theta r\,, \qquad \frac{1}{p_0} = \frac{1-\Theta}{p_1} + \frac{\Theta}{u}\,, \qquad \frac{1}{q_0} = \frac{1-\Theta}{q_1} + \frac{\Theta}{v}\,. \tag{62}$$

There is still a lot of freedom. We wish to apply Cor. 2 with respect to the pair $(s^r_{u,v}f(\Omega), s^0_{p_1,q_1}f(\Omega))$. This requires the validity of the following inequalities:

$$\begin{aligned}
r > \frac{1}{u} - \frac{1}{p_1} &\iff t > \frac{\Theta}{u} - \frac{\Theta}{p_1} = \frac{1}{p_0} - \frac{1-\Theta}{p_1} - \frac{\Theta}{p_1} = \frac{1}{p_0} - \frac{1}{p_1}\\
r > \frac{1}{u} - \frac{1}{q_1} &\iff t > \frac{\Theta}{u} - \frac{\Theta}{q_1} = \frac{1}{p_0} - \frac{1-\Theta}{p_1} - \frac{\Theta}{q_1} = \frac{1}{p_0} - \frac{1}{p_1} + \Theta\Big(\frac{1}{p_1} - \frac{1}{q_1}\Big)\\
r > \frac{1}{v} - \frac{1}{p_1} &\iff t > \frac{\Theta}{v} - \frac{\Theta}{p_1} = \frac{1}{q_0} - \frac{1-\Theta}{q_1} - \frac{\Theta}{p_1} = \frac{1}{q_0} - \frac{1}{q_1} + \Theta\Big(\frac{1}{q_1} - \frac{1}{p_1}\Big)\\
r > \frac{1}{v} - \frac{1}{q_1} &\iff t > \frac{\Theta}{v} - \frac{\Theta}{q_1} = \frac{1}{q_0} - \frac{1-\Theta}{q_1} - \frac{\Theta}{q_1} = \frac{1}{q_0} - \frac{1}{q_1}\,.
\end{aligned}$$

Except the second inequality all others are guaranteed by our assumptions. Observe, if $p_1 = q_1$ then all four inequalities are fulfilled and we obtain

$$r > \frac{1}{\min(u,v)} - \frac{1}{\max(p_1,q_1)} .$$

Let $p_1 < q_1$. Temporarily we also assume $u \neq p_1$ and $v \neq q_1$. Reformulating the second inequality we arrive at the condition

$$\frac{t - \frac{1}{p_0} + \frac{1}{p_1}}{\frac{1}{p_1} - \frac{1}{q_1}} > \Theta = \frac{\frac{1}{p_0} - \frac{1}{p_1}}{\frac{1}{u} - \frac{1}{p_1}} = \frac{\frac{1}{q_0} - \frac{1}{q_1}}{\frac{1}{v} - \frac{1}{q_1}} , \tag{63}$$

where we also used (62). We shall use the abbreviations

$$\Theta_p := \frac{\frac{1}{p_0} - \frac{1}{p_1}}{\frac{1}{u} - \frac{1}{p_1}} \qquad \text{and} \qquad \Theta_q := \frac{\frac{1}{q_0} - \frac{1}{q_1}}{\frac{1}{v} - \frac{1}{q_1}} .$$

Further, let $\mathcal{R}(\Theta_w)$ be the set of all possible values of Θ_w, $w \in \{p,q\}$ (p_0, p_1, q_0, q_1 are fixed). Now we continue with a discussion of these quantities Θ_p and Θ_q. Of course, if $p_0 < p_1$, then $u < p_0$ follows. Because of

$$\lim_{u \downarrow 0} \frac{\frac{1}{p_0} - \frac{1}{p_1}}{\frac{1}{u} - \frac{1}{p_1}} = 0$$

we obtain that $\mathcal{R}(\Theta_p) = (0,1)$ in this case. Now, let $p_1 < p_0$ and hence $p_0 < u$. Then we find

$$\lim_{u \uparrow \infty} \frac{\frac{1}{p_0} - \frac{1}{p_1}}{\frac{1}{u} - \frac{1}{p_1}} = 1 - \frac{p_1}{p_0}$$

and hence $\mathcal{R}(\Theta_p) = (1 - \frac{p_1}{p_0}, 1)$. Observe

$$\frac{t - \frac{1}{p_0} + \frac{1}{p_1}}{\frac{1}{p_1} - \frac{1}{q_1}} > 1 - \frac{p_1}{p_0} \qquad \Longleftrightarrow \qquad t > \frac{1}{q_1}\Big(\frac{p_1}{p_0} - 1\Big) .$$

Since $\frac{p_1}{p_0} - 1 < 0$ the right-hand side in the last inequality is ≤ 0 and for this reason the inequality is obviously true. In other words, if we choose u sufficiently large, then (63) is satisfied with Θ replaced by Θ_p. Finally, if $p_0 = p_1$ then $u = p_0$ follows and $\mathcal{R}(\Theta_p) = (0,1)$. Next we repeat this arguments with respect to Θ_q. It follows

$$\mathcal{R}(\Theta_q) = \begin{cases} (0,1) & \text{if} \quad q_0 \leq q_1 , \\ 1 - \frac{q_1}{q_0} & \text{if} \quad q_1 < q_0 . \end{cases}$$

In case $q_1 < q_0$ we have

$$\frac{t - \frac{1}{p_0} + \frac{1}{p_1}}{\frac{1}{p_1} - \frac{1}{q_1}} > 1 - \frac{q_1}{q_0} \qquad \Longleftrightarrow \qquad t > \frac{1}{p_0} - \frac{q_1}{p_1}\frac{1}{p_1} + \frac{1}{q_0} - \frac{1}{q_1} .$$

Observe $q_1 > p_1$ and $\frac{1}{q_0} - \frac{1}{q_1} < 0$. Hence, the inequality on the right-hand side follows from $t > \frac{1}{p_0} - \frac{1}{p_1}$. As above, if we choose v sufficiently large, then (63) is satisfied with

Θ given by Θ_q.

Summarizing, we can always choose u, v (depending on p_0, p_1, q_0, q_1) such that $\Theta = \Theta_p = \Theta_q$ and (63) is fulfilled. This implies

$$r > \frac{1}{\min(u,v)} - \frac{1}{\max(p_1,q_1)}$$

also in case $p_1 < q_1$.

Substep 1.2. Hence, by Substep 1.1, we can apply Cor. 2 and find

$$s^r_{u,v}f(\Omega) \hookrightarrow \mathcal{A}_\infty^{r,-(d-1)(r-\frac{1}{v}+\frac{1}{q_1})}\big(s^0_{p_1,q_1}f(\Omega),\mathcal{B}\big)\,.$$

Next we use Thm. 3 and conclude by obvious monotonicity properties of the real method

$$\begin{aligned} s^t_{p_0,q_0}f(\Omega) &\hookrightarrow \Big(s^0_{p_1,q_1}f(\Omega), s^r_{u,v}f(\Omega)\Big)_{\Theta,\infty} \\ &\hookrightarrow \Big(s^0_{p_1,q_1}f(\Omega), \mathcal{A}_\infty^{r,-(d-1)(r-\frac{1}{v}+\frac{1}{q_1})}\big(s^0_{p_1,q_1}f(\Omega),\mathcal{B}\big)\Big)_{\Theta,\infty} \\ &= \mathcal{A}_\infty^{t,-(d-1)\left(t-\frac{1}{q_0}+\frac{1}{q_1}\right)}\big(s^0_{p_1,q_1}f(\Omega),\mathcal{B}\big)\,. \end{aligned}$$

The last line follows from Prop. 5(i) and the choice of the parameters.

Step 2. Let the parameters be as in (61). This case is covered by Cor. 4. ■

5.5 The widths of best m-term approximation in the case of two Besov sequence spaces

We shall use the same strategy as in Subsection 5.4. This time we have a complete solution.

Theorem 5. *Let*

$$t > \max\Big(0, \frac{1}{p_0} - \frac{1}{p_1}\Big)\,. \tag{64}$$

Then we have

$$\sigma_m\Big(s^t_{p_0,q_0}b(\Omega), s^0_{p_1,q_1}b(\Omega),\mathcal{B}\Big) \asymp m^{-t}\big(\log m\big)^{(d-1)\left(t-\frac{1}{q_0}+\frac{1}{q_1}\right)_+}, \quad m \geq 2\,. \tag{65}$$

Proof. The estimates from below can be found in Proposition 9. The proof of the estimates from above is a bit sophisticated and requires a splitting into several cases.

Step 1. If

$$\max\Big(0, \frac{1}{p_0} - \frac{1}{p_1}\Big) < t \leq \frac{1}{q_0} - \frac{1}{q_1}$$

then the claim follows from Cor. 5.

Step 2. If t is as in (60) and $p_1 \leq q_1$, then the arguments from Step 1 of the proof of

Thm. 4 carry over without changes.

Step 3. We assume

$$p_0 \le p_1\,, \qquad 0 \le \frac{1}{q_0} - \frac{1}{q_1} < t \qquad \text{and} \qquad q_1 < p_1\,.$$

As always we choose $0 < u, v \le \infty$, $r > 0$, and $0 < \Theta < 1$ according to (62). Since $u \le p_0 \le p_1$ and $v \le q_0 \le q_1$ we can choose Θ arbitrarily small. Then the condition

$$r > \frac{1}{\min(u,v)} - \frac{1}{\max(p_1,q_1)} = \frac{1}{\min(u,v)} - \frac{1}{p_1}$$

can be fulfilled. This follows from

$$\begin{aligned} r > \frac{1}{u} - \frac{1}{p_1} \quad &\Longleftrightarrow \quad t > \frac{1}{p_0} - \frac{1}{p_1}\,, \\ r > \frac{1}{v} - \frac{1}{p_1} \quad &\Longleftrightarrow \quad t > \frac{1}{q_0} - \frac{1}{q_1} + \Theta\Big(\frac{1}{q_1} - \frac{1}{p_1}\Big)\,. \end{aligned}$$

Corollary 2 implies

$$s^r_{u,v}b(\Omega) \hookrightarrow \mathcal{A}_\infty^{r,-(d-1)(r-\frac{1}{v}+\frac{1}{q_1})}\big(s^0_{p_1,q_1}b(\Omega),\mathcal{B}\big)\,.$$

Hence, using Thm. 3 we conclude

$$\begin{aligned} s^t_{p_0,q_0}b(\Omega) \quad &\hookrightarrow \quad \Big(s^0_{p_1,q_1}b(\Omega)\,, s^r_{u,v}b(\Omega)\Big)_{\Theta,\infty} \\ &\hookrightarrow \quad \Big(s^0_{p_1,q_1}b(\Omega)\,, \mathcal{A}_\infty^{r,-(d-1)(r-\frac{1}{v}+\frac{1}{q_1})}\big(s^0_{p_1,q_1}b(\Omega),\mathcal{B}\big)\Big)_{\Theta,\infty} \\ &= \quad \mathcal{A}_\infty^{t,-(d-1)\left(t-\frac{1}{q_0}+\frac{1}{q_1}\right)}\big(s^0_{p_1,q_1}b(\Omega),\mathcal{B}\big)\,. \end{aligned}$$

The last line follows from Prop. 5(i) and the choice of the parameters.

Step 4. We assume

$$p_1 < p_0\,, \qquad 0 \le \frac{1}{q_0} - \frac{1}{q_1} < t \qquad \text{and} \qquad q_1 < p_1\,.$$

This time we choose $u = p_0$ and v and r as before. The condition with respect to r, $r > \frac{1}{v} - \frac{1}{p_1}$, can be satisfied as in Step 3 (by choosing Θ small). Furthermore, $r > \frac{1}{p_0} - \frac{1}{p_1}$ follows immediately from $r > t$. Hence, with Θ sufficiently small we obtain

$$s^r_{p_0,v}b(\Omega) \hookrightarrow \mathcal{A}_\infty^{r,-(d-1)(r-\frac{1}{v}+\frac{1}{q_1})}\big(s^0_{p_1,q_1}b(\Omega),\mathcal{B}\big) \qquad \text{(see Cor. 2)}\,,$$

and consequently

$$\begin{aligned} s^t_{p_0,q_0}b(\Omega) \quad &\hookrightarrow \quad \Big(s^0_{p_0,q_1}b(\Omega)\,, s^r_{p_0,v}b(\Omega)\Big)_{\Theta,\infty} \qquad \text{(see Thm. 3)} \\ &\hookrightarrow \quad \Big(s^0_{p_1,q_1}b(\Omega)\,, \mathcal{A}_\infty^{r,-(d-1)(r-\frac{1}{v}+\frac{1}{q_1})}\big(s^0_{p_1,q_1}b(\Omega),\mathcal{B}\big)\Big)_{\Theta,\infty} \\ &= \quad \mathcal{A}_\infty^{t,-(d-1)\left(t-\frac{1}{q_0}+\frac{1}{q_1}\right)}\big(s^0_{p_1,q_1}b(\Omega),\mathcal{B}\big)\,, \end{aligned}$$

where we additionally used $s^0_{p_0,q_1}b(\Omega) \hookrightarrow s^0_{p_1,q_1}b(\Omega)$, see Lemma 1. Summarizing Step 3 and Step 4 we have proved (65) if $q_0 \le q_1$.

Step 5. It remains to consider

$$q_1 < \min(q_0, p_1) \qquad \text{and} \qquad \max\Big(0, \frac{1}{p_0} - \frac{1}{p_1}\Big) < t\,.$$

Here we need a further splitting into the following cases: Case A: $q_1 \le p_0 \le q_0$; Case B: $q_1 < p_1 \le q_0 < p_0$; Case C: $q_1 < q_0 \le \min(p_0, p_1)$; Case D: $p_0 \le q_1 < \min(q_0, p_1)$.

Case A: We choose some α s.t.

$$0 < \alpha < t - \max\Big(0, \frac{1}{p_0} - \frac{1}{p_1}\Big)\,.$$

Lemma 2 implies the chain of continuous embeddings

$$s^t_{p_0,q_0}b(\Omega) \hookrightarrow s^{t-\alpha}_{p_0,q_1}b(\Omega) \hookrightarrow s^0_{p_1,q_1}b(\Omega)\,.$$

Since, by assumption of Case A,

$$\alpha > 0 = \frac{1}{\min(p_0, q_0)} - \frac{1}{\max(p_0, q_1)}$$

we derive from Cor. 2 in combination with a lifting argument (compare with Cor. 2 in [25])

$$s^t_{p_0,q_0}b(\Omega) \hookrightarrow \mathcal{A}_\infty^{\alpha,-(d-1)(\alpha-\frac{1}{q_0}+\frac{1}{q_1})}\big(s^{t-\alpha}_{p_0,q_1}b(\Omega), \mathcal{B}\big)\,.$$

On the other hand, as a consequence of Steps 3 and 4 (see the last sentence in Step 4)

$$s^{t-\alpha}_{p_0,q_1}b(\Omega) \hookrightarrow \mathcal{A}_\infty^{t-\alpha,-(d-1)(t-\alpha)}\big(s^0_{p_1,q_1}b(\Omega), \mathcal{B}\big)\,.$$

Next we employ the reiteration theorem for approximation spaces, see Prop. 6. We obtain

$$\begin{aligned}
s^t_{p_0,q_0}b(\Omega) &\hookrightarrow \mathcal{A}_\infty^{\alpha,-(d-1)(\alpha-\frac{1}{q_0}+\frac{1}{q_1})}\big(s^{t-\alpha}_{p_0,q_1}b(\Omega), \mathcal{B}\big) \\
&\hookrightarrow \mathcal{A}_\infty^{\alpha,-(d-1)(\alpha-\frac{1}{q_0}+\frac{1}{q_1})}\Big(\mathcal{A}_\infty^{t-\alpha,-(d-1)(t-\alpha)}\big(s^0_{p_1,q_1}b(\Omega), \mathcal{B}\big), \mathcal{B}\Big) \\
&= \mathcal{A}_\infty^{t,-(d-1)(t-\frac{1}{q_0}+\frac{1}{q_1})}\big(s^0_{p_1,q_1}b(\Omega), \mathcal{B}\big)\,.
\end{aligned}$$

Case B: Let α be as in Case A. Lemma 2 implies the chain of continuous embeddings

$$s^t_{p_0,q_0}b(\Omega) \hookrightarrow s^\alpha_{p_1,q_0}b(\Omega) \hookrightarrow s^0_{p_1,q_1}b(\Omega)\,.$$

Cor. 2 yields

$$s^\alpha_{p_1,q_0}b(\Omega) \hookrightarrow \mathcal{A}_\infty^{\alpha,-(d-1)(\alpha-\frac{1}{q_0}+\frac{1}{q_1})}\big(s^0_{p_1,q_1}b(\Omega), \mathcal{B}\big)\,.$$

Furthermore, as a consequence of Steps 3 and 4, complemented by a lifting argument, see Cor. 2 in [25], we find

$$s^t_{p_0,q_0}b(\Omega) \hookrightarrow \mathcal{A}_\infty^{t-\alpha,-(d-1)(t-\alpha)}\big(s^\alpha_{p_1,q_0}b(\Omega), \mathcal{B}\big)\,.$$

In the same manner as in Case A the reiteration theorem (Prop. 6) yields the claim.

Case C: Again let α be as in Case A. Lemma 2 implies the chain of continuous embeddings

$$s^t_{p_0,q_0}b(\Omega) \hookrightarrow s^{t-\alpha}_{p_0,p_0}b(\Omega) \hookrightarrow s^0_{p_1,q_1}b(\Omega)\,.$$

Steps 3 and 4 yield

$$s^t_{p_0,q_0}b(\Omega) \hookrightarrow \mathcal{A}_\infty^{\alpha,-(d-1)(\alpha-\frac{1}{q_0}+\frac{1}{p_0})}\big(s^{t-\alpha}_{p_0,p_0}b(\Omega),\mathcal{B}\big)\,.$$

On the other hand, using Cor. 2, we find

$$s^{t-\alpha}_{p_0,p_0}b(\Omega) \hookrightarrow \mathcal{A}_\infty^{t-\alpha,-(d-1)(t-\alpha-\frac{1}{p_0}+\frac{1}{q_1})}\big(s^0_{p_1,q_1}b(\Omega),\mathcal{B}\big)\,.$$

Now we proceed as in Cases A,B.

Case D: In this case a direct application of Cor. 2 yields (65). The proof is complete. ■

6 Best m-term approximation and spaces of dominating mixed smoothness

In this section we shall formulate and prove results about the asymptotic behaviour of the widths of best m-term approximation for various pairs of spaces of dominating mixed smoothness.

Corollary 6. *We suppose*

$$r_0 - r_1 > \max\Big(0, \frac{1}{p_0} - \frac{1}{p_1}\Big)\,.$$

Furthermore, we assume that Φ satisfies the conditions in Prop. 4 with respect to $S^{r_0}_{p_0,q_0}B(\mathbb{R}^d)$ as well as with respect to $S^{r_1}_{p_1,q_1}B(\mathbb{R}^d)$. Then we have

$$\sigma_m\Big(S^{r_0}_{p_0,q_0}B(\Omega), S^{r_1}_{p_1,q_1}B(\Omega), \Phi_\Omega\Big) \asymp m^{-r_0+r_1}\big(\log m\big)^{(d-1)\big(r_0-r_1-\frac{1}{q_0}+\frac{1}{q_1}\big)_+}\,, \qquad m \geq 2\,.$$

Proof. In case $r_1 = 0$ the proof follows from Thm. 5 in combination with Prop. 4 and the comments in Subsection 3.2.3. The case $r_1 \neq 0$ can be traced to $r_1 = 0$ by a more or less obvious lifting argument, compare with Cor. 2 in [25]. ■

In an analogous way we could treat pairs of Lizorkin-Triebel spaces of dominating mixed smoothness. But here we prefer to concentrate on the best m-term approximation for pairs $(S^{r_0}_{p_0,q_0}F(\Omega), S^{r_1}_{p_1}H(\Omega))$.

Corollary 7. *We suppose* $p_0 \le q_0$, $1 < p_1 < \infty$ *and*

$$r_0 - r_1 > \max\Big(0, \frac{1}{p_0} - \frac{1}{p_1}\Big).$$

Furthermore, we assume that Φ *satisfies the conditions in Prop. 3 with respect to* $S^{r_0}_{p_0,q_0}F(\mathbb{R}^d)$ *as well as with respect to* $S^{r_1}_{p_1,2}F(\mathbb{R}^d)$. *Then we have*

$$\sigma_m\Big(S^{r_0}_{p_0,q_0}F(\Omega), S^{r_1}_{p_1}H(\Omega), \Phi_\Omega\Big) \asymp m^{-r_0+r_1}\big(\log m\big)^{(d-1)(r_0-r_1-\frac{1}{q_0}+\frac{1}{2})_+}, \quad m \ge 2.$$

Proof. Recall the Littlewood-Paley assertion

$$S^{r_1}_{p_1,2}F(\mathbb{R}^d) = S^{r_1}_{p_1}H(\mathbb{R}^d), \qquad 1 < p_1 < \infty, \tag{66}$$

for which we refer to [28], [31, 1.5.6], [39] and [41, Prop. 2.3.1, Thm. 2.3.1]. This identity carries over to domains just by definition. Now we can argue as in the proof of Cor. 6. For simplicity we assume $r_1 = 0$. If $p_1 \ge 2$, then

$$r_0 > \max\Big(0, \frac{1}{p_0} - \frac{1}{p_1}\Big) = \max\Big(0, \frac{1}{\min(p_0, q_0)} - \frac{1}{\max(p_1, 2)}\Big)$$

and we may apply Cor. 2 in combination with Prop. 3. If $p_1 < 2$, then either

$$\max\Big(0, \frac{1}{p_0} - \frac{1}{p_1}\Big) < r_0 \le \frac{1}{q_0} - \frac{1}{2}$$

or

$$\max\Big(0, \frac{1}{p_0} - \frac{1}{p_1}, \frac{1}{q_0} - \frac{1}{2}\Big) < r_0.$$

In both situations we can employ Thm. 4. The first case is covered by (61) and the second one by (60). ■

Remark 17. Since $S^{r_0}_{p_0,p_0}B(\Omega) = S^{r_0}_{p_0,p_0}F(\Omega)$ the Cor. 7 implies Theorem 2.

Corollary 8. *We suppose* $2 \le p_1 < \infty$ *and*

$$r_0 > \max\Big(0, \frac{1}{p_0} - \frac{1}{p_1}\Big).$$

Furthermore, we assume that Φ *satisfies the conditions in Prop. 4 with respect to* $S^{r_0}_{p_0,q_0}B(\mathbb{R}^d)$ *and the conditions in Prop. 3 with respect to* $S^0_{p_1,2}F(\mathbb{R}^d)$. *Then we have*

$$\sigma_m\Big(S^{r_0}_{p_0,q_0}B(\Omega), L_{p_1}(\Omega), \Phi_\Omega\Big) \asymp m^{-r_0}\big(\log m\big)^{(d-1)(r_0-\frac{1}{q_0}+\frac{1}{2})_+}, \quad m \ge 2.$$

Proof. We shall use the continuous embedding $S^0_{p_1,2}B(\Omega) \hookrightarrow L_{p_1}(\Omega)$, Cor. 6 and Prop. 9 in combination with Prop. 3, 4. ■

Proof of Thm. 1. The claim follows from the quoted Littlewood-Paley assertions, Thm. 4 in combination with Prop. 3 and the comments in Subsection 3.2.3 since $q_0 = q_1 = 2$. ■

We shall consider one final special case.

Corollary 9. *We suppose*

$$r_0 > R(p_0,p_1) := \begin{cases} \max\left(0, \frac{1}{p_0} - \frac{1}{p_1}\right), & \text{if} \quad 2 \le p_1 < \infty\,, \\ \max\left(0, \frac{1}{p_0} - \frac{1}{2}\right), & \text{if} \quad p_1 < 2\,. \end{cases}$$

Furthermore, we assume that Φ *satisfies the conditions in Prop. 4 with respect to* $S^{r_0}_{p_0,\infty}B(\mathbb{R}^d)$ *and the conditions in Prop. 3 with respect to* $S^0_{p_1,2}F(\mathbb{R}^d)$*. Then we have*

$$\sigma_m\Big(S^{r_0}_{p_0,\infty}B(\Omega), L_{p_1}(\Omega), \Phi_\Omega\Big) \asymp m^{-r_0}\big(\log m\big)^{(d-1)(r_0+\frac{1}{2})}\,, \qquad m \ge 2\,.$$

Proof. This assertion follows from Prop. 9 and either Cor. 6 using the continuous embedding $S^0_{p_1,2}B(\Omega) \hookrightarrow L_{p_1}(\Omega)$ or from Cor. 2, in combination with Prop. 3, 4. ■

Remark 18. *This result is the nonperiodic counterpart of a result by Temlyakov [48]. Comparing his condition*

$$r_0 > r(p_0,p_1) := \begin{cases} \max\left(0, \frac{1}{p_0} - \frac{1}{p_1}\right), & \text{if} \quad 2 \le p_1 < \infty\,, \\ \left(\max\left(\frac{2}{p_0}, \frac{2}{p_1}\right) - 1\right)\frac{1}{p_1}\,, & \text{if} \quad p_1 < 2\,, \end{cases}$$

to the one in Cor. 9, we find that in case $p_1 < 2$ *we always have* $R(p_0,p_1) < r(p_0,p_1)$*, i.e. the above condition is the weaker one (though still not optimal).*

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