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„Extraktion quantifizierbarer Information aus komplexen Systemen"

Weak Order for the Discretization of the Stochastic Heat Equation Driven by Impulsive Noise

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Felix Lindner René L. Schilling

Abstract

We study the approximation of the distribution of X_T, where $(X_t)_{t\in[0,T]}$ is a Hilbert space valued stochastic process that solves a linear parabolic stochastic partial differential equation driven by an impulsive space time noise,

$$dX_t + AX_t\,dt = Q^{1/2}\,dZ_t, \quad X_0 = x_0 \in H, \quad t \in [0,T].$$

Here $(Z_t)_{t\in[0,T]}$ is an impulsive cylindrical process and Q is the covariance operator of the noise; we assume that $A^{-\alpha}$ has finite trace for some $\alpha > 0$ and that $A^{\beta}Q$ is bounded for some $\beta \in (\alpha - 1, \alpha]$.

A discretized solution $(X_h^n)_{n\in\{0,1,\dots,N\}}$ is defined via the finite element method in space (parameter $h > 0$) and a θ-method in time (parameter $\Delta t = T/N$). For $\varphi \in C_b^2(H;\mathbb{R})$ we show an integral representation for the error $|\mathbb{E}\varphi(X_h^N) - \mathbb{E}\varphi(X_T)|$ and prove that

$$|\mathbb{E}\varphi(X_h^N) - \mathbb{E}\varphi(X_T)| = O(h^{2\gamma} + (\Delta t)^{\gamma})$$

where $\gamma < 1 - \alpha + \beta$. This is the same order of convergence as in the case of a Gaussian space time noise, which has been obtained in a paper by A. Debussche and J. Printems [8].

Our result also holds for a combination of impulsive and Gaussian space time noise.

1 Introduction

In this paper, we study the weak order of convergence of numerical approximations of the solutions of a certain class of linear parabolic stochastic partial differential equations (SPDEs, for short) driven by impulsive space time noise. Unlike the strong order of convergence which measures the pathwise approximation of the true solution by a numerical one, the weak order is concerned with the approximation of the law of the true solution at a fixed time. There are not many works in literature about the weak approximation of the solutions of SPDEs (see [2], [7], [8], [9], [13]) and, to our knowledge, only SPDEs driven by Gaussian noise have been considered in this context so far. This work extends the paper [8] by A. Debussche and J. Printems, where the following Hilbert space valued stochastic differential equation is considered:

$$dX_t + AX_t\,dt = Q^{1/2}\,dW_t, \quad X_0 = x_0 \in H, \quad t \in [0,T]. \tag{1}$$

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Here $A : D(A) \subset H \to H$ is a unbounded strictly positive definite self-adjoint operator whose domain $D(A)$ is compactly embedded in H; $Q : H \to H$ is a bounded nonnegative definite symmetric operator and $(W_t)_{t\in[0,T]}$ is a cylindrical Wiener process on H, $T \in (0,\infty)$. A standard reference for this setting is [5].
If we set $H := L^2(\mathcal{O}) = L^2(\mathcal{O}, \mathcal{B}(\mathcal{O}), d\xi)$, $\mathcal{O} \subset \mathbb{R}^d$ open and bounded, and $(A, D(A)) := (-\Delta,\, H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}))$, then (1) is an abstract formulation of the stochastic heat equation with Dirichlet boundary conditions

$$\left.\begin{aligned} \frac{\partial X(t,\xi)}{\partial t} - \Delta X(t,\xi) &= \dot{\eta}(t,\xi), && (t,\xi) \in [0,T]\times\mathcal{O}, \\ X(\cdot,\cdot) &= 0 && \text{on } [0,T]\times\partial\mathcal{O}, \\ X(0,\cdot) &= x_0 && \text{on } \mathcal{O}. \end{aligned}\right\} \tag{2}$$

Here $\dot{\eta} = \frac{\partial\eta}{\partial t}$ is a generalized random function which can be described as the time derivative (in a distributional sense, cf. [25], [29]) of a real-valued generalized Gaussian process, formally written as

$$\eta(t,\xi) = \int_{\mathcal{O}} q_0(\xi,\zeta) W(t,\zeta)\, d\zeta, \tag{3}$$

where $(W(t,\cdot))_{t\in[0,T]}$ is a cylindrical Wiener process on $L^2(\mathcal{O})$ and q_0 is a positive semidefinite symmetric (generalized) function on $\mathcal{O}\times\mathcal{O}$. The operator Q is given by $Qx(\xi) = \int_{\mathcal{O}} q(\xi,\zeta)x(\zeta)\, d\zeta$ with $q(\xi,\zeta) = \int_{\mathcal{O}} q_0(\xi,\tau)q_0(\tau,\zeta)\, d\tau$ describing the spatial correlation of the noise η, cf. [24], Ch. 4.9.2 and Example 14.26.

Throughout this article, let $H := L^2(\mathcal{O})$. We consider the equation

$$dX_t + AX_t\, dt = Q^{1/2}\, dZ_t, \quad X_0 = x_0 \in H, \quad t \in [0,T], \tag{4}$$

where A and Q are as above and $(Z_t)_{t\in[0,T]} = (Z(t,\cdot))_{t\in[0,T]}$ is an impulsive cylindrical process on H, see Section 2 for the definition. This is an abstract version of problem (2) with $W(t,\zeta)$ replaced by $Z(t,\zeta)$ in the formal definition (3) of the noise η.

In [8], a discretization $(X_h^n)_{n\in\{1,\dots,N\}}$ of the solution $(X_t)_{t\in[0,T]}$ of equation (1) is obtained by the finite element method in space (parameter $h > 0$) and a θ-method in time (parameter $\Delta t = T/N$). Under the assumption that $A^{-\alpha}$ is a finite trace operator for some $\alpha > 0$ and that $A^\beta Q$ is bounded for some $\beta \in (\alpha - 1, \alpha]$, it is shown there that for functions $\varphi \in C_b^2(H,\mathbb{R})$,

$$|\mathbb{E}\varphi(X_h^N) - \mathbb{E}\varphi(X_T)| \le C \cdot (h^{2\gamma} + (\Delta t)^\gamma) \tag{5}$$

for any $\gamma < 1 - \alpha + \beta \le 1$.

In this paper, we consider the analoguous discretization of the solution of (4) and make the same assumptions on the operators A and Q as in [8]. We give a representation formula for the error (Theorem 4) and, under some integrability condition on the jump size intensity ν of the cylindrical impulsive process $(Z_t)_{t\in[0,T]}$, we show that (5) holds also for the solution $(X_t)_{t\in[0,T]}$ in (4) and the corresponding discretization (Theorem 7).

SPDEs driven by impulsive noise (or Poisson noise) have been considered, e.g. in [1], [14], [18], [20], [22], [23]. The monograph [24] gives a good overview about SPDEs driven by Lévy noise. In [15] and [16], numerical approximations in time and space of SPDEs driven by Poisson random measures are investigated and the strong error is estimated. Of course, this especially implies an estimate for the weak approximation error. A difference to our result is that we look at impulsive noise which is white in time and coloured in space whereas in [15] and [16] a class of SPDEs driven by Poisson random measures which correspond to impulsive space time white noise is considered. Our motivation for this paper was to show that the techniques applied in [8] with respect to the cylindrical Wiener process also work for certain jump processes.

The main technical difference between (4) and (1) lies in the fact that the impulsive cylindrical process $(Z_t)_{t\in[0,T]}$ is a purely discontinuous Hilbert space valued martingale, while the cylindrical Wiener process $(W_t)_{t\in[0,T]}$ is continuous. As a consequence, the main tools for estimating the weak order of convergence for the numerical scheme — the Itô formula and (connected with it) the backward Kolmogorov equations for certain processes associated with the solutions of the SPDEs and their discretizations — are completely different for (4) and (1). The main task therefore is to find manageable expression for the approximation error, which allows estimates using techniques similar to those in [8].

Let us finally note that (5) remains true for the solution $(X_t)_{t\in[0,T]}$ of

$$dX_t + AX_t\,dt = Q_0^{1/2}\,dW_t + Q_1^{1/2}\,dZ_t, \quad X_0 = x_0 \in H, \quad t \in [0,T],$$

and the corresponding discretization. Here, the covariance operators Q_0 and Q_1 are assumed to have the same properties as Q above.

2 Impulsive cylindrical process

Let $\mathcal{O} \subset \mathbb{R}^d$ be open and bounded, $T \in (0,\infty)$, and consider the product space $[0,T]\times\mathcal{O}\times\mathbb{R}$ equipped with the Borel σ-algebra $\mathcal{B}([0,T]\times\mathcal{O}\times\mathbb{R})$. The generic element in $[0,\infty)\times\mathcal{O}\times\mathbb{R}$ is denoted by (t,ξ,σ) or (s,ξ,σ). Let ν be a sigma-finite measure on $\mathbb{R}$ and π a Poisson random measure on $[0,\infty)\times\mathcal{O}\times\mathbb{R}$ with reference measure $dt\,d\xi\,\nu(d\sigma)$. By $\hat{\pi}$ we denote the compensated Poisson random measure, i.e.

$$\hat{\pi}(dt,d\xi,d\sigma) = \pi(dt,d\xi,d\sigma) - dt\,d\xi\,\nu(d\sigma).$$

Let π be defined on a complete probability space $(\Omega,\mathcal{A},\mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual hypotheses, cf. [21]. We assume that $\pi([0,t]\times B)$ is $(\mathcal{F}_t)$-measurable for all $t\in[0,T]$, $B\in\mathcal{B}(\mathcal{O}\times\mathbb{R})$ and that $\pi((s,t]\times B)$ is independent of $\mathcal{F}_s$ for $0\le s<t\le T$, $B\in\mathcal{B}(\mathcal{O}\times\mathbb{R})$. Furthermore, we assume that π is of the form

$$\pi(A)(\omega) = \sum_{j=1}^{\infty} \delta_{(T_j(\omega),\Xi_j(\omega),\Sigma_j(\omega))}(A), \qquad A\in\mathcal{B}([0,T]\times\mathcal{O}\times\mathbb{R}),\ \omega\in\Omega, \tag{6}$$

for a properly chosen sequence $\big((T_j,\Xi_j,\Sigma_j)\big)_{j\in\mathbb{N}}$ of random elements in $[0,T]\times\mathcal{O}\times\mathbb{R}$, cf. [24], Chapter 6.

For general impulsive cylindrical processes, ν is a Lévy measure, i.e. $\nu(\{0\})=0$ and $\int_{\mathbb{R}}\min\big(\sigma^2,1\big)\,\nu(d\sigma)<\infty$, cf. [24], Ch. 7.2. In addition, we will assume that $\int_{|\sigma|\ge1}\sigma^2\,\nu(d\sigma)<\infty$ which is equivalent to saying that

$$\int_{\mathbb{R}}\sigma^2\,\nu(d\sigma)<\infty, \tag{7}$$

Under this condition the random variables $Z_t^{(k)}$ and Z_t defined below have finite second moments. (This is, e.g. the case if ν is a Lévy measure with bounded support.)

To fix notation, let us give a brief survey of L^2-integration w.r.t. $\hat{\pi}$ for deterministic integrands. Let $f : [0,T] \times \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ be a simple function, i.e.

$$f = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k},$$

where $n \in \mathbb{N}$, $a_k \in \mathbb{R}$ and $A_k \in \mathcal{B}([0,T] \times \mathcal{O} \times \mathbb{R})$, such that

$$\int_0^T \int_{\mathcal{O}} \int_{\mathbb{R}} \mathbb{1}_{A_k}(t,\xi,\sigma)\, \nu(d\sigma)\, d\xi\, dt < \infty,$$

for $k = 1, \ldots, n$.
For simple functions f the stochastic integral w.r.t. $\hat{\pi}$ is defined by

$$\int_0^T \int_{\mathcal{O}\times\mathbb{R}} f(t,\xi,\sigma)\, \hat{\pi}(dt, d\xi, d\sigma) := \sum_{k=1}^{n} a_k \hat{\pi}(A_k).$$

Since

$$\mathbb{E}\left|\int_0^T \int_{\mathcal{O}\times\mathbb{R}} f(t,\xi,\sigma)\, \hat{\pi}(dt, d\xi, d\sigma)\right|^2 = \int_0^T \int_{\mathcal{O}\times\mathbb{R}} |f(t,\xi,\sigma)|^2\, dt\, d\xi\, \nu(d\sigma),$$

the stochastic integral can be uniquely extended to an isometric linear operator mapping $L^2([0,T] \times \mathcal{O} \times \mathbb{R}, ds\, d\xi\, \nu(d\sigma); \mathbb{R})$ to $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R})$. Later, we will also consider $\hat{\pi}$-integrals of H-valued square integrable functions which are defined similarly. In the course of the proof of our result, we will have to deal with L^1-integrals of stochastic integrands against the (not compensated) random measure π; see [24], Sections 6.2, 8.7 and [29], Chapter 2, for a detailed exposition of the various integrals.
As usual, for $t \in [0,T]$ and $f \in L^2([0,T] \times \mathcal{O} \times \mathbb{R}, ds\, d\xi\, \nu(d\sigma); \mathbb{R})$, we define

$$\int_0^t \int_{\mathcal{O}\times\mathbb{R}} f(s,\xi,\sigma)\, \hat{\pi}(ds, d\xi, d\sigma) := \int_0^T \int_{\mathcal{O}\times\mathbb{R}} \mathbb{1}_{[0,t]}(s) f(s,\xi,\sigma)\, \hat{\pi}(ds, d\xi, d\sigma).$$

Now we can to define the impulsive cylindrical process. Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of H and for $k \in \mathbb{N}$, $t \in [0,T]$, let

$$Z_t^{(k)} := Z(t, e_k) := \int_0^t \int_{\mathcal{O}\times\mathbb{R}} e_k(\xi) \cdot \sigma\, \hat{\pi}(ds, d\xi, d\sigma).$$

Note that the processes $(Z_t^{(k)})_{t\in[0,T]}$, $k \in \mathbb{N}$, are real-valued, square integrable Lévy processes which are also martingales.
Now let U be a further Hilbert space such that H is densely embedded in U and such that the embedding is Hilbert-Schmidt, e.g. $U = H^{-\frac{d}{2}-\epsilon}(\mathcal{O})$ for some $\epsilon > 0$. In [24], Chapter 7, it is shown that

$$Z_t := L^2(\Omega, \mathcal{A}, \mathbb{P}; U)\text{-}\lim_{n\uparrow\infty} \sum_{k=1}^{n} Z_t^{(k)} e_k, \qquad t \in [0,T],$$

defines an U-valued $L^2(\mathbb{P})$-Lévy-martingale with reproducing kernel Hilbert space $H = L^2(\mathcal{O})$; $(Z_t)_{t\in[0,T]}$ is called *impulsive cylindrical process on $L^2(\mathcal{O})$ with jump size intensity ν.*

As for the following examples, compare [22], [23].

Example 1. Let $\beta \in (0,2)$, $\tau \in (0,\infty)$ and consider the jump size intensity

$$\nu(d\sigma) = \frac{1}{\sigma^{1+\beta}} \mathbb{1}_{[0,\tau]}(\sigma)\, d\sigma.$$

Then, for every $B \in \mathcal{B}(\mathcal{O})$, the one-dimensional process

$$\big(Z(t,\mathbb{1}_B)\big)_{t\in[0,T]} = \left(\int_0^t \int_{\mathcal{O}\times\mathbb{R}} \mathbb{1}_B(\xi)\cdot\sigma\,\hat{\pi}(ds,d\xi,d\sigma)\right)_{t\in[0,T]}$$

can be characterized in terms of one-sided β-stable Lévy processes as follows:
If $\beta \in (1,2)$, then $\big(Z(t,\mathbb{1}_B)\big)_{t\in[0,T]}$ is equivalent to the process obtained by removing all jumps greater than τ from a one-sided β-stable Lévy process $(L_t)_{t\in[0,T]}$ with Laplace transform

$$\mathbb{E}e^{-rL_t} = \exp\left\{-t|B|c_\beta\cdot r^\beta\right\} = \exp\left\{-t|B|\int_0^\infty \big(1-e^{-r\sigma}-r\sigma\big)\frac{d\sigma}{\sigma^{1+\beta}}\right\}, \qquad r>0,$$

and then adding the shift $t\mapsto t|B|\int_\tau^\infty \sigma\frac{d\sigma}{\sigma^{1+\beta}} = t|B|\frac{\tau^{1-\beta}}{\beta-1}$. Here $|B|$ denotes the Lebesgue measure of $B\in\mathcal{B}(\mathcal{O})$.
If $\beta \in (0,1)$, $\big(Z(t,\mathbb{1}_B)\big)_{t\in[0,T]}$ is equivalent to the process obtained by removing all jumps greater than τ from a positive one-sided β-stable Lévy process $(L_t)_{t\in[0,T]}$ with Laplace transform

$$\mathbb{E}e^{-rL_t} = \exp\left\{-t|B|c_\beta\cdot r^\beta\right\} = \exp\left\{-t|B|\int_0^\infty \big(1-e^{-r\sigma}\big)\frac{d\sigma}{\sigma^{1+\beta}}\right\}, \qquad r>0,$$

and then subtracting the shift $t\mapsto \mathbb{E}\int_0^t\int_{\mathcal{O}}\int_{\mathbb{R}}\mathbb{1}_B(\xi)\cdot\sigma\,\pi(ds,d\xi,d\sigma) = t|B|\frac{\tau^{1-\beta}}{1-\beta}$.

Example 2. Let $\alpha\in(0,2)$, $\tau\in(0,\infty)$ and set

$$\nu(d\sigma) = \frac{1}{|\sigma|^{1+\alpha}}\mathbb{1}_{[-\tau,\tau]}(\sigma)\,d\sigma.$$

Then, for $B\in\mathcal{B}(\mathcal{O})$, we have

$$\begin{aligned} Z(t,\mathbb{1}_B) &= \int_0^t\int_{\mathcal{O}\times\mathbb{R}}\mathbb{1}_B(\xi)\cdot\sigma\,\hat{\pi}(ds,d\xi,d\sigma) \\ &= L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R})\text{-}\lim_{\epsilon\searrow 0}\int_0^t\int_{\mathcal{O}\times\{|\sigma|\geq\epsilon\}}\mathbb{1}_B(\xi)\cdot\sigma\,\pi(ds,d\xi,d\sigma), \end{aligned} \tag{8}$$

and $\big(Z(t,\mathbb{1}_B)\big)_{t\in[0,T]}$ is equivalent to the process obtained by removing all jumps of absolute value greater than τ from a symmetric α-stable Lévy process $(L_t)_{t\in[0,T]}$ with Fourier transform

$$\mathbb{E}e^{irL_t} = \exp\left\{-t|B|C_\alpha\cdot r^\alpha\right\} = \exp\left\{-t|B|\int_{\mathbb{R}}(1-\cos(r\sigma))\frac{d\sigma}{|\sigma|^{1+\alpha}}\right\}, \qquad r\in\mathbb{R},$$

i.e.

$$\mathbb{E}e^{irZ(t,\mathbb{1}_B)} = \exp\left\{-t|B|\int_{-\tau}^{\tau}(1-\cos(r\sigma))\frac{d\sigma}{|\sigma|^{1+\alpha}}\right\}, \qquad r\in\mathbb{R}.$$

The second equality in (8) holds since ν is symmetric and therefore

$$\int_0^t\int_{\mathcal{O}\times\{|\sigma|\geq\epsilon\}}\mathbb{1}_B(\xi)\cdot\sigma\,ds\,d\xi\,\nu(d\sigma) = 0, \qquad \epsilon>0,$$

so that the integrals w.r.t. $\hat{\pi}$ and π coincide for integrands in $L^2([0,T]\times\mathcal{O}\times\mathbb{R},\,ds\,d\xi\,\nu(d\sigma);\mathbb{R})\cap L^1([0,T]\times\mathcal{O}\times\mathbb{R},\,ds\,d\xi\,\nu(d\sigma);\mathbb{R})$.

For the estimate of the weak order of approximation (Theorem 7), we have to make a further restriction on the jump size intensity ν by assuming that

$$\int_{\mathbb{R}} \max\left(|\sigma|, \sigma^2\right) \nu(d\sigma) < \infty. \tag{9}$$

This is, e.g. fulfilled if $\beta \in (0,1)$ and $\alpha \in (0,1)$ in Examples 1 and 2.

3 Notation, assumptions and preliminary results

The inner product and norm of $H = L^2(\mathcal{O})$ are denoted by $\langle \cdot, \cdot \rangle_H$ and $|\cdot|_H$, respectively; $Q : H \to H$ is a bounded nonnegative definite symmetric operator and $A : D(A) \subset H \to H$ a (unbounded) strictly positive definite self-adjoint operator whose domain $D(A)$ (endowed with the graph norm $|\cdot|_H + |A\cdot|_H$) is compactly embedded in H. Therefore, the spectrum consists of a sequence $(\lambda_k)_{k\in\mathbb{N}} \subset (0,\infty)$. We assume that the eigenvalues are ordered increasingly, $\lambda_1 \le \lambda_2 \le \ldots$, including multiplicities. By $(\tilde{e}_k)_{k\in\mathbb{N}}$ we denote the corresponding orthonormal basis of eigenvectors. For any $s \ge 0$ we set

$$\begin{aligned} D(A^s) &:= \left\{ u = \sum_{k=1}^{\infty} \langle u, \tilde{e}_k \rangle_H \tilde{e}_k \in H \,:\, \sum_{k=1}^{\infty} \lambda_k^{2s} \langle u, \tilde{e}_k \rangle_H^2 < \infty \right\}, \\ A^s u &:= \sum_{k=1}^{\infty} \lambda_k^s \langle u, \tilde{e}_k \rangle_H \tilde{e}_k, \qquad u \in D(A^s), \end{aligned}$$

so that $D(A^s)$ endowed with the graph norm $|\cdot|_H + |A^s\cdot|_H$ is a Hilbert space. Furthermore we define $D(A^{-s})$ for $s \ge 0$ as the completion of H with respect to the norm $|\cdot|_H + |A^{-s}\cdot|_H$, defined on H by $|A^{-s}u|_H^2 = \sum_{k=1}^{\infty} \lambda_k^{-2s} \langle u, \tilde{e}_k \rangle_H^2$.

Let $(V_h)_{h>0}$ be a family of finite dimensional subspaces of $V := D(A^{1/2})$ parameterized by a small parameter $h > 0$, given by the finite element method. As standard references for the finite element method we mention [4] and [27]. By P_h we denote the orthogonal projector from H onto V_h with respect to the inner product $\langle \cdot, \cdot \rangle_H$ and Π_h is the orthogonal projector from H onto V_h with respect to the inner product $\langle A^{1/2}\cdot, A^{1/2}\cdot \rangle_H$. For any $h > 0$, we define a positive symmetric bounded linear operator $A_h : V_h \to V_h$ by

$$\langle A_h u_h, v_h \rangle_H = \langle A^{1/2} u_h, A^{1/2} v_h \rangle_H \qquad \forall (u_h, v_h) \in V_h \times V_h.$$

As in [8], let $(S(t))_{t\ge0}$ denote the C_0-semigroup of contractive operators on H generated by $-A$ and let $(S_h(t))_{t\ge0}$ denote the semigroup of operators on V_h generated by $-A_h$.

We assume that the finite dimensional spaces V_h admit the following two estimates, which are classical properties for standard finite element spaces.

Assumption:
For all $q \in [0,2]$ there exist constants $\kappa_1 > 0$, $\kappa_2 > 0$ independent from h such that for all $t > 0$

$$\|S_h(t)P_h - S(t)\|_{L(H)} \le \kappa_1 h^q t^{-q/2}, \tag{10}$$

$$\|S_h(t)P_h - S(t)\|_{L(H, D(A^{1/2}))} \le \kappa_2 h t^{-1}. \tag{11}$$

In [28], Theorem 3.5, and [17], Theorem 4.1, the above estimates (10) and (11) are shown for the case $(A, D(A)) = (-\Delta,\, H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}))$ under the assumption that

$$|\Pi_h v - v|_H \le \kappa_0 h^s |A^{s/2} v|_H, \tag{12}$$

$$|A^{1/2}(\Pi_h v - v)|_H \quad \le \quad \kappa_0 h^{s-1} |A^{s/2} v|_H \tag{13}$$

for all $s \in [1,2]$, $v \in D(A^{s/2})$ and some constant $\kappa_0 > 0$. Finite elements satisfying (12) and (13) are well known, like P_k triangular finite elements on a convex polygonal domain or Q_k rectangular finite elements on a rectangular domain, $k \ge 1$, see [4], [27].

The main assumptions concerning the operator A and the covariance operator Q of the noise are the following: There exist real numbers

$$\alpha > 0, \quad \beta \in (\alpha - 1, \alpha] \tag{14}$$

such that

$$\mathrm{Tr}(A^{-\alpha}) = \sum_{n=1}^{\infty} \lambda_n^{-\alpha} < \infty , \tag{15}$$

$$A^{\beta} Q \in L(H). \tag{16}$$

Notice that (16) implies for any $\lambda \in [0,1]$

$$A^{\lambda\beta} Q^{\lambda} \in L(H) \ \text{ and } \ \|A^{\lambda\beta} Q^{\lambda}\|_{L(H)} \le \|A^{\beta} Q\|_{L(H)}^{\lambda}. \tag{17}$$

According to [24], Chapter 9, equation (4) has a unique (predictable) *weak solution* which is given as the *mild solution*

$$X_t = S(t)x_0 + \int_0^t S(t-s) Q^{1/2}\, dZ_s, \qquad t \in [0,T], \tag{18}$$

provided that

$$\|S(t)Q^{1/2}\|_{(\mathrm{HS})} \in L^2([0,T], dt), \tag{19}$$

where $\|\cdot\|_{(\mathrm{HS})}$ is the Hilbert-Schmidt norm. It is shown in [8] that (14), (15) and (16) are sufficient conditions for (19).

We proceed with some further notation: By $C_b^k(H) = C_b^k(H;\mathbb{R})$ we denote the space of all k-times continuously Fréchet-differentiable real valued functions on H which are bounded together with their derivatives. For $\phi \in C_b^1(H)$ and $x \in H$ the first-order derivative $D\phi(x)$ of ϕ in x is identified with its gradient and thus considered as an element in H. Similarly, for $\phi \in C_b^2(H)$ and $x \in H$, the second-order derivative $D^2\varphi(x)$ is seen as an element in $L(H) = L(H,H)$, the space of all linear and bounded operators on H.
Let $L_{(\mathrm{HS})}(H) = L_{(\mathrm{HS})}(H,H)$ denote the Hilbert space of all Hilbert-Schmidt operators on H. It is a subspace of $L(H)$. Given an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of H, the scalar product in $L_{(\mathrm{HS})}(H)$ of operators $T \in L_{(\mathrm{HS})}(H)$, $S \in L_{(\mathrm{HS})}(H)$ is given by

$$\langle T, S\rangle_{(\mathrm{HS})} := \sum_{k\in\mathbb{N}} \langle Se_k, Te_k\rangle_H.$$

The corresponding Hilbert-Schmidt norm is denoted by $\|\cdot\|_{(\mathrm{HS})}$.

Finally, we use $\mathcal{M}_T^2(H)$ for the space of all right continuous $L^2(\mathbb{P})$-martingales $M = (M_t)_{t\in[0,T]}$ with values in H. $([\![M]\!]_t)_{t\in[0,T]}$ denotes the tensor quadratic variation (or operator square bracket) of $M \in \mathcal{M}_T^2(H)$; $([\![M]\!]_t^c)_{t\in[0,T]}$ denotes the continuous part. Note that for $\mathbb{P}$-almost every $\omega \in \Omega$, for every $t \in [0,T]$, $[\![M]\!]_t(\omega)$ and $[\![M]\!]_t^c(\omega)$ belong to the space of nuclear operators on H, a subspace of $L_{(\mathrm{HS})}(H)$ which is continuously embedded in $L_{(\mathrm{HS})}(H)$, cf. [21].

4 Approximation scheme

In order to approximate the mild solution

$$X_t = S(t)x_0 + \int_0^t S(t-s)Q^{1/2}\,dZ_s, \qquad t \in [0,T],$$

of equation (4), we adapt the numerical scheme described in [8], with $(W_t)_{t\in[0,T]}$ replaced by the jump process $(Z_t)_{t\in[0,T]}$.
Given an integer $N \geq 1$, set $\Delta t = T/N$ and $t_n = n\Delta t$, $n = 0,\dots,N$. For any $h > 0$, the approximations X_h^n of X_{t_n} in V_h, $n = 0,\dots,N$, are defined as those V_h-valued random variables which satisfy for all $v_h \in V_h$

$$\left\langle X_h^{n+1} - X_h^n\,,\, v_h\right\rangle_H + \Delta t\left\langle A^{1/2}\left(\theta X_h^{n+1} + (1-\theta)X_h^n\right),\, A^{1/2}v_h\right\rangle_H = \left\langle Q^{1/2}Z_{t_{n+1}} - Q^{1/2}Z_{t_n}\,,\, v_h\right\rangle_H \tag{20}$$

and

$$\langle X_h^0, v_h\rangle_H = \langle x_0, v_h\rangle_H, \tag{21}$$

with

$$\theta \in\ (1/2, 1]. \tag{22}$$

Here the expression $\left\langle Q^{1/2}Z_{t_{n+1}} - Q^{1/2}Z_{t_n}\,,\, v_h\right\rangle_H$ is the usual shorthand for

$$L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R})\text{-}\lim_{M\to\infty}\sum_{k=1}^{M}\left(Z(t_{n+1},e_k) - Z(t_n,e_k)\right)\cdot\left\langle Q^{1/2}e_k\,,\, v_h\right\rangle_H, \tag{23}$$

$(e_k)_{k\in\mathbb{N}}$ being an arbitrary orthonormal basis of H. In particular, one can choose $(e_k)_{k\in\mathbb{N}}$ as an orthonormal basis consisting solely of eigenvectors of Q and of elements of the kernel of Q. Note that $(X_h^n)_{n\in\{0,\dots,N\}}$ is a discretization of $(X_t)_{t\in[0,T]}$, both in time and space. Solving the equations above leads to

$$X_h^n \;=\; S_{h,\Delta t}^n P_h x_0 + \sum_{k=0}^{n-1} S_{h,\Delta t}^{n-k-1} T_{h,\Delta t} P_h Q^{1/2}(Z_{t_{k+1}} - Z_{t_k}), \tag{24}$$

for $n \in \{0,\dots,N\}$ and $h > 0$, where

$$\begin{aligned} S_{h,\Delta t} &:= (I + \theta\Delta t A_h)^{-1}(I - (1-\theta)\Delta t A_h),\\ T_{h,\Delta t} &:= (I + \theta\Delta t A_h)^{-1}. \end{aligned}$$

Remark 3. In this paper, we are interested in the weak order of the scheme (20)-(21). Theorem 7 below gives an estimate for the difference $|\mathbb{E}\varphi(X_h^N) - \mathbb{E}\varphi(X_T)|$ for suitable real valued functions φ. Obviously, for the numerical implementation one has to truncate the infinite sum in (23). That is, one has to replace the right hand side of (20) by

$$\left\langle \left(Q^{(M)}\right)^{1/2} Z_{t_{n+1}} - \left(Q^{(M)}\right)^{1/2} Z_{t_n}\,,\, v_h\right\rangle_H = \sum_{k=1}^{M}\left(Z(t_{n+1},e_k) - Z(t_n,e_k)\right)\cdot\left\langle Q^{1/2}e_k\,,\, v_h\right\rangle_H,$$

for some $M \in \mathbb{N}$. Here, the operator $Q^{(M)}$ is defined by

$$Q^{(M)}u = \sum_{k=1}^{M}\langle u, e_k\rangle_H Q e_k, \quad u = \sum_{k=1}^{\infty}\langle u, e_k\rangle_H e_k \in H.$$

Let $\big(X_h^{n,(M)}\big)_{n\in\{0,\dots,N\}}$ be the solution of the truncated scheme, i.e.

$$X_h^{n,(M)} = S_{h,\Delta t}^n P_h x_0 + \sum_{k=0}^{n-1} S_{h,\Delta t}^{n-k-1} T_{h,\Delta t} P_h \big(Q^{(M)}\big)^{1/2} (Z_{t_{k+1}} - Z_{t_k}),$$

and let

$$X_t^{(M)} = S(t)x_0 + \int_0^t S(t-s)\big(Q^{(M)}\big)^{1/2}\, dZ_s, \qquad t \in [0,T].$$

Then, in the setting of Theorem 7 below, it is not hard to see that

$$\Big|\mathbb{E}\varphi\big(X_h^{N,(M)}\big) - \mathbb{E}\varphi\big(X_T^{(M)}\big)\Big| \le C \cdot (h^{2\gamma} + (\Delta t)^\gamma),$$

where $C > 0$ and $\gamma > 0$ are the same numbers that appear in the upper bound for the error $|\mathbb{E}\varphi(X_h^N) - \mathbb{E}\varphi(X_T)|$ in Theorem 7. In particular, the constant C does not depend on $M \in \mathbb{N}$. Therefore, the error $\big|\mathbb{E}\varphi\big(X_h^{N,(M)}\big) - \mathbb{E}\varphi\big(X_T\big)\big|$, which is relevant for numerical simulations, can be estimated by

$$\Big|\mathbb{E}\varphi\big(X_h^{N,(M)}\big) - \mathbb{E}\varphi\big(X_T\big)\Big| \le C \cdot \big(h^{2\gamma} + (\Delta t)^\gamma\big) + \Big|\mathbb{E}\varphi\big(X_T^{(M)}\big) - \mathbb{E}\varphi\big(X_T\big)\Big|.$$

It will be useful to introduce some further notation. For each $h > 0$, we will also consider the following spatial discretization of $(X_t)_{t\in[0,T]}$

$$X_{h,t} = S_h(t)P_h x_0 + \int_0^t S_h(t-s)P_h Q^{1/2}\, dZ_s, \qquad t \in [0,T], \tag{25}$$

and the two auxiliary processes

$$\begin{aligned} Y_{h,t} &= S_h(T)P_h x_0 + \int_0^t S_h(T-s)P_h Q^{1/2}\, dZ_s \\ &= S_h(T)P_h x_0 + \int_0^t \Phi(s)\, dZ_s \qquad t \in [0,T], \end{aligned} \tag{26}$$

$$\begin{aligned} \bar{Y}_{h,t} &= S_{h,\Delta t}^N P_h x_0 + \int_0^t \sum_{k=0}^{N-1} S_{h,\Delta t}^{N-k-1} T_{h,\Delta t} \mathbb{1}_{(t_k,t_{k+1}]}(s) P_h Q^{1/2} dZ_s \\ &= S_{h,\Delta t}^N P_h x_0 + \int_0^t \Gamma(s)\, dZ_s \qquad t \in [0,T]. \end{aligned} \tag{27}$$

To ease notation, we have used

$$\begin{aligned} \Phi(s) &:= S_h(T-s)P_h Q^{1/2}, \\ \Gamma(s) &:= \sum_{k=0}^{N-1} S_{h,\Delta t}^{N-k-1} T_{h,\Delta t} \mathbb{1}_{(t_k,t_{k+1}]}(s) P_h Q^{1/2}. \end{aligned}$$

Moreover, it will be convenient to set

$$\tilde{\Phi}(s) := \Phi(T-s).$$

Before we come to the results, we let us rewrite the stochastic integrals in a way that fits to our purposes. The integrals with respect to dZ_s can be written as $\hat{\pi}$-integrals. To this end, let us define mappings E, F, G, $\tilde{F}$ from $[0,T] \times \mathcal{O} \times \mathbb{R}$ into H, by

$$E(s,\xi,\sigma) := \sum_{k=1}^\infty e_k(\xi)\sigma S(T-s)Q^{1/2} e_k, \tag{28}$$

$$F(s,\xi,\sigma) := \sum_{k=1}^{\infty} e_k(\xi)\sigma\Phi(s)e_k, \tag{29}$$

$$G(s,\xi,\sigma) := \sum_{k=1}^{\infty} e_k(\xi)\sigma\Gamma(s)e_k, \tag{30}$$

$$\tilde{F}(s,\xi,\sigma) := \sum_{k=1}^{\infty} e_k(\xi)\sigma\tilde{\Phi}(s)e_k, \tag{31}$$

where $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of H and the infinite sums are limits in the space $L^2([0,T]\times\mathcal{O}\times\mathbb{R},\, ds\,d\xi\,\nu(d\sigma);\,H)$. These limits exist since the operator valued functions $s\mapsto S(s)Q^{1/2}$, $s\mapsto\Phi(s)$ and $s\mapsto\Gamma(s)$ belong to $L^2\big([0,T],\,ds;\,L_{(\mathrm{HS})}(H)\big)$ and because of the integrability assumption (7). As usual, we do not distinguish between measurable mappings $[0,T]\times\mathcal{O}\times\mathbb{R}\to H$ and their $ds\,d\xi\,\nu(d\sigma)$-equivalence classes. Notice that the following equalities hold:

$$\begin{aligned}
\int_0^t S(T-s)Q^{1/2}\,dZ_s &= \int_0^t\int_{\mathcal{O}\times\mathbb{R}} E(s,\xi,\sigma)\,\hat{\pi}(ds,d\xi,d\sigma),\\
\int_0^t \Phi(s)\,dZ_s &= \int_0^t\int_{\mathcal{O}\times\mathbb{R}} F(s,\xi,\sigma)\,\hat{\pi}(ds,d\xi,d\sigma),\\
\int_0^t \Gamma(s)\,dZ_s &= \int_0^t\int_{\mathcal{O}\times\mathbb{R}} G(s,\xi,\sigma)\,\hat{\pi}(ds,d\xi,d\sigma),\\
\int_0^t \tilde{\Phi}(s)\,dZ_s &= \int_0^t\int_{\mathcal{O}\times\mathbb{R}} \tilde{F}(s,\xi,\sigma)\,\hat{\pi}(ds,d\xi,d\sigma).
\end{aligned}$$

Indeed, if we consider for example $\int_0^t \Phi(s)\,dZ_s$, it is easy to verify that

$$\begin{aligned}
\int_0^t \Phi(s)\,dZ_s &= L^2(\Omega,\mathcal{A},\mathbb{P};H)\text{-}\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^t \Phi(s)e_k\,dZ_s^{(k)}\\
&= L^2(\Omega,\mathcal{A},\mathbb{P};H)\text{-}\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^t\int_{\mathcal{O}\times\mathbb{R}} e_k(\xi)\sigma\Phi(s)e_k\,\hat{\pi}(ds,d\xi,d\sigma)\\
&= \int_0^t\int_{\mathcal{O}\times\mathbb{R}}\left(\sum_{k=1}^{\infty} e_k(\xi)\sigma\Phi(s)e_k\right)\hat{\pi}(ds,d\xi,d\sigma),
\end{aligned}$$

where the infinite sum in the last integral is a limit in $L^2([0,T]\times\mathcal{O}\times\mathbb{R},\, ds\,d\xi\,\nu(d\sigma);\,H)$.

5 Error expansion

In this section, for suitable functions φ defined on H, we state and prove representation formulas for the time discretization error $\mathbb{E}\varphi(X_h^N)-\mathbb{E}\varphi(X_{h,T})$ and the space discretization error $\mathbb{E}\varphi(X_{h,T})-\mathbb{E}\varphi(X_T)$. The errors are represented in terms of the functions $v_h:[0,T]\times H\to\mathbb{R}$ and $v_h:[0,T]\times H\to\mathbb{R}$ defined by

$$v_h(t,x) := \mathbb{E}\varphi\left(x+\int_{T-t}^{T} S_h(T-r)P_hQ^{1/2}dZ_r\right), \tag{32}$$

$$v(t,x) := \mathbb{E}\varphi\left(x+\int_{T-t}^{T} S(T-r)Q^{1/2}dZ_r\right). \tag{33}$$

For the reader's convenience, we summarize all equations, definitions and assumptions which will be needed in the formulation of our error representation theorem below:

Summary/Assumptions:

- The equation we are interested in is
$$dX_t + AX_t\,dt = Q^{1/2}\,dZ_t, \quad X_0 = x_0 \in H, \quad t \in [0,T]. \tag{4}$$

- *Assumption:* The jump size intensity ν of $(Z_t)_{t\in[0,T]}$ satisfies
$$\int_{\mathbb{R}} \sigma^2\,\nu(d\sigma) < \infty. \tag{7}$$

- *Assumption:* The mild solution of equation (4) exists, i.e.
$$\|S(t)Q^{1/2}\|_{(\mathrm{HS})} \in L^2([0,T],dt) \tag{19}$$
(This ensures that the weak solution exists and is equal to the mild solution.)

- We consider the following space-time- and space-discretizations of the weak solution of (4)
$$X_h^n = S_{h,\Delta t}^n P_h x_0 + \sum_{k=0}^{n-1} S_{h,\Delta t}^{n-k-1} T_{h,\Delta t} P_h Q^{1/2}(Z_{t_{k+1}} - Z_{t_k}), \quad n \in \{0,\dots,N\}, \tag{24}$$
$$X_{h,t} = S_h(t)P_h x_0 + \int_0^t S_h(t-s)P_h Q^{1/2}\,dZ_s, \qquad t \in [0,T]. \tag{25}$$

Theorem 4. *Assume that* (7) *and* (19) *hold. Let* $\varphi \in C_b^2(H)$ *such that for each* $x \in H$ *the derivative* $D^2\varphi(x)$ *is an element of* $L_{(HS)}(H)$ *and that the mapping* $x \mapsto D^2\varphi(x) \in L_{(HS)}(H)$ *is uniformly continuous on any bounded subset of* H. *Let* $T \geq 1$ *and* $(X_t)_{t\in[0,T]}$ *be the* H*-valued stochastic process solution of equation* (4). *For any* $N \geq 1$ *and* $h > 0$, *let* $(X_h^n)_{n\in\{0,\dots,N\}}$ *be given by* (24) *and let* $(X_{h,t})_{t\in[0,T]}$ *be as in* (25).
Then, the following error expansions hold:

$$\begin{aligned}
\mathbb{E}\varphi(X_h^N) - \mathbb{E}\varphi(X_{h,T}) &= \big\{v_h(T, S_{h,\Delta t}^N P_h x_0) - v_h(T, S_h(T)P_h x_0)\big\} \\
&\quad + \mathbb{E}\int_0^T \int_{\mathcal{O}\times\mathbb{R}} \Big\{ v_h\big(T-t, \bar{Y}_{h,t-} + G(t,\xi,\sigma)\big) - v_h\big(T-t, \bar{Y}_{h,t-} + F(t,\xi,\sigma)\big) \\
&\qquad + \big\langle D_x v_h(T-t, \bar{Y}_{h,t-}),\, F(t,\xi,\sigma) - G(t,\xi,\sigma)\big\rangle_H \Big\}\,dt\,d\xi\,\nu(d\sigma) \\
&=: I + II,
\end{aligned} \tag{34}$$

$$\begin{aligned}
\mathbb{E}\varphi(X_{h,T}) - \mathbb{E}\varphi(X_T) &= \{v(T, S_h(T)P_h x_0) - v(T, S(T)x_0)\} \\
&\quad + \mathbb{E}\int_0^T \int_{\mathcal{O}\times\mathbb{R}} \Big\{ v\big(T-t, Y_{h,t-} + F(t,\xi,\sigma)\big) - v\big(T-t, Y_{h,t-} + E(t,\xi,\sigma)\big) \\
&\qquad + \big\langle D_x v(T-t, Y_{h,t-}),\, E(t,\xi,\sigma) - F(t,\xi,\sigma)\big\rangle_H \Big\}\,dt\,d\xi\,\nu(d\sigma) \\
&=: III + IV.
\end{aligned} \tag{35}$$

Remark 5. In the following proof of Theorem 4, the uniform continuity of the mapping $H \ni x \mapsto D^2\varphi(x) \in L_{(\mathrm{HS})}(H)$ on bounded subsets of H is needed to be able to apply Itô's formula in infinite dimensions, see [21], Section 27. This assumption is always fulfilled in finite dimensions and in literature it is sometimes forgotten to mention in the infinite-dimensional case.

Proof of Theorem 4. We begin with the time discretization error $\mathbb{E}\varphi(X_h^N) - \mathbb{E}\varphi(X_{h,T})$. Due to the definition of v_h we have

$$\mathbb{E}\varphi(X_{h,T}) = v_h(T, S_h(T)P_h x_0), \tag{36}$$

$$\mathbb{E}\varphi(X_h^N) = \mathbb{E}\varphi(\bar{Y}_{h,T}) = \mathbb{E}v_h(0, \bar{Y}_{h,T}). \tag{37}$$

Applying Itô's formula to the function $(t,x) \mapsto v_h(T-t,x)$ and the (càdlàg) martingale $(\bar{Y}_{h,t})_{t\in[0,T]}$ yields

$$\begin{aligned} v_h(0,\bar{Y}_{h,T}) = v_h(T,\bar{Y}_{h,0}) &- \int_0^T \frac{\partial v_h}{\partial t}(T-t,\bar{Y}_{h,t-})\,dt \\ &+ \int_0^T \big\langle D_x v_h(T-t,\bar{Y}_{h,t-}),\, d\bar{Y}_{h,t}\big\rangle_H + \frac{1}{2}\int_0^T \big\langle D_x^2 v_h(T-t,\bar{Y}_{h,t-}),\, d[\![\bar{Y}_{h,\cdot}]\!]_t^c\big\rangle_{(\mathrm{HS})} \\ &+ \sum_{t\le T}\Big\{ v_h(T-t,\bar{Y}_{h,t}) - v_h(T-t,\bar{Y}_{h,t-}) - \big\langle D_x v_h(T-t,\bar{Y}_{h,t-}),\, \Delta\bar{Y}_{h,t}\big\rangle_H \Big\}. \end{aligned} \tag{38}$$

A process is called càdlàg (continu à droite et limites à gauche) if almost every path is right-continuous and has finite left limits. We have used the standard notation $\bar{Y}_{h,t-} = \lim_{s\nearrow t}\bar{Y}_{h,s}$ and $\Delta\bar{Y}_{h,t} = \bar{Y}_{h,t} - \bar{Y}_{h,t-}$ which make sense for every càdlàg process. We will now consider the mean values of the terms on the right hand side of (38) separately:

(i) $\mathbb{E}v_h(T,\bar{Y}_{h,0}) = \mathbb{E}v_h(T, S_{h,\Delta t}^N P_h x_0) = v_h(T, S_{h,\Delta t}^N P_h x_0)$.

(ii) $\mathbb{E}\int_0^T \big\langle D_x v_h(T-t,\bar{Y}_{h,t-}),\, d\bar{Y}_{h,t}\big\rangle_H = 0$ because $\big(\int_0^t \big\langle D_x v_h(T-s,\bar{Y}_{h,s-}),\, d\bar{Y}_{h,s}\big\rangle_H\big)_{t\in[0,T]}$ is a martingale starting in 0.

(iii) $\big([\![\bar{Y}_{h,\cdot}]\!]_t^c\big)_{t\in[0,T]} = 0$ because $(\bar{Y}_{h,t})_{t\in[0,T]}$ is a purely discontinuous martingale. This follows because $(\bar{Y}_{h,t})_{t\in[0,T]}$ is the $\mathcal{M}_T^2(H)$-limit of finite sums of its coordinate processes which can be easily identified as purely discontinuous martingales. Since the space of all purely discontinuous $L^2(\mathbb{P})$-martingales $\mathcal{M}_T^{2,d}(H)$ is closed in $\mathcal{M}_T^2(H)$, the process $(\bar{Y}_{h,t})_{t\in[0,T]}$ is also purely discontinuous (cf. [21], Chapter 4). Hence $\big([\![\bar{Y}_{h,\cdot}]\!]_t^c\big)_{t\in[0,T]} = 0$, and therefore

$$\frac{1}{2}\mathbb{E}\int_0^T \big\langle D_x^2 v_h(T-t,\bar{Y}_{h,t-}),\, d[\![\bar{Y}_{h,\cdot}]\!]_t^c\big\rangle_{(\mathrm{HS})} = 0.$$

(iv) Concerning the jump term

$$\mathbb{E}\sum_{t\le T}\Big\{ v_h(T-t,\bar{Y}_{h,t}) - v_h(T-t,\bar{Y}_{h,t-}) - \big\langle D_x v_h(T-t,\bar{Y}_{h,t-}),\, \Delta\bar{Y}_{h,t}\big\rangle_H \Big\} \tag{39}$$

we give only a brief sketch how to deal with it; the full argument is postponed to the appendix.

Let μ be the jump counting measure of $(\bar{Y}_{h,t})_{t\in[0,T]}$, i.e.

$$\mu((0,t]\times B)(\omega) = \sum_{s\le t}\mathbb{1}_B(\Delta\bar{Y}_{h,s}(\omega)), \qquad t\in(0,T],\ B\in\mathcal{B}(H),\ \omega\in\Omega.$$

The basic idea is to write the sum in (39) as a (pathwise) integral with respect to μ and then to consider the random measure μ as an image measure of π with respect to the function G defined by (30):

$$\mathbb{E}\sum_{t\le T}\Big\{ v_h(T-t,\bar{Y}_{h,t}) - v_h(T-t,\bar{Y}_{h,t-}) - \big\langle D_x v_h(T-t,\bar{Y}_{h,t-}),\, \Delta\bar{Y}_{h,t}\big\rangle_H \Big\}$$

$$= \mathbb{E}\int_0^T \int_H \Big\{v_h(T-t,\bar{Y}_{h,t-}+y) - v_h(T-t,\bar{Y}_{h,t-}) - \big\langle D_x v_h(T-t,\bar{Y}_{h,t-}),\, y\big\rangle_H \Big\}\,\mu(dt,dy)$$

$$= \mathbb{E}\int_0^T \int_{\mathcal{O}\times\mathbb{R}} \Big\{v_h\big(T-t,\bar{Y}_{h,t-}+G(t,\xi,\sigma)\big) - v_h(T-t,\bar{Y}_{h,t-}) - \big\langle D_x v_h(T-t,\bar{Y}_{h,t-}),\, G(t,\xi,\sigma)\big\rangle_H \Big\}\,\pi(dt,d\xi,d\sigma). \tag{40}$$

Here the last term is the expectation of an stochastic L^1-integral with respect to the random measure π, cf. [24], Section 8.7. Since $dt\,d\xi\,\nu(d\sigma)$ is the compensator of π,

$$\mathbb{E}\int_0^T \int_{\mathcal{O}\times\mathbb{R}} \Big\{v_h\big(T-t,\bar{Y}_{h,t-}+G(t,\xi,\sigma)\big) - v_h(T-t,\bar{Y}_{h,t-}) - \big\langle D_x v_h(T-t,\bar{Y}_{h,t-}),\, G(t,\xi,\sigma)\big\rangle_H \Big\}\,\pi(dt,d\xi,d\sigma)$$

$$= \mathbb{E}\int_0^T \int_{\mathcal{O}\times\mathbb{R}} \Big\{v_h\big(T-t,\bar{Y}_{h,t-}+G(t,\xi,\sigma)\big) - v_h(T-t,\bar{Y}_{h,t-}) - \big\langle D_x v_h(T-t,\bar{Y}_{h,t-}),\, G(t,\xi,\sigma)\big\rangle_H \Big\}\,dt\,d\xi\,\nu(d\sigma).$$

(v) Going back to the usual construction of the stochastic integral, one can see that the laws of the random variables

$$\int_{T-t}^T S_h(T-s)P_hQ^{1/2}\,dZ_s \quad \text{and} \quad \int_0^t S_h(s)P_hQ^{1/2}\,dZ_s$$

are equal. Consequently

$$\begin{aligned} v_h(t,x) &= \mathbb{E}\varphi\Big(x+\int_0^t S_h(s)P_hQ^{1/2}dZ_s\Big) \\ &= \mathbb{E}\varphi\Big(x+\int_0^t \tilde{\Phi}(s)\,dZ_s\Big), \end{aligned}$$

where we have used the notation of Section 4 $\tilde{\Phi}(s)=\Phi(T-s)$, $\Phi(s)=S_h(T-s)P_hQ^{1/2}$ for $s\in[0,T]$. Now we apply the Itô formula to the function $H\ni y\mapsto\varphi(x+y)\in\mathbb{R}$ and the (càdlàg) martingale $\big(\int_0^t\tilde{\Phi}(r)\,dZ_r\big)_{t\in[0,T]}$:

$$\begin{aligned} \varphi\Big(x+\int_0^t\tilde{\Phi}(s)\,dZ_s\Big) &= \varphi(x)+\int_0^t\Big\langle D\varphi\Big(x+\int_0^{r-}\tilde{\Phi}(s)\,dZ_s\Big),\,\tilde{\Phi}(r)\,dZ_r\Big\rangle_H \\ &\quad+\frac{1}{2}\int_0^t\Big\langle D^2\varphi\Big(x+\int_0^{r-}\tilde{\Phi}(s)\,dZ_s\Big),\,\tilde{\Phi}(r)^{\otimes 2}\,d[\![Z]\!]_r^c\Big\rangle_{(\mathrm{HS})} \\ &\quad+\sum_{r\le t}\Big\{\varphi\Big(x+\int_0^r\tilde{\Phi}(s)\,dZ_s\Big)-\varphi\Big(x+\int_0^{r-}\tilde{\Phi}(s)\,dZ_s\Big) \\ &\qquad-\Big\langle D\varphi\Big(x+\int_0^{r-}\tilde{\Phi}(s)\,dZ_s\Big),\,\Delta\int_0^r\tilde{\Phi}(s)\,dZ_s\Big\rangle_H\Big\}. \end{aligned}$$

Here, $\int_0^{r-}\tilde{\Phi}(s)\,dZ_s$ denotes the (pathwise) left limit $\lim_{q\nearrow r}\int_0^q\tilde{\Phi}(s)\,dZ_s$ and the expression $\tilde{\Phi}(r)^{\otimes 2}$ in the second line of the formula stands for the operator $L_{(\mathrm{HS})}(H)\to L_{(\mathrm{HS})}(H)$, $T\mapsto\tilde{\Phi}(r)T\tilde{\Phi}(r)^*$. Taking the expectation on both sides of the equation, reasoning as in (iv) and differentiating with respect to t yields

$$\frac{\partial}{\partial t}v_h(t,x) = \int_{\mathcal{O}\times\mathbb{R}}\Big\{v_h\big(t,x+\tilde{F}(t,\xi,\sigma)\big)-v_h(t,x) - \big\langle Dv_h(t,x),\,\tilde{F}(t,\xi,\sigma)\big\rangle_H\Big\}\,d\xi\,\nu(d\sigma). \tag{41}$$

Finally we get (34) if we combine (36) - (38) and (i) - (v).

Now consider the spatial error $\mathbb{E}\varphi(X_{h,T}) - \mathbb{E}\varphi(X_T)$. The definition of the function v implies

$$\begin{aligned}\mathbb{E}\varphi(X_T) &= v(T, S(T)x_0),\\ \mathbb{E}\varphi(X_{h,T}) &= \mathbb{E}v(0, Y_{h,T}).\end{aligned}$$

As above, we can apply Itô's formula to $(t,x) \mapsto v(T-t,x)$ and $(Y_{h,t})_{t\in[0,T]}$ to get

$$\begin{aligned} v(0, Y_{h,T}) &= v(T, Y_{h,0}) - \int_0^T \frac{\partial v}{\partial t}(T-t, Y_{h,t-})\, dt \\ &\quad + \int_0^T \left\langle D_x v(T-t, Y_{h,t-}),\, dY_{h,t}\right\rangle_H \\ &\quad + \frac{1}{2}\int_0^T \left\langle D_x^2 v(T-t, Y_{h,t-}),\, d[\![Y_{h,\cdot}]\!]_t^c\right\rangle_{(\mathrm{HS})} \\ &\quad + \sum_{t\le T}\Big\{ v(T-t, Y_{h,t}) - v(T-t, Y_{h,t-}) - \left\langle D_x v(T-t, Y_{h,t-}),\, \Delta Y_{h,t}\right\rangle_H \Big\}. \end{aligned}$$

This can be used, similar to the argumentation leading to (34), to verify (35). □

Remark 6. The proof of Theorem 4 reveals that we can extend Theorem 4 to equations of the form

$$dX_t + AX_t\, dt = Q_0^{1/2}\, dW_t + Q_1^{1/2}\, dZ_t, \quad X_0 = x_0 \in H, \quad t \in [0,T], \tag{42}$$

where $(W_t)_{t\in[0,T]}$ is a cylindrical Wiener process on H which is independent of $(Z_t)_{t\in[0,T]}$ and the covariance operators Q_0 and Q_1 are, just as Q above, bounded, nonnegative definite, symmetric and satisfy (14), (16). The solution $(X_t)_{t\in[0,T]}$ of (42) is given by

$$X_t = S(t)x_0 + \int_0^t S(t-s)Q_1^{1/2}\, dW_s + \int_0^t S(t-s)Q_2^{1/2}\, dZ_s, \qquad t \in [0,T],$$

and the discretizations are

$$\begin{aligned} X_h^n &= S_{h,\Delta t}^n P_h x_0 + \sum_{k=0}^{n-1} S_{h,\Delta t}^{n-k-1} T_{h,\Delta t} P_h Q_0^{1/2}(W_{t_{k+1}} - W_{t_k}) \\ &\quad + \sum_{k=0}^{n-1} S_{h,\Delta t}^{n-k-1} T_{h,\Delta t} P_h Q_1^{1/2}(Z_{t_{k+1}} - Z_{t_k}), \qquad h>0,\ n \in \{0,\dots,N\}, \\ X_{h,t} &= S_h(t)P_h x_0 + \int_0^t S_h(t-s)P_h Q_0^{1/2}\, dW_s \\ &\quad + \int_0^t S_h(t-s)P_h Q_1^{1/2}\, dZ_s, \qquad t \in [0,T]. \end{aligned}$$

Assuming for simplicity that $Q_1 = Q_2$, the time discretization error $\mathbb{E}\varphi(X_h^N) - \mathbb{E}\varphi(X_{h,T})$ and the spatial error $\mathbb{E}\varphi(X_{h,T}) - \mathbb{E}\varphi(X_T)$ are obtained by adding the terms

$$\frac{1}{2}\,\mathbb{E}\int_0^T \mathrm{Tr}\Big\{\Gamma(t)^*\, D_x^2 v_h(T-t, \bar{Y}_{h,t-})\, \Gamma(t) \,-\, \Phi(t)^*\, D_x^2 v_h(T-t, \bar{Y}_{h,t-})\Phi(t)\Big\}\, dt \tag{43}$$

and

$$\frac{1}{2}\,\mathbb{E}\int_0^T \mathrm{Tr}\Big\{\Phi(t)^*\, D_x^2 v(T-t,Y_{h,t-})\,\Phi(t) - \big(S(T-t)Q_1^{1/2}\big)^* D_x^2 v(T-t,Y_{h,t-})\big(S(T-t)Q_1^{1/2}\big)\Big\}\,dt \tag{44}$$

to the right hand side of (34) and (35), respectively. (Here one has to replace Q by Q_1 in the definition of $\Phi(t)$ and $\Gamma(t)$.)

6 Weak order of convergence

In this section, we show an estimate for the weak order of the convergence of X_h^N to X_T as h tends to zero and N tends to infinity, given the integrability condition (9). The proof is based on the error expansions in the last section.
For the convenience of the readers we summarize the assumptions made in the theorem below.

Assumptions:

- The jump size intensity ν of $(Z_t)_{t\in[0,T]}$ satisfies

$$\int_{\mathbb{R}} \max\big(|\sigma|,\sigma^2\big)\,\nu(d\sigma) < \infty. \tag{9}$$

- The finite element spaces $V_h,\ h>0$, are such that for all $q\in[0,2]$ there exist constants $\kappa_1>0,\ \kappa_2>0$ independent from h such that for all $t>0$

$$\|S_h(t)P_h - S(t)\|_{L(H)} \le \kappa_1 h^q t^{-q/2}, \tag{10}$$

$$\|S_h(t)P_h - S(t)\|_{L(H,D(A^{1/2}))} \le \kappa_2 h t^{-1}. \tag{11}$$

- There exist real numbers α and β with

$$\alpha>0,\quad \beta\in(\alpha-1,\alpha], \tag{14}$$

$$\mathrm{Tr}(A^{-\alpha}) = \sum_{n=1}^{\infty}\lambda_n^{-\alpha} < \infty\ , \tag{15}$$

$$A^\beta Q \in L(H). \tag{16}$$

(Note that this particularly implies assumption (19) needed for Theorem 4.)

Theorem 7. *Assume (9), (10), (11), (14), (15) and (16) listed above. Let $\varphi\in C_b^2(H)$ such that for each $x\in H$ the derivative $D^2\varphi(x)$ is an element of $L_{(HS)}(H)$ and that the mapping $x\mapsto D^2\varphi(x)\in L_{(HS)}(H)$ is uniformly continuous on any bounded subset of H. Let $T\ge 1$ and $(X_t)_{t\in[0,T]}$ be the H-valued stochastic process solution of*

$$dX_t + AX_t\,dt = Q^{1/2}\,dZ_t,\quad X_0 = x_0\in H,\quad t\in[0,T]. \tag{4}$$

For any $N\ge 1$ and $h>0$, let $(X_h^n)_{n\in\{0,\dots,N\}}$ be given by (24) and let $(X_{h,t})_{t\in[0,T]}$ be as in (25). Then there exists a constant $C=C(T,\varphi)>0$ which does not depend on h and N such that for any $\gamma<1-\alpha+\beta\le 1$, the following inequality holds

$$|\mathbb{E}\varphi(X_h^N) - \mathbb{E}\varphi(X_T)| \le C\cdot(h^{2\gamma}+(\Delta t)^\gamma),$$

where $\Delta t = T/N \le 1$.

Proof. As in Theorem 4, we split the error into the time discretization error and the spatial error,

$$\mathbb{E}\varphi(X_h^N) - \mathbb{E}\varphi(X_T) = \Big\{\mathbb{E}\varphi(X_h^N) - \mathbb{E}\varphi(X_{h,T})\Big\} + \Big\{\mathbb{E}\varphi(X_{h,T}) - \mathbb{E}\varphi(X_T)\Big\}.$$

We will estimate each term separately. Throughout this proof, C denotes a positive (and finite) constant that may change from line to line.

Consider the time discretization error given by (34), $\mathbb{E}\varphi(X_h^N) - \mathbb{E}\varphi(X_{h,T}) = I + II$. Clearly, due to the definition of v_h, (32),

$$|I| \le \|\varphi\|_{C_b^1(H)} \left\|S_{h,\Delta t}^N P_h - S_h(T)P_h\right\|_{L(H)} |x_0|_H.$$

Using spectral calculus and (22), $\left\|S_{h,\Delta t}^N P_h - S_h(T)P_h\right\|_{L(H)}$ can be bounded uniformly with respect to h. To be more precise, for $T \ge 1$ we have

$$\begin{aligned}
\left\|S_{h,\Delta t}^N P_h - S_h(T)P_h\right\|_{L(H)} &\le \sup_{\lambda>0} \left| e^{-N\lambda\Delta t} - \left(\frac{1-(1-\theta)\lambda\Delta t}{1+\theta\lambda\Delta t}\right)^N \right| \\
&= \sup_{r>0} \left| e^{-Nr} - \left(\frac{1-(1-\theta)r}{1+\theta r}\right)^N \right| \\
&\le \frac{C}{N} \le C\cdot\Delta T,
\end{aligned}$$

cf. [19], p. 921, Theorem 1.1 for the penultimate estimate. One obtains

$$|I| \le C\cdot\Delta t. \tag{45}$$

Concerning II, we apply the mean value theorem and the Cauchy-Schwarz inequality and obtain

$$\begin{aligned}
II &\le \int_0^T \int_{\mathcal{O}\times\mathbb{R}} 2\|D_x v_h\|_{C_b(H)} |F(t,\xi,\sigma) - G(t,\xi,\sigma)| \, dt\, d\xi\, \nu(d\sigma) \\
&= 2\|D_x v_h\|_{C_b(H)} \int_{\mathbb{R}} |\sigma|\,\nu(d\sigma) \int_0^T \int_{\mathcal{O}} \left| \sum_{k=1}^\infty e_k(\xi)\big(\Phi(t)-\Gamma(t)\big)e_k \right|_H dt\, d\xi,
\end{aligned} \tag{46}$$

where the expression $\sum_{k=1}^\infty e_k(\xi)\big(\Phi(t)-\Gamma(t)\big)e_k$ has to be understood as the value at (t,ξ) of a $dt\,d\xi$-version of

$$L^2([0,T]\times\mathcal{O}, dt\,d\xi;\, H)\text{-}\lim_{N\to\infty}\left((t,\xi)\mapsto \sum_{k=1}^N e_k(\xi)\big(\Phi(t)-\Gamma(t)\big)e_k\right).$$

Next, let $\gamma > 0$ and $\gamma_1 > 0$ such that $0 < \gamma < \gamma_1 < 1-\alpha+\beta \le 1$. Using

$$\sum_{k=1}^\infty e_k(\xi)\big(\Phi(t)-\Gamma(t)\big)e_k = \\
\left(S_h(T-t) - \sum_{n=0}^{N-1} \mathbb{1}_{(t_n,t_{n+1}]}(t) S_{h,\Delta t}^{N-n-1} T_{h,\Delta t}\right) A_h^{(1-\gamma_1)/2} \sum_{k=1}^\infty e_k(\xi) A_h^{(\gamma_1-1)/2} P_h Q^{1/2} e_k,$$

(46) can be estimated by

$$II \le C \int_0^T \left\| \left(S_h(T-t) - \sum_{n=0}^{N-1} \mathbb{1}_{(t_n,t_{n+1}]}(t) S_{h,\Delta t}^{N-n-1} T_{h,\Delta t}\right) A_h^{(1-\gamma_1)/2} \right\|_{L(V_h)} dt$$

$$\times \int_{\mathcal{O}} \left| \sum_{k=1}^{\infty} e_k(\xi) A_h^{(\gamma_1-1)/2} P_h Q^{1/2} e_k \right|_H d\xi$$

$$\le C \int_0^T \left\| \left(S_h(T-t) - \sum_{n=0}^{N-1} \mathbb{1}_{(t_n,t_{n+1}]}(t) S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} \right) A_h^{(1-\gamma_1)/2} \right\|_{L(V_h)} dt$$

$$\times \left(\int_{\mathcal{O}} \mathbb{1}_{\mathcal{O}}(\xi)\, d\xi \right)^{1/2} \left(\int_{\mathcal{O}} \left| \sum_{k=1}^{\infty} e_k(\xi) A_h^{(\gamma_1-1)/2} P_h Q^{1/2} e_k \right|_H^2 d\xi \right)^{1/2}$$

$$= C \, \| A_h^{(\gamma_1-1)/2} P_h Q^{1/2} \|_{(\mathrm{HS})}$$

$$\times \int_0^T \left\| \left(S_h(T-t) - \sum_{n=0}^{N-1} \mathbb{1}_{(t_n,t_{n+1}]}(t) S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} \right) A_h^{(1-\gamma_1)/2} \right\|_{L(V_h)} dt. \tag{47}$$

Following [8], $\|A_h^{(\gamma_1-1)/2} P_h Q^{1/2}\|_{(\mathrm{HS})}$ can be bounded from above by some constant $C > 0$ which does not depend on the choice of $h > 0$:

$$\|A_h^{(\gamma_1-1)/2} P_h Q^{1/2}\|_{(\mathrm{HS})} \le C. \tag{48}$$

This is essentially due to (15) and (16). Furthermore, again following [8],

$$\left\| \left(S_h(T-t) - \sum_{n=0}^{N-1} \mathbb{1}_{(t_n,t_{n+1}]}(t) S_{h,\Delta t}^{N-n-1} T_{h,\Delta t} \right) A_h^{(1-\gamma_1)/2} \right\|_{L(V_h)}$$

$$\le \begin{cases} C\Delta t^{\gamma} \sum_{n=0}^{N-2} \mathbb{1}_{(t_n,t_{n+1}]}(t)((N-n-1)\Delta t)^{-((1-\gamma_1)/2+\gamma)} & t \in (0, t_{N-1}] \\ C(T-t)^{(\gamma_1-1)/2}, & t \in (t_{N-1}, T], \end{cases} \tag{49}$$

which follows directly from spectral calculus. A combination of (47), (48) and (49) yields

$$\begin{aligned} II &\le C \cdot \left(\Delta t^{\gamma} \int_0^{t_{N-1}} \sum_{n=0}^{N-2} \mathbb{1}_{(t_n,t_{n+1}]}(t)((N-n-1)\Delta t)^{-((1-\gamma_1)/2+\gamma)}\, dt + \int_{t_{N-1}}^T (T-t)^{(\gamma_1-1)/2}\, dt \right) \\ &\le C \cdot \Delta t^{\gamma}, \end{aligned} \tag{50}$$

as $(1-\gamma_1)/2 + \gamma \in (0,1)$ and $(T-t)^{(\gamma_1-1)/2} \le (T-t)^{-((1-\gamma_1)/2+\gamma)} \Delta t^{\gamma}$ for $t \in (t_{N-1}, T]$.

Finally (34), (45) and (50) add up to

$$\left| \mathbb{E}\varphi(X_h^N) - \mathbb{E}\varphi(X_{h,T}) \right| \le C\Delta t^{\gamma} \tag{51}$$

for all $T \ge 1$ and $\Delta t \le 1$.

Now we turn to the spatial error given by (35), $\mathbb{E}\varphi(X_{h,T}) - \mathbb{E}\varphi(X_T) = III + IV$. Firstly, according to the definition of v, (33), and resulting from (10) with $q = 2\gamma < 2$,

$$\begin{aligned} |III| &\le \|\varphi\|_{C_b^2(H)} \, \|S_h(T)P_h - S(T)\|_{L(H)} \, |x_0|_H \\ &\le \|\varphi\|_{C_b^2(H)} \left(\kappa_1 h^{2\gamma} T^{-\gamma} \right) |x_0|_H \;=\; C \cdot h^{2\gamma}. \end{aligned} \tag{52}$$

Considering IV, we apply the mean value theorem and the Cauchy-Schwarz inequality and obtain

$$|IV| \le \int_0^T \int_{\mathcal{O}\times\mathbb{R}} 2\|D_x v\|_{C_b(H)} \, |E(t,\xi,\sigma) - F(t,\xi,\sigma)| \, dt\, d\xi\, \nu(d\sigma)$$

$$= \; 2\|D_x v\|_{C_b(H)} \int_{\mathbb{R}} |\sigma|\, \nu(d\sigma)$$
$$\times \int_0^T \int_{\mathcal{O}} \Big| \sum_{k=1}^{\infty} e_k(\xi)\big(S(T-t) - S_h(T-t)P_h\big)Q^{1/2} e_k \Big|_H \, dt\, d\xi, \tag{53}$$

where the expression $\sum_{k=1}^{\infty} e_k(\xi)\big(S(T-t) - S_h(T-t)P_h\big)Q^{1/2}e_k$ has to be understood as the value at (t,ξ) of a $dt\,d\xi$-version of

$$L^2([0,T]\times\mathcal{O}, dt\, d\xi;\, H)\text{-}\lim_{N\to\infty}\Big((t,\xi)\mapsto \sum_{k=1}^{N} e_k(\xi)\big(S(T-t)-S_h(T-t)P_h\big)Q^{1/2}e_k\Big).$$

Pick $\gamma_1 > 0$ such that $0 < \gamma < \gamma_1 < 1-\alpha+\beta \le 1$. Because of (17) we know that $A^{\beta/2}Q^{1/2} \in L(H)$. Furthermore, the fact that $1-\gamma_1+\beta > \alpha$ implies $A^{-(1-\gamma_1+\beta)/2} \in L_{(\mathrm{HS})}(H)$. It is also not hard to see that the operator $(S(T-t)-S_h(T-t)P_h)A^{(1-\gamma_1)/2}$ has a continuous extension defined on the whole space H. Therefore we may write

$$\sum_{k=1}^{\infty} e_k(\xi)\big(S(T-t)-S_h(T-t)P_h\big)Q^{1/2}e_k$$
$$= \; (S(T-t)-S_h(T-t)P_h)\, A^{(1-\gamma_1)/2} \sum_{k=1}^{\infty} e_k(\xi) A^{-(1-\gamma_1+\beta)/2} A^{\beta/2} Q^{1/2} e_k.$$

A further application of the Cauchy-Schwarz inequality in (53) gives

$$\begin{aligned}
|IV| \;&\le\; C\int_0^T \Big\|\big(S(T-t)-S_h(T-t)P_h\big)A^{(1-\gamma_1)/2}\Big\|_{L(H)} dt \\
&\qquad \times \Big(\int_{\mathcal{O}} \mathbb{1}_{\mathcal{O}}(\xi)\, d\xi\Big)^{1/2} \Big(\int_{\mathcal{O}} \Big|\sum_{k=1}^{\infty} e_k(\xi) A^{-(1-\gamma_1+\beta)/2} A^{\beta/2} Q^{1/2} e_k\Big|_H^2 d\xi\Big)^{1/2} \\
&=\; C\,\big\|A^{-(1-\gamma_1+\beta)/2}A^{\beta/2}Q^{1/2}\big\|_{(\mathrm{HS})} \int_0^T \Big\|\big(S(T-t)-S_h(T-t)P_h\big)A^{(1-\gamma_1)/2}\Big\|_{L(H)} dt \\
&=\; C\int_0^T \Big\|\big(S(T-t)-S_h(T-t)P_h\big)A^{(1-\gamma_1)/2}\Big\|_{L(H)} dt.
\end{aligned} \tag{54}$$

We are now going to show that for all $t\in[0,T]$

$$\Big\|\Big(S(T-t)-S_h(T-t)P_h\Big)A^{(1-\gamma_1)/2}\Big\|_{L(H)} \le C\, h^{2\gamma}\Big(t^{-\left(\gamma_1(\gamma_1-1)/(2\gamma)+1\right)} + t^{-\left((1-\gamma_1)/2+\gamma\right)}\Big). \tag{55}$$

This will be done in several steps.

(i) Due to the self adjointness of $S(T-t)$, $S_h(T-t)P_h$ and $A^{(1-\gamma_1)/2}$,

$$\begin{aligned}
\Big\|\big(S(T-t)-S_h(T-t)P_h\big)A^{(1-\gamma_1)/2}\Big\|_{L(H)} \;&=\; \Big\|\Big(\big(S(T-t)-S_h(T-t)P_h\big)A^{(1-\gamma_1)/2}\Big)^*\Big\|_{L(H)} \\
&=\; \Big\|A^{(1-\gamma_1)/2}\big(S(T-t)-S_h(T-t)P_h\big)\Big\|_{L(H)}.
\end{aligned}$$

Here the operator $A^{(1-\gamma_1)/2}\big(S(T-t)-S_h(T-t)P_h\big)$ is properly defined since $S(t)H\subset D(A^{\lambda})$, $t>0$, $\lambda\in\mathbb{R}$ and $V_h\subset D(A^{1/2})\subset D(A^{(1-\gamma_1)/2})$.

(ii) Due to the Hölder inequality we have for all $x \in D(A^{1/2})$

$$\left|A^{(1-\gamma_1)/2}x\right|_H \le \left|A^{1/2}x\right|_H^{1-\gamma_1} \cdot |x|_H^{\gamma_1}.$$

Consequently,

$$\begin{aligned}
&\left\|A^{(1-\gamma_1)/2}\big(S(T-t) - S_h(T-t)P_h\big)\right\|_{L(H)} \\
\le\ & \|S(T-t) - S_h(T-t)P_h\|_{L(H,D(A^{1/2}))}^{1-\gamma_1} \cdot \|S(T-t) - S_h(T-t)P_h\|_{L(H)}^{\gamma_1}. \qquad (56)
\end{aligned}$$

(iii) If $2\gamma + \gamma_1 \ge 1$, then $(2\gamma + \gamma_1 - 1)/\gamma \in [0,2]$. In this case, we combine (56) with (11) and (10), $q = (2\gamma + \gamma_1 - 1)/\gamma$. Provided that $h \le 1$, one gets

$$\begin{aligned}
&\left\|A^{(1-\gamma_1)/2}\left(S(T-t) - S_h(T-t)P_h\right)\right\|_{L(H)} \\
\le\ & C\left(h^{1-\gamma_1}t^{\gamma_1-1}\right)\left(h^{(2\gamma+\gamma_1-1)\gamma_1/\gamma}t^{-(2\gamma+\gamma_1-1)\gamma_1/(2\gamma)}\right) \\
\le\ & Ch^{2\gamma}t^{-\left(\gamma_1(\gamma_1-1)/(2\gamma)+1\right)}.
\end{aligned}$$

(iv) If $2\gamma + \gamma_1 < 1$, we first combine (56) with (11) and (10) choosing $q = 0$ and get

$$\left\|A^{(1-\gamma_1)/2}\left(S(T-t) - S_h(T-t)P_h\right)\right\|_{L(H)} \le C\,h^{1-\gamma_1}(T-t)^{-(1-\gamma_1)}. \qquad (57)$$

Secondly, again using (10) with $q = 0$, one derives

$$\begin{aligned}
&\left\|A^{(1-\gamma_1)/2}\left(S(T-t) - S_h(T-t)P_h\right)\right\|_{L(H)} \\
\le\ & \|S(T-t) - S_h(T-t)P_h\|_{L(H,D(A^{1/2}))}^{1-\gamma_1}\|S(T-t) - S_h(T-t)P_h\|_{L(H)}^{\gamma_1} \\
\le\ & C\,\|S(T-t) - S_h(T-t)P_h\|_{L(H,D(A^{1/2}))}^{1-\gamma_1} \\
\le\ & C\left(\|A^{1/2}S(T-t)\|_{L(H)} + \|A^{1/2}S_h(T-t)P_h\|_{L(H)}\right)^{1-\gamma_1} \\
=\ & C\left(\|A^{1/2}S(T-t)\|_{L(H)} + \|A_h^{1/2}S_h(T-t)P_h\|_{L(H)}\right)^{1-\gamma_1} \\
\le\ & C\,(T-t)^{-(1-\gamma_1)/2}. \qquad (58)
\end{aligned}$$

In the last step, the inequality

$$\sup_{x\ge 0} x^{\epsilon}e^{-tx} \le \left(\frac{\epsilon}{e}\right)^{\epsilon} t^{-\epsilon}, \qquad t > 0,\ \epsilon > 0$$

has been used.

Now let $\lambda := 2\gamma/(1-\gamma_1) \in (0,1)$. Interpolating (57) and (58) yields

$$\begin{aligned}
&\left\|A^{(1-\gamma_1)/2}\left(S(T-t) - S_h(T-t)P_h\right)\right\|_{L(H)} \\
\le\ & C\left(h^{1-\gamma_1}(T-t)^{-(1-\gamma_1)}\right)^{\lambda}\left((T-t)^{-(1-\gamma_1)/2}\right)^{1-\lambda} \\
=\ & C\,h^{2\gamma}(T-t)^{-((1-\gamma_1)/2+\gamma)}
\end{aligned}$$

Combining (i), (ii) and (iii) yields (55).

As $\gamma_1(\gamma_1 - 1)/(2\gamma) + 1 < 1$ and $(1-\gamma_1)/2 + \gamma < 1$, we can integrate (55) with respect to the time variable t. Then (35), (52), (54) and (55) add up to

$$|\mathbb{E}\varphi(X_{h,T}) - \mathbb{E}\varphi(X_T)| \le C\,h^{2\gamma}.$$

□

Remark 8. (i) The proof of Theorem 7 can easily be combined with the proof in [8] in order to obtain the same result for the equation

$$dX_t + AX_t\,dt = Q_0^{1/2}\,dW_t + Q_1^{1/2}\,dZ_t, \quad X_0 = x_0 \in H, \quad t \in [0,T], \tag{59}$$

where $(W_t)_{t\in[0,T]}$ is a cylindrical Wiener process on H which is independent of $(Z_t)_{t\in[0,T]}$ and where the covariance operators Q_0 and Q_1 are bounded, nonnegative definite, symmetric and satisfy (14), (16). The corresponding discretization and error expansion has been described in Remark 1.

Of course, the result also holds for the equation

$$dX_t + AX_t\,dt = Q_0^{1/2}dW_t + Q_1^{1/2}\,dZ_{1,t} + Q_2^{1/2}\,dZ_{2,t}, \quad X_0 = x_0 \in H, \quad t \in [0,T], \tag{60}$$

where $(Z_{1,t})_{t\in[0,T]}$ and $(Z_{2,t})_{t\in[0,T]}$ are impulsive cylindrical processes satisfying the condition required of $(Z_t)_{t\in[0,T]}$ (see (9) above), the processes $(W_t)_{t\in[0,T]}$, $(Z_{1,t})_{t\in[0,T]}$, $(Z_{2,t})_{t\in[0,T]}$ are independent, and the covariance operators Q_0, Q_1 and Q_2 are bounded, nonnegative definite, symmetric and satisfy (14), (16). For example, one could consider impulsive cylindrical processes described by the jump size intensities

$$\nu_i(d\sigma) = \frac{1}{|\sigma|^{1+\alpha_i}}\mathbb{1}_{[-\tau,\tau]}(\sigma)\,d\sigma, \quad i = 1,\,2,$$

with indices of stability $0 < \alpha_1 < \alpha_2 < 1$.

(ii) One might try to avoid the integrability assumption (9) in the proof of Theorem 7 by rewriting the terms II and IV in (34) and (35) using Taylor's theorem. Obviously,

$$\begin{aligned} II &= \mathbb{E}\int_0^T\int_{\mathcal{O}\times\mathbb{R}}\Big\{\big\langle G(t,\xi,\sigma)\,,\,D_x^2 v_h\big(T-t,\bar{Y}_{h,t-}+\theta G(t,\xi,\sigma)\big)G(t,\xi,\sigma)\big\rangle_H \\ &\qquad -\big\langle F(t,\xi,\sigma)\,,\,D_x^2 v_h\big(T-t,\bar{Y}_{h,t-}+\theta F(t,\xi,\sigma)\big)F(t,\xi,\sigma)\big\rangle_H\Big\}\,dt\,d\xi\,\nu(d\sigma), \\ IV &= \mathbb{E}\int_0^T\int_{\mathcal{O}\times\mathbb{R}}\Big\{\big\langle F(t,\xi,\sigma)\,,\,D_x^2 v\big(T-t,Y_{h,t-}+\vartheta F(t,\xi,\sigma)\big)F(t,\xi,\sigma)\big\rangle_H \\ &\qquad -\big\langle E(t,\xi,\sigma)\,,\,D_x^2 v\big(T-t,Y_{h,t-}+\vartheta E(t,\xi,\sigma)\big)E(t,\xi,\sigma)\big\rangle_H\Big\}\,dt\,d\xi\,\nu(d\sigma), \end{aligned}$$

where $\theta = \theta(t,\xi,\omega) \in (0,1)$ and $\vartheta = \vartheta(t,\xi,\omega) \in (0,1)$. However, the integrands appearing here cannot be estimated analogously to the estimates of the integrands in the terms (43) and (44) in [8], which appear in the case of Gaussian noise. The reason is, that in the case of impulsive noise one has to estimate the difference of the second derivatives. As one needs integrability of suitable upper bounds for the integrands, this leads to profound technical complications.

7 Appendix

Here we give a detailed proof for equality (40). We have to show

$$\begin{aligned} &\mathbb{E}\sum_{t\le T}\Big\{v_h(T-t,\bar{Y}_{h,t}) - v_h(T-t,\bar{Y}_{h,t-}) - \big\langle D_x v_h(T-t,\bar{Y}_{h,t-}),\,\Delta\bar{Y}_{h,t}\big\rangle_H\Big\} \\ &= \mathbb{E}\int_0^T\int_{\mathcal{O}\times\mathbb{R}}\Big\{v_h\big(T-t,\bar{Y}_{h,t-}+G(t,\xi,\sigma)\big) - v_h(T-t,\bar{Y}_{h,t-}) \\ &\qquad - \big\langle D_x v_h(T-t,\bar{Y}_{h,t-}),\,G(t,\xi,\sigma)\big\rangle_H\Big\}\,\pi(dt,d\xi,d\sigma). \end{aligned} \tag{40}$$

We have already seen that

$$\bar{Y}_{h,t} - S^N_{h,\Delta t} P_h x_0 = \int_0^t \Gamma(s) dZ_s = \int_0^t \int_{\mathcal{O}\times\mathbb{R}} G(s,\xi,\sigma)\,\hat{\pi}(ds,d\xi,d\sigma),$$

with $G(s,\xi,\sigma) = \sum_{k=1}^\infty e_k(\xi)\sigma\Gamma(s)e_k$, where the the infinite sum is a limit in the space $L^2([0,T]\times\mathcal{O}\times\mathbb{R},\, ds\,d\xi\,\nu(d\sigma);\, H)$. By a standard monotone class argument, there exist simple functions

$$G_n(s,\xi,\sigma) = \sum_{k=1}^{m(n)} a_{n,k} \mathbb{1}_{A_{n,k}}(s,\xi,\sigma), \qquad (s,\xi,\sigma) \in [0,T]\times\mathcal{O}\times\mathbb{R},$$

where $n\in\mathbb{N}$, $m(n)\in\mathbb{N}$, $a_{n,k}\in H$ and $A_{n,k}\in\mathcal{B}([0,T]\times\mathcal{O}\times\mathbb{R})$ for $k=1,\dots,m(n)$ with

$$\int_0^T \int_{\mathcal{O}\times\mathbb{R}} \mathbb{1}_{A_{n,k}}(t,\xi,\sigma)\,dt\,d\xi\,\nu(d\sigma) < \infty,$$

such that G_n converges to G in $L^2([0,T]\times\mathcal{O}\times\mathbb{R},\, ds\,d\xi\,\nu(d\sigma);\, H)$ if n tends to infinity. Note that the processes

$$\left(\int_0^t \int_{\mathcal{O}\times\mathbb{R}} G(s,\xi,\sigma)\,\hat{\pi}(ds,d\xi,d\sigma)\right)_{t\in[0,T]} \quad \text{and} \quad \left(\int_0^t \int_{\mathcal{O}\times\mathbb{R}} G_n(s,\xi,\sigma)\,\hat{\pi}(ds,d\xi,d\sigma)\right)_{t\in[0,T]},$$

$n\in\mathbb{N}$, are martingales and hence have càdlàg modifications. Applying Doob's inequality for submartingales, we have

$$\begin{aligned}
\mathbb{E}\sup_{t\le T}&\left|\int_0^t \int_{\mathcal{O}\times\mathbb{R}} G(s,\xi,\sigma)\,\hat{\pi}(ds,d\xi,d\sigma) - \int_0^t \int_{\mathcal{O}\times\mathbb{R}} G_n(s,\xi,\sigma)\,\hat{\pi}(ds,d\xi,d\sigma)\right|_H^2 \\
&\le 4\,\mathbb{E}\left|\int_0^T \int_{\mathcal{O}\times\mathbb{R}} \left(G(s,\xi,\sigma) - G_n(s,\xi,\sigma)\right)\,\hat{\pi}(ds,d\xi,d\sigma)\right|_H^2 \\
&= 4\,|G-G_n|^2_{L^2([0,T]\times\mathcal{O}\times\mathbb{R},\, ds\,d\xi\,\nu(d\sigma);\, H)}
\end{aligned} \tag{61}$$

and the last term goes to zero as n tends to infinity.
Remember that the Poisson random measure π on $[0,T]\times\mathcal{O}\times\mathbb{R}$ with given compensator $dt\,d\xi\,\nu(d\sigma)$ is given by

$$\pi = \sum_{j=1}^\infty \delta_{(T_j,\Xi_j,\Sigma_j)},$$

where $\left((T_j,\Xi_j,\Sigma_j)\right)_{j\in\mathbb{N}}$ is a properly chosen sequence of random elements in $[0,T]\times\mathcal{O}\times\mathbb{R}$, cf. [24], Chapter 6. Since the functions G_n, $n\in\mathbb{N}$, are simple functions, the stochastic integral of G_n with respect to $\hat{\pi}$ is just an ω-wise integral with respect to

$$\sum_{j=1}^\infty \delta_{(T_j(\omega),\Xi_j(\omega),\Sigma_j(\omega))}(dt,d\xi,d\sigma) - dt\,d\xi\,\nu(d\sigma).$$

Therefore, for every $n\in\mathbb{N}$, the càdlàg process

$$\left(\int_0^t \int_{\mathcal{O}\times\mathbb{R}} G_n(s,\xi,\sigma)\,\hat{\pi}(ds,d\xi,d\sigma)\right)_{t\in[0,T]}$$

has the jumps $G_n\big(T_j(\omega),\Xi_j(\omega),\Sigma_j(\omega)\big)$, $j\in\mathbb{N}$, $T_j(\omega)\le T$, occurring at the jump times $T_j(\omega)$, $j\in\mathbb{N}$, $T_j(\omega)\le T$, for almost every $\omega\in\Omega$.
The convergence $G_n\to G$, $n\to\infty$, in $L^2([0,T]\times\mathcal{O}\times\mathbb{R},\, dt\,d\xi\,\nu(d\sigma);\, H)$ implies the convergence $G_n(t,\xi,\sigma)\to G(t,\xi,\sigma)$, $n\to\infty$, in H for $ds\,d\xi\,\nu(d\sigma)$-almost every $(t,\xi,\sigma)\in[0,T]\times\mathcal{O}\times\mathbb{R}$. This, the uniform convergence following from (61) and the fact that the laws of the random vectors (T_j,Ξ_j,Σ_j), $j\in\mathbb{N}$, are absolutely continuous with respect to $dt\,d\xi\,\nu(d\sigma)$ imply that, for almost every $\omega\in\Omega$, the jumps of the process $\big(\bar Y_{h,t}\big)_{t\in[0,T]}$ occur at the jump times $T_j(\omega)$, $j\in\mathbb{N}$, $T_j(\omega)\le T$ and are exactly $G\big(T_j(\omega),\Xi_j(\omega),\Sigma_j(\omega)\big)$, $j\in\mathbb{N}$, $T_j(\omega)\le T$.
Consequently, we have

$$\begin{aligned}
&\mathbb{E}\sum_{t\le T}\Big\{v_h(T-t,\bar Y_{h,t})-v_h(T-t,\bar Y_{h,t-})-\big\langle D_xv_h(T-t,\bar Y_{h,t-}),\,\Delta\bar Y_{h,t}\big\rangle_H\Big\}\\
&=\mathbb{E}\sum_{j\in\mathbb{N},\,T_j\le T}\Big\{v_h\big(T-T_j,\bar Y_{h,T_j-}+G(T_j,\Xi_j,\Sigma_j)\big)-v_h(T-T_j,\bar Y_{h,T_j-})\\
&\qquad\qquad-\big\langle D_xv_h(T-T_j,\bar Y_{h,T_j-}),\,G(T_j,\Xi_j,\Sigma_j)\big\rangle_H\Big\}.
\end{aligned}$$

With similar arguments, one can show that

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_{\mathcal{O}\times\mathbb{R}}\Big\{v_h\big(T-t,\bar Y_{h,t-}+G(t,\xi,\sigma)\big)-v_h(T-t,\bar Y_{h,t-})\\
&\qquad\qquad-\big\langle D_xv_h(T-t,\bar Y_{h,t-}),\,G(t,\xi,\sigma)\big\rangle_H\Big\}\,\pi(dt,d\xi,d\sigma)\\
&=\mathbb{E}\sum_{j\in\mathbb{N},\,T_j\le T}\Big\{v_h\big(T-T_j,\bar Y_{h,T_j-}+G(T_j,\Xi_j,\Sigma_j)\big)-v_h(T-T_j,\bar Y_{h,T_j-})\\
&\qquad\qquad-\big\langle D_xv_h(T-T_j,\bar Y_{h,T_j-}),\,G(T_j,\Xi_j,\Sigma_j)\big\rangle_H\Big\}.
\end{aligned}$$

This finishes the proof of (40).

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References

[1] D. Applebaum, J. L. Wu, Stochastic partial differential equations driven by Lévy space-time white noise, *Random Oper. Stoch. Equ.* **3** (2000), 245–261.

[2] A. de Bouard, A. Debussche, Weak and strong order of convergence of a semidiscrete scheme for the stochastic nonlinear Schrödinger equation, *Appl. Math. Optim.* **54** (2006), 369–399.

[3] J. H. Bramble, A. H. Schatz, V. Thomée, L. B. Wahlbin, Some convergence estimates for semidiscrete Galerkin type approximations for parabolic equations, *SIAM J. Numer. Anal.* **14** (1977), 218–241.

[4] P. G. Ciarlet, *The finite element method for elliptic problems*, North-Holland, 1978.

[5] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, Encyclopedia of Mathematics and Its Applications 44, Cambridge University Press, 1992.

[6] C. Prévôt, M. Röckner, *A concise course on stochastic partial differential equations*, Lecture Notes in Mathematics 1905, Springer, 2007.

[7] A. Debussche, Weak approximation of stochastic partial differential equations: The non linear case, preprint, 2008.

[8] A. Debussche, J. Printems, Weak order for the discretization of the stochastic heat equation, *Math. Comp.* **78** (2009), 845–863.

[9] M. Geissert, M. Kovács, S. Larsson, Rate of weak convergence of the finite element method for the stochastic heat equation with additive noise, *BIT* **49** (2009), 343–356.

[10] I. Gyöngy, A. Millet, On discretization schemes for stochastic evolution equations, *Potential Anal.* **23** (2005), 99–134.

[11] I. Gyöngy, A. Millet, Rate of convergence of implicit approximations for stochastic evolution equations, in: P. H. Baxendale, S. V. Lototsky (eds.), *Stochastic Differential Equations: Theory and Applications. A Volume in Honor of Professor Boris L. Rozovskii*, Interdisciplinary Mathematical Siences, Vol. 2, World Scientific, 2007, 281–310.

[12] I, Gyöngy, A. Millet, Rate of convergence of space time approximations for stochastic evolution equations, *Potential Anal.* **30** (2009), 29–64.

[13] E. Hausenblas, Weak approximation of stochastic partial differential equations, in: U. Capar, A. Üstünel (eds.), *Stochastic analysis and related topics VIII. Silivri workshop*, Progress in Probability, Birkhäuser, 2003.

[14] E. Hausenblas, Existence, uniqueness and regularity of parabolic SPDEs driven by Poisson random measure, *Electron. J. Probab.* **10** (2005), 1496–1546.

[15] E. Hausenblas, I. Marchis, The numerical approximation of parabolic stochastic partial differential equations driven by Poisson random measure, *BIT* **46** (2006), 773–811.

[16] E. Hausenblas, Finite element approximation of stochastic partial differential equations driven by Poisson random measures of jump type, *SIAM J. Numer. Anal.* **46** (2008), 437–471.

[17] C. Johnson, S. Larsson, V. Thomée, B. Wahlbin, Error estimates for spatially discrete approximations of semilinear parabolic equations with non smooth initial data, *Math. Comp.* **49** (1987), 331–357.

[18] C. Knoche, SPDEs in infinite dimension with Poisson noise, *C. R. Math. Acad. Sci. Paris, Ser. I* **339** (2004), 647–652.

[19] M.-N. Le Roux, Semidiscretization in time for parabolic problems, *Math. Comp.* **33** (1979), 919–931.

[20] C. Marinelli, C. Prévot, M. Röckner, Regular dependence on initial data for stochastic convolution equations with multiplicative Poisson noise, Preprint (2008).

[21] M. Métivier, *Semimartingales - a course on stochastic processes*, de Gruyter Studies in Mathematics 2, de Gruyter, 1982.

[22] C. Mueller, The heat equation with Lévy noise, *Stochastic Process. Appl.* **74** (1998), 67–82.

[23] L. Mytnik, Stochastic partial differential equation driven by stable noise, *Probab. Theory Related Fields* **123** (2002), 157–201.

[24] S. Peszat, J. Zabczyk, *Stochastic partial differential equations with Lévy noise*, Encyclopedia of Mathematics and Its Applications 113, Cambridge University Press, 2007.

[25] Yu. A. Rozanov, *Random fields and stochastic partial differential equations*, Kluwer Academic Publishers, 1998.

[26] K. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge studies in advanced mathematics 68, Cambridge University Press, 2005.

[27] G. Strang, G. J. Fix, *An analysis of the finite element method*, Prentice-Hall, 1973.

[28] V. Thomée, *Galerkin finite element methods for parabolic problems*, Springer, 1997.

[29] J. B. Walsh, An introduction to stochastic partial differential equations, in: *École d'été de probabilités de Saint-Flour XIV - 1984*, Lecture Notes in Mathematics 1180, Springer, 1986, 265–439.

[30] J. B. Walsh, Finite element methods for parabolic stochastic PDEs, *Potential Anal.* **23** (2005), 1–43.

[31] Y. Yan, Galerkin finite element methods for stochastic partial differential equations, *SIAM J. Numer. Anal.* **43** (2005), 1363–1384.

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[22] M. Hansen and W. Sickel. Best m-term Approximation and Lizorkin-Triebel Spaces. Preprint 22, DFG-SPP 1324, August 2009.

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[24] S. Dereich and F. Heidenreich. A Multilevel Monte Carlo Algorithm for Lévy Driven Stochastic Differential Equations. Preprint 24, DFG-SPP 1324, August 2009.

[25] S. Dahlke, M. Fornasier, and T. Raasch. Multilevel Preconditioning for Adaptive Sparse Optimization. Preprint 25, DFG-SPP 1324, August 2009.

[26] S. Dereich. Multilevel Monte Carlo Algorithms for Lévy-driven SDEs with Gaussian Correction. Preprint 26, DFG-SPP 1324, August 2009.

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[30] D. Conte and C. Lubich. An Error Analysis of the Multi-configuration Time-dependent Hartree Method of Quantum Dynamics. Preprint 30, DFG-SPP 1324, November 2009.

[31] C. E. Powell and E. Ullmann. Preconditioning Stochastic Galerkin Saddle Point Problems. Preprint 31, DFG-SPP 1324, November 2009.

[32] O. G. Ernst and E. Ullmann. Stochastic Galerkin Matrices. Preprint 32, DFG-SPP 1324, November 2009.

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