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Regularization With Non-convex Separable Constraints

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Regularization with non-convex separable constraints

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Abstract

We consider regularization of nonlinear ill-posed problems with constraints which are non-convex. As a special case we consider separable constraints, i.e. the regularization takes place in a sequence space and the constraint acts on each sequence element with a possibly non-convex function. We derive conditions under which such a constraint provides a regularization. Moreover, we derive estimates for the error and obtain convergence rates for vanishing noise level. Our assumptions especially cover the example of regularization with a sparsity constraint in which the p-th power with $0 < p \leq 1$ is used and we present other examples as well. In particular we derive error estimates for the error measured in the quasi-norms and obtain a convergence rate of $\mathcal{O}(\delta)$ for the error measured in $\|\cdot\|_p$.

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Keywords: Regularization theory, non-convex constraints

1 Introduction

Recent years have seen impressive progress in regularization theory for inverse problems in Banach spaces. Starting from total variation regularization [37] and maximum entropy regularization [16, 18] the focus has moved to general Tikhonov functionals with convex constraints [1, 9, 23, 25, 29, 32, 33] with special cases like Besov-space constraints [28] and sparsity constraints [10, 14, 21, 26, 27, 30]. Also iterative methods have been analyzed [35]. A different issue of importance in the solution of the corresponding minimization problems, see e.g. [2–5, 15, 19, 22, 31] for sparsity constraints, [11, 24] for total variation and [6, 8, 13, 36] for general Banach space constraints. Even more recently quasi-norms or in general non-convex penalties have been considered. Inspired by the use of non-convex constraints for sparse recovery and compressed sensing [12] the regularizing properties of ℓ^p penalties with $p < 1$ have been analyzed [20, 38].

In this paper we focus on a special class of non-convex constraints, namely constraints which are separable, and derive sufficient and some necessary conditions for regularization. The paper is organized as follows. In Section 2

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we collect some more or less known results about regularization with general constraints. In Section 3 we analyze the regularizing properties of separable constraints and in Section 4 we derive error estimates under appropriate source conditions.

2 Regularization with non-convex constraints

The framework we will consider will be the following: let $F : \mathrm{dom}(F) \subset X \to Y$ be a non-linear linear mapping between its domain, which lies in a Banach space X, and a Banach space Y and we exclude the case $\mathrm{dom}(F) = \emptyset$ to avoid trivialities. We start from the operator equation $F(f) = g$ which is potentially ill posed. We denote with $g \in \mathrm{range}\, F$ exact data and with g^δ noisy data with known noise level $\|g - g^\delta\| \le \delta$. In the following we consider generalized Tikhonov regularization, i.e. the regularized problem consists in minimizing

$$T_\alpha(f) = \frac{1}{q} \left\|F(f) - g^\delta\right\|^q + \alpha R(f)$$

where $0 < q < \infty$, $\alpha > 0$ is the regularization parameter and $R : X \to [0, \infty]$ is a regularization functional with values in the extended half line. With $\mathrm{dom}(R)$ we denote the set of elements f such that $R(f)$ is finite. Note that we will not assume R to be convex. We denote a minimizer of T_α with $f^{\alpha,\delta}$ i.e.

$$f^{\alpha,\delta} \in \underset{f}{\mathrm{argmin}}\ T_\alpha(f). \tag{1}$$

Hence, we dropped the dependence on q, R and g^δ.

Similar to [9, 25] we will use minimum-R-solutions of $F(f) = g$, i.e. an element $f^\dagger$ such that

$$R(f^\dagger) = \min\{R(f) \mid F(f) = g\}.$$

Following the lines of [25] we establish convergence of non-convex regularization. We need the following list of assumptions.

Assumption 2.1. Let

(a) X and Y be reflexive Banach spaces.

(b) $F : \mathrm{dom}(F) \subset X \to Y$ be weakly sequentially closed and bounded on bounded sets.

(c) $R : X \to [0, \infty]$ be proper, i.e.

$$\mathrm{dom}(R) \neq \emptyset.$$

(d) $R : X \to [0, \infty]$ be coercive, i.e.

$$\|f\| \to \infty \implies R(f) \to \infty.$$

(e) $R : X \to [0, \infty]$ be weakly lower-semicontinuous, i.e.

$$f^n \rightharpoonup f \text{ in } X \implies R(f) \le \liminf R(f^n).$$

(f) $R : X \to [0, \infty]$ obey the so-called *Kadec-property* (see [34]), i.e.

$$f^n \rightharpoonup f \text{ in } X \text{ and } R(f^n) \to R(f) \quad \Longrightarrow \quad f^n \to f \text{ in } X. \tag{2}$$

(g) the set $\operatorname{dom}(F) \cap \operatorname{dom}(R)$ be non-empty.

The Assumptions 2.1 (a)–(e) together with (g) guarantee the existence of a minimizing element of T_α, see [17]. If we assume that at least one solution of the equation $F(f) = g$ is in $\operatorname{dom}(R)$ we conclude the existence of a minimum-R-solution as the next proposition shows.

Proposition 2.2 (Existence of minimum-R-solutions)**.** *Let Assumption 2.1 (a)–(e) be fulfilled and let $g \in Y$ such that $\{F(f) = g\} \cap \operatorname{dom}(R) \neq \emptyset$. Then there exists a minimum-R-solution of $F(f) = g$.*

Proof. Suppose there is no minimum-R-solution. Because $g \in \operatorname{range} F$ there is a sequence (f^n) such that

$$R(f^n) \to c \text{ but } R(f) > c \text{ for all } f \text{ such that } F(f) = g. \tag{3}$$

Hence, for sufficiently large n it follows that $R(f^n) < 2c$. Due to Assumption 2.1 (d) the sequence (f^n) is bounded and due to Assumption 2.1 (a) there is a weakly convergent subsequence, also denoted by (f^n), with limit $\tilde{f}$. From Assumption 2.1 (e) we conclude that $R(\tilde{f}) \leq \liminf_{n\to\infty} R(f^n) = c$. Since $(F(f^n))$ is also bounded, we may extract another subsequence such that $(F(f^n))$ converges weakly to some $\tilde{g}$. Since F is weakly sequentially closed we have $F(\tilde{f}) = \tilde{g}$. Together with $R(\tilde{f}) \leq c$ this contradicts (3). □

The next step is to show weak subsequential convergence of $f^{\alpha,\delta}$ for an appropriate parameter choice $\alpha(\delta)$ and $\delta \to 0$.

Theorem 2.3 (Weak regularization)**.** *Let Assumption 2.1 (a)–(e) be fulfilled and assume that $g \in \operatorname{range} F$ such that $\{F(f) = g\} \cap \operatorname{dom}(R) \neq \emptyset$. Then, for a parameter choice with*

$$\alpha \to 0, \ \frac{\delta^q}{\alpha} \to 0 \textit{ for } \delta \to 0$$

there exists a subsequence of $(f^{\alpha,\delta})$ which converges weakly to a minimum-R-solution $f^\dagger$ of $F(f) = g$. Moreover it holds that $R(f^{\alpha,\delta}) \to R(f^\dagger)$.

Proof. The minimum-R-solution $f^\dagger$ exists due to Proposition 2.2. Due to the minimizing property of $f^{\alpha,\delta}$ it holds that

$$\frac{1}{q} \left\| F(f^{\alpha,\delta}) - g^\delta \right\|^q + \alpha R(f^{\alpha,\delta}) \leq \frac{1}{q}\delta^q + \alpha R(f^\dagger).$$

From the parameter choice it follows that

$$\lim_{\delta\to 0} F(f^{\alpha,\delta}) = g \tag{4}$$

and

$$\limsup_{\delta\to 0} R(f^{\alpha,\delta}) \leq R(f^\dagger). \tag{5}$$

Due to Assumption 2.1 (d) the $f^{\alpha,\delta}$ are bounded and according to Assumption 2.1 (a) there is a weakly convergent subsequence (f^n), with weak limit $\tilde{f}$.

Moreover, Assumption 2.1 (b) states that F is weakly sequentially closed and hence, it follows from (4) that $F(\tilde{f}) = g$. Assumption 2.1 (e) together with (5) gives

$$R(\tilde{f}) \leq \liminf_{n\to\infty} R(f^n) \leq R(f^\dagger) \leq R(f)$$

for all f such that $F(f) = g$, since $f^\dagger$ is a minimum-R-solution. Taking $f = \tilde{f}$ shows $R(\tilde{f}) = R(f^\dagger)$ and hence, $\tilde{f}$ is a minimum-R-solution. Moreover, it follows $R(f^n) \to R(f^\dagger)$ and also $R(f^{\alpha,\delta}) \to R(f^\dagger)$ since the latter holds for every weakly convergent subsequence. □

It is an easy corollary that the Kadec property (2) gives strong convergence.

Corollary 2.4 (Strong regularization)**.** *Under the assumptions of Theorem 2.3 and Assumption 2.1 (f) it follows that* $(f^{\alpha,\delta})$ *has a subsequence which converges strongly to a minimum-R-solution.*

3 Regularization with separable constraints

In this section we apply the results of Section 2 to regularization with separable constraints. A motivation for separable constraints comes from so-called sparsity constraints which have been proposed in [14]. There, the authors assumed X to be a Hilbert space and (ψ^k) be an orthonormal basis of X. Regularization shall now be obtained by imposing that the sought after solution f has a sparse representation in this basis, i.e. $f = \sum u_k\psi^k$ where only finitely many u_k are non-zero. In [14], sparsity of the regularized solution was obtained by using the separable constraint $R(f) = \sum w_k\phi(|\langle f | \psi^k\rangle|)$ with $\phi(s) = s^p$ with $1 \leq p \leq 2$ and $w_k \geq w_0 > 0$. Analysis of regularizing properties of this convex regularization can be found in [14, 21, 26, 30]. In this section we investigate the case for general ϕ and state conditions under which R provides a regularization. We apply the results to the non-convex case $0 < p < 1$ (see [20, 38] and also [26] for preliminary results).

In the remainder of the section we use the following notation. Let (ψ^k) be a linearly independent system in the Banach space X. With $B : \ell^2 \to X$ we denote the synthesis operator, i.e. for $u = (u_k) \in \ell^2$ we write $Bu = \sum_k u_k\psi^k$. Further we assume that B is bounded. We then rewrite the ill-posed problem with the operator $K = F \circ B : \mathrm{dom}(K) \subset \ell^2 \to Y$ where $\mathrm{dom}(K) = B^{-1}(\mathrm{dom}(F))$. Hence, the generalized Tikhonov functional has the form

$$T_\alpha(u) = \frac{1}{q}\left\|K(u) - g^\delta\right\|^q + \alpha R(u)$$

with a general separable penalty term of the form

$$R(u) = \sum_k \phi(|u_k|). \qquad (6)$$

The goal is, to apply the results of Section 2. Note that if F fulfills Assumption 2.1 (b) the same holds for K since B is linear and bounded and hence, weak-to-weak continuous. To apply the results of Section 2 it remains to show Assumption 2.1 (c)–(f) (i.e. properness, coercivity, weak lower-semicontinuity, and the Kadec property).

Lemma 3.1 (A sufficient condition for properness of R)**.** *If $\phi(0) = 0$ then R is proper.*

Proof. It holds $R(0) = 0$. □

Lemma 3.2 (A sufficient condition for coercivity of R)**.** *Let $\phi : [0,\infty[\to [0,\infty[$ be continuous and fulfill*

(a) For every $\epsilon > 0$ there exists $\eta > 0$ such that

$$x \leq \epsilon \implies \phi(x) \geq \eta x^2.$$

(b) It holds that $x \to \infty$ implies $\phi(x) \to \infty$.

Then $R : \ell^2 \to [0,\infty]$ defined by (6) *is coercive.*

Proof. Let us assume that R is not coercive. Then there is a sequence (u^n) and a constant C such that $\|u^n\|_2 \to \infty$ but $R(u^n) \leq C$. First we consider the case when there is $\epsilon > 0$ such that $|u_k^n| \leq \epsilon$ for every k, n. Then we immediately have by (a)

$$R(u^n) = \sum \phi(|u_k^n|) \geq \eta \|u^n\|_2^2 \to \infty \quad \text{for} \quad n \to \infty$$

which is a contradiction.

Now consider that $|u_k^n|$ is unbounded. Hence there exist sequences (n_l), (k_l) such that $|u_{k_l}^{n_l}| \to \infty$. Since $|u_{k_l}^{n_l}| \leq \|u^{n_l}\|_2$ there cannot be infinitely many k_l such that n_l is constant and hence, $n_l \to \infty$. Moreover we may assume that n_l is strictly increasing. Now, due to (b), we again get a contradiction:

$$R(u^{n_l}) \geq \phi(|u_{k_l}^{n_l}|) \to \infty \quad \text{for} \quad l \to \infty.$$

□

Remark 3.3. Condition (a) from Lemma 3.2 is fulfilled if, for example, $\phi(x) \geq \eta x^2$ for x smaller than some ϵ and strictly bounded away from zero for $x > \epsilon$.

Lemma 3.4 (A sufficient condition for weak lower-semicontinuity of R)**.** *Let $\phi : [0,\infty[\to [0,\infty[$ be lower-semicontinuous. Then $R : \ell^2 \to [0,\infty]$ defined by* (6) *is weakly lower-semicontinuous.*

Proof. Let $u^n \rightharpoonup u$. Then it holds $u_k^n \to u_k$ for all k. Now, by lower-semicontinuity of ϕ and Fatou's lemma

$$\begin{aligned} R(u) &= \textstyle\sum_k \phi(|u_k|) \\ &\leq \textstyle\sum_k \liminf_{n\to\infty} \phi(|u_k^n|) \leq \liminf_{n\to\infty} \sum_k \phi(|u_k^n|) = \liminf_{n\to\infty} R(u^n). \end{aligned}$$

□

Lemma 3.5 (A sufficient condition for R to obey the Kadec property)**.** *Let $\phi : [0,\infty[\to [0,\infty[$. Assume that for every $0 \leq x, y < L_1$ with $|x - y| < L_2$ there exists $C(L_1, L_2) > 0$ such that*

$$|x^2 - y^2| \leq C(L_1, L_2)|\phi(x) - \phi(y)|.$$

Then $R : \ell^2 \to [0,\infty]$ defined by (6) *obeys the Kadec property.*

Proof. Let $u^n \rightharpoonup u$ in ℓ^2 and $R(u^n) \to R(u)$. It is enough to show, that $\|u^n\| \to \|u\|$ (since this implies strong convergence in the Hilbert space ℓ^2). Due to weak convergence we have for all k that $u_k^n \to u_k$ and hence, for large n it holds that $|u_k^n|, |u_k| < L_1$ and $|u_k^n - u_k| < L_2$ uniformly in k. By assumption there is a constant $C > 0$ such that

$$\big|\|u^n\|^2 - \|u\|^2\big| \leq C \sum_k \big|\phi(|u_k^n|) - \phi(|u_k|)\big|. \tag{7}$$

Define now $v_k^n = \min(\phi(|u_k^n|), \phi(|u_k|))$. Obviously it holds that $v_k^n \to \phi(|u_k|)$ and since $\sum_k \phi(|u_k|) = R(u)$ is finite, it follows by the dominated convergence theorem that $\sum_k v_k^n \to R(u)$. Observe now that due to $R(u^n) \to R(u)$ it holds that

$$\begin{aligned}\sum_k |\phi(|u_k^n|) - \phi(|u_k|)| &= \textstyle\sum_k \phi(|u_k^n|) + \phi(|u_k|) - 2v_k^n \\ &= R(u^n) + R(u) - 2\textstyle\sum_k v_k^n \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Combining this with (7) proves the claim. □

Remark 3.6. In the case of invertible ϕ we may rephrase the condition in Lemma 3.5 as:

$$|(\phi^{-1}(s))^2 - (\phi^{-1}(t))^2| \leq C|s - t|.$$

Hence, we may say that R has the Kadec property if the mapping $s \mapsto (\phi^{-1}(s))^2$ is locally Lipschitz continuous.

Remark 3.7. If we assume that for some $p \geq 1$ the inequality

$$|x^p - y^p| \leq C(L_1, L_2)|\phi(x) - \phi(y)|$$

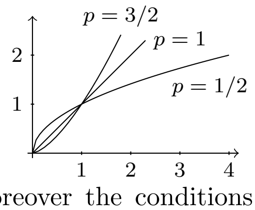
is valid, it follows that

$$u^n \rightharpoonup u \text{ in } \ell^2 \text{ and } R(u^n) \to R(u) \implies u^n \to u \text{ in } \ell^p.$$

The Lemmas 3.1, 3.2, 3.4, and 3.5 provide conditions on a function ϕ such that R defined by (6) can be used for regularization. Now we give some examples of functions ϕ:

Example 3.8 (ℓ^p norms). Let $0 < p \leq 2$ and define

$$\phi(x) = x^p.$$

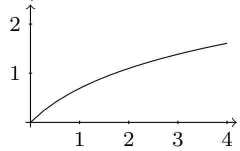


The continuity of ϕ is obvious. Moreover the conditions of Lemma 3.2 are trivially fulfilled. By Remark 3.6 we also get that $R(u) = \sum |u_k|^p$ obeys the Kadec property and hence, ℓ^p-norms provide strong subsequential regularization and we reproduced the corresponding results from [20, 38].

We present two more examples of functions ϕ which provide subsequential regularization.

Example 3.9 (A slowly growing function). Let

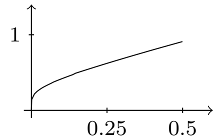
$$\phi(x) = \log(x+1).$$



This function is clearly continuous and non-convex but monotonically increasing to ∞ (but this slower than every power x^p with $p > 0$). Moreover, condition (a) from Lemma 3.2 is trivially fulfilled (see e.g. Remark 3.3). Furthermore the Kadec property is fulfilled by Remark 3.6 since the mapping $s \mapsto (\phi^{-1}(s))^2 = (\exp(s) - 1)^2$ is locally Lipschitz continuous. Hence, we see that the functional $R(u) = \sum \phi(|u_k|)$ provides strong subsequential regularization.

Example 3.10 (A sharp cusp in 0). Let

$$\phi(x) = -\frac{1}{\log(\frac{x}{1+x})}.$$



This function grows asymptotically like x but has a sharp cusp at 0 (namely it is sharper than the cusp of x^p for every $p > 0$ in the sense that for every $p > 0$ and $\eta > 0$ there is an $\epsilon > 0$ such that $\phi(x) > \eta x^p$ for $x < \epsilon$).

Again all assumptions of Lemma 3.2, Lemma 3.4 and Lemma 3.5 are fulfilled and hence, the functional $R(u) = \sum \phi(|u_k|)$ also provides strong subsequential regularization.

Before we turn to estimates for the error $u^{\alpha,\delta} - u^\dagger$ we provide some necessary conditions for ϕ.

Proposition 3.11. *Let* $\phi : [0,\infty[\to [0,\infty[$ *and let* $R : \ell^2 \to [0,\infty]$ *be defined by* (6)*. Then it holds:*

(a) If R is proper and ϕ is monotonically increasing then $\phi(0) = 0$.

(b) If R is weakly lower-semicontinuous, then ϕ is lower semicontinuous.

(c) If R is coercive then it holds:

$$\phi(x) \to \infty \quad \text{for} \quad x \to \infty,$$
$$\text{for each } \beta > 2: \ \frac{\phi(x)}{x^\beta} \to \infty \quad \text{for} \quad x \to 0.$$

Proof. **(a):** For a proper R there is some u such that $R(u) < \infty$. By monotonicity we have $R(u) = \sum \phi(|u_k|) \geq \sum \phi(0)$ which shows that $\phi(0) = 0$ has to hold.

(b): Consider the first unit vector $e = (1, 0, \dots)$ and the sequence $x_n e$ with $x_n \to x$. It follows

$$\phi(x) = R(xe) \leq \liminf R(x_n e) = \liminf \phi(x_n).$$

(c): Similarly to (b) one sees that $\phi(x) \to \infty$ for $x \to \infty$. Now let $\beta > 2$ and consider the sequence

$$u^n = n^{-1/\beta}(\underbrace{1, \dots, 1}_{n \text{ times}}, 0, \dots).$$

It holds

$$\|u^n\|^2 = n^{-2/\beta} n = n^{1-2/\beta} \to \infty \quad \text{for } n \to \infty.$$

Since R is coercive it follows

$$R(u^n) = \phi(n^{-1/\beta})n \to \infty \quad \text{for } n \to \infty$$

which implies the claim.

□

Remark 3.12. The statement (b) is precisely the converse of Lemma 3.4 and hence weak lower-semicontinuity of R and lower semicontinuity of ϕ are equivalent. However, Lemma 3.2 and Proposition 3.11 do not provide a characterization of coercivity of R in terms of ϕ.

4 Error estimates

To refine the regularization results from the previous section and derive error estimates for the total error $u^{\alpha,\delta} - u^\dagger$ one has to make additional assumptions on $u^\dagger$. Resembling the approach taken in [20] we derive two different error estimates for different measures of the error. To this end we first derive estimates which have been proven to be useful in [21]. For simplicity we restrict ourselves to a bounded linear operator K (see Remark 4.6 below). The first estimate is an estimate for $\left\|u^{\alpha,\delta} - u^\dagger\right\|_1$:

Lemma 4.1. *Let ϕ fulfill the following conditions:*

(a) For every $\lambda > 0$ there exists a $C_1 > 0$ such that it holds that

$$x > \lambda \implies \phi(x) - \phi(y) \le C_1|x - y|.$$

(b) For every $M > 0$ there exists $C_2 > 0$ such that for $x \le M$ it holds that $x \le C_2\phi(x)$.

Let furthermore R be defined by (6), and assume that $u^\dagger$ is sparse. Moreover let $K : \ell^2 \to Y$ be linear and bounded and assume that the canonical basis vectors e_k fulfill

$$e_k \in \operatorname{range} K^* \quad \textit{whenever} \quad u_k^\dagger \ne 0.$$

Then there exists a constant $C > 0$ such that

$$\left\|u - u^\dagger\right\|_1 \le C_2(R(u) - R(u^\dagger)) + C\left\|K(u - u^\dagger)\right\|.$$

Proof. We define the index set $I = \{k \mid u_k^\dagger \ne 0\}$ and estimate with the help of Condition (b):

$$\begin{aligned}\sum_k |u_k - u_k^\dagger| &= \textstyle\sum_{k\notin I} |u_k| + \sum_{k\in I} |u_k - u_k^\dagger| \\ &\le C_2 \textstyle\sum_{k\notin I} \big(\phi(|u_k|) - \phi(|u_k^\dagger|)\big) + \sum_{k\in I} |u_k - u_k^\dagger| \\ &= C_2 \textstyle\sum_{k\in\mathbf{N}} \big(\phi(|u_k|) - \phi(|u_k^\dagger|)\big) \\ &\quad + \textstyle\sum_{k\in I} \big(|u_k - u_k^\dagger| + C_2(\phi(|u_k^\dagger|) - \phi(|u_k|))\big)\end{aligned}$$

Since there is a $\lambda > 0$ such that we have $|u_k^\dagger| > \lambda$ for $k \in I$, we apply Condition (a) and get

$$\sum_k |u_k - u_k^\dagger| \le C_2 \sum_{k\in\mathbf{N}} \big(\phi(|u_k|) - \phi(|u_k^\dagger|)\big) + (1 + C_2C_1) \sum_{k\in I} |u_k - u_k^\dagger|. \qquad (8)$$

Moreover, $e_k \in \operatorname{range} K^*$ whenever $k \in I$, so there exist d_k such that $K^* d_k = e_k$ and we have for $k \in I$ and every $w \in \ell^2$:

$$\begin{aligned} |w_k| = |\,\langle w|\, e_k\rangle\,| \quad &= |\,\langle Kw|\, d_k\rangle\,| \\ &\le \|d_k\|\,\|Kw\| \le \max\nolimits_{k\in I} \|d_k\|\,\|Kw\| = C_3\,\|Kw\|\,. \end{aligned}$$

Using this we conclude from (8):

$$\sum_k |u_k - u_k^\dagger| \le C_2 \sum_{k\in\mathbf{N}} \big(\phi(|u_k|) - \phi(|u_k^\dagger|)\big) + (1 + C_2C_1)|I|C_3\,\big\|K(u - u^\dagger)\big\|$$

where $|I|$ denotes the size of the index set I. □

Remark 4.2. Note that Condition (a) from Lemma 4.1 is fulfilled if the function ϕ is Lipschitz continuous for $x > \lambda$ which is for example case for all p-th powers with $0 < p \le 1$ and also for the slowly growing function from Example 3.9 and the function with the sharp cusp from Example 3.10. Moreover, Condition (b) is also fulfilled for p-th powers with $p \le 1$ and the functions from Example 3.9 and 3.10 since they are concave.

Now we formulate a different estimate for the term $R(u - u^\dagger)$. This measure for the difference may be stronger than the ℓ^1 norm and this is reflected in a different estimate.

Lemma 4.3. *Let ϕ fulfill the following conditions:*

(a) ϕ is monotonically increasing and $\phi(0) = 0$.

(b) For every $\lambda \ge 0$ there exists $C_1 \ge 1$ such that it holds

$$x > \lambda \implies \phi(x) - \phi(y) \le C_1\phi(|x - y|). \qquad (9)$$

(c) For every $c > 0$ there exists $M > 0$ such that for every $x > 0$ it holds

$$\phi(cx) \le M\phi(x).$$

Let furthermore R be defined by (6), and assume that $u^\dagger$ is sparse. Moreover let $K : \ell^2 \to Y$ be linear and bounded and assume that the canonical basis vectors e_k fulfill

$$e_k \in \operatorname{range} K^* \quad \textit{whenever} \quad u_k^\dagger \ne 0.$$

Then there exists a constant $C > 0$ such that

$$R(u - u^\dagger) \le R(u) - R(u^\dagger) + C\phi(\big\|K(u - u^\dagger)\big\|).$$

Proof. Similar to Lemma 4.1 we define the index set $I = \{k \mid u_k^\dagger \neq 0\}$ and observe:

$$\sum_k \phi(|u_k - u_k^\dagger|) = \sum_{k\notin I} \phi(|u_k|) - \phi(|u_k^\dagger|) + \sum_{k\in I} \phi(|u_k - u_k^\dagger|)$$

Since we have $|u_k^\dagger| > \lambda$ for $k \in I$ we apply (9) and get

$$\sum_k \phi(|u_k - u_k^\dagger|) \leq \sum_k \phi(|u_k|) - \phi(|u_k^\dagger|) + (1 + C_1) \sum_{k\in I} \phi(|u_k - u_k^\dagger|). \qquad (10)$$

Again similar to Lemma 4.1 we have for $k \in I$ and every $w \in \ell^2$:

$$|w_k| \leq C_3 \|Kw\|.$$

Conditions (a) and (c) give

$$\phi(|w_k|) \leq C_4 \phi(\|Kw\|).$$

Using this, we get from (10)

$$\sum_k \phi(|u_k - u_k^\dagger|) \leq \sum_k \phi(|u_k|) - \phi(|u_k^\dagger|) + (1 + C_1)C_4|I|\phi(\left\|K(u - u^\dagger)\right\|).$$

□

Remark 4.4. One may check that the conditions on ϕ from Lemma 4.3 also hold for p-th powers with $p \leq 1$ and the functions from Example 3.9 and 3.10. (To see that (c) is fulfilled note that ϕ is monotonically increasing while ϕ' is monotonically decreasing in all cases. Hence, we may take $M = 1$ for $c \leq 1$ and $M = c$ for $c > 1$).

Remark 4.5 (Relation to the FBI property). We remark on the condition "$e_k \in \operatorname{range} K^*$ for $u_k^\dagger \neq 0$". Since $\operatorname{range} K^* \subset (\ker K)^\perp$ we know that $e_k \in (\ker K)^\perp$ and hence, the operator K acts injectively on $\operatorname{span}\{e_k \,|\, u_k^\dagger \neq 0\}$. This means that the operator K fulfills some kind of "finite basis injectivity condition" from [6].

Remark 4.6 (Nonlinear operators). Both Lemma 4.1 and Lemma 4.3 can be generalized to non-linear K. For Lemma 4.1 we may assume that the nonlinearity fulfills the estimate

$$R(u) - R(u^\dagger) \geq \gamma_1 \left\|K'(u^\dagger)(u - u^\dagger)\right\| - \gamma_2 \left\|K(u) - K(u^\dagger)\right\| \qquad (11)$$

locally around $u^\dagger$ for some constants $\gamma_1, \gamma_2 > 0$ similar to [20]. Similarly we may assume for Lemma 4.3 that

$$R(u) - R(u^\dagger) \geq \gamma_1 \phi(\left\|K'(u^\dagger)(u - u^\dagger)\right\|) - \gamma_2 \phi(\left\|K(u) - K(u^\dagger)\right\|). \qquad (12)$$

Then the condition for the canonical basis vector e_k reads as $e_k \in \operatorname{range} K'(u^\dagger)^*$ whenever $u_k^\dagger \neq 0$.

Both Lemma 4.1 and Lemma 4.3 lead to error estimates for $u^{\alpha,\delta} - u^\dagger$. While Lemma 4.1 gives an estimate for $\left\|u^{\alpha,\delta} - u^\dagger\right\|_1$, Lemma 4.3 will lead to an estimate for $R(u^{\alpha,\delta} - u^\dagger)$.

The following theorem resembles [21, Proposition 8]:

Theorem 4.7. *Let $K : \ell^2 \to Y$ and $\phi : [0,\infty[\to [0,\infty[$ fulfill the conditions of Lemma 4.1 and let R be defined by (6). Moreover, assume $q \geq 1$ and let $u^\dagger$ be a minimum-R-solution of $Ku = g$ which is sparse. For g^δ with $\left\|g - g^\delta\right\| \leq \delta$ define*

$$u^{\alpha,\delta} \in \operatorname*{argmin}_u \frac{1}{q} \left\|Ku - g^\delta\right\|^q + \alpha R(u).$$

Then for $q > 1$ it holds that

$$\left\|u_k^{\alpha,\delta} - u_k^\dagger\right\|_1 \leq \frac{C_2}{q}\frac{\delta^q}{\alpha} + \frac{(q-1)C^{q/(q-1)}}{qC_2^{q'-1}}\alpha^{1/(q-1)} + C\delta$$

while for $q = 1$ we obtain for $\alpha < C_2/C$

$$\left\|u^{\alpha,\delta} - u^\dagger\right\|_1 \leq \frac{(1 + \alpha C/C_2)C_2\delta}{\alpha}.$$

Proof. The proof follows [21, Proposition 8] and is included for the sake of completeness. Due to the minimization property of $u^{\alpha,\delta}$ and because of $\left\|Ku^\dagger - g^\delta\right\| \leq \delta$ we conclude

$$\frac{1}{q} \left\|Ku^{\alpha,\delta} - g^\delta\right\|^q + \alpha \sum_k \phi(|u_k^{\alpha,\delta}|) \leq \frac{\delta^q}{q} + \alpha \sum_k \phi(|u_k^\dagger|). \tag{13}$$

From Lemma 4.1 we deduce

$$\begin{aligned}\frac{\delta^q}{q} &\geq \alpha\Big(\textstyle\sum_k \phi(|u_k^{\alpha,\delta}|) - \sum_k \phi(|u_k^\dagger|)\Big) + \frac{1}{q}\left\|Ku^{\alpha,\delta} - g^\delta\right\|^q \\ &\geq \tfrac{\alpha}{C_2}\left\|u^{\alpha,\delta} - u^\dagger\right\|_1 - \tfrac{\alpha C}{C_2}\left\|K(u^{\alpha,\delta} - u^\dagger)\right\| + \tfrac{1}{q}\left\|Ku^{\alpha,\delta} - g^\delta\right\|^q \\ &\geq \tfrac{\alpha}{C_2}\left\|u^{\alpha,\delta} - u^\dagger\right\|_1 - \tfrac{\alpha C}{C_2}(\left\|Ku^{\alpha,\delta} - g^\delta\right\| + \delta) + \tfrac{1}{q}\left\|Ku^{\alpha,\delta} - g^\delta\right\|^q.\end{aligned}$$

For $q = 1$ this implies the claim. For $q > 1$ we use Young's inequality ($ab \leq a^r/r + b^{r'}/r'$) with $r = q$ and $r' = q/(q-1)$ to obtain

$$\frac{\alpha C}{C_2}\left\|Ku^{\alpha,\delta} - g^\delta\right\| \leq \frac{(q-1)(\alpha C/C_2)^{q/(q-1)}}{q} + \frac{1}{q}\left\|Ku^{\alpha,\delta} - g^\delta\right\|^q.$$

Combining this with the preceding inequality proves the claim. □

As already noticed in [25] the case $q = 1$ provides exact penalization, i.e. without noise ($\delta = 0$) the regularization is exact if α is small enough.

The next theorem gives an error estimate for $R(u^{\alpha,\delta} - u^\dagger)$.

Theorem 4.8. *Let $K : \ell^2 \to Y$ and $\phi : [0,\infty[\to [0,\infty[$ fulfill the conditions of Lemma 4.3 and let R be defined by (6). Moreover, assume that $q > 0$ and that for $0 < p \leq q$ it holds that $\phi(x) \leq C_5 x^p$. Let furthermore $u^\dagger$ be a minimum-R-solution of $Ku = g$ which is sparse. For g^δ with $\left\|g - g^\delta\right\| \leq \delta$ define*

$$u^{\alpha,\delta} \in \operatorname*{argmin}_u \frac{1}{q} \left\|Ku - g^\delta\right\|^q + \alpha R(u).$$

Then for $q > p$ it holds

$$R(u^{\alpha,\delta} - u^\dagger) \leq \frac{\delta^q}{\alpha q} + (CC_1C_5)^{\frac{q}{q-p}} p^{\frac{p}{q-p}} \frac{q-p}{q} \alpha^{\frac{p}{q-p}} + CC_1C_5\delta^p$$

while for $q = p$ we obtain for $\alpha \le 1/(qCC_1C_5)$ that

$$R(u^{\alpha,\delta} - u^\dagger) \le \frac{(q^{-1} + \alpha CC_1C_5)\delta^q}{\alpha}.$$

Proof. Due to the minimization property of $u^{\alpha,\delta}$ and because of $\|Ku^\dagger - g^\delta\| \le \delta$ we conclude

$$\frac{1}{q}\left\|Ku^{\alpha,\delta} - g^\delta\right\|^q + \alpha\sum_k \phi(|u_k^{\alpha,\delta}|) \le \frac{\delta^q}{q} + \alpha\sum_k \phi(|u_k^\dagger|).$$

From Lemma 4.3 we deduce

$$\begin{aligned}\frac{\delta^q}{q} &\ge \alpha\Big(\textstyle\sum_k \phi(|u_k^{\alpha,\delta}|) - \sum_k \phi(|u_k^\dagger|)\Big) + \frac{1}{q}\left\|Ku^{\alpha,\delta} - g^\delta\right\|^q \\ &\ge \alpha\textstyle\sum_k \phi(|u_k^{\alpha,\delta} - u_k^\dagger|) - \alpha C\phi(\left\|K(u^{\alpha,\delta} - u^\dagger)\right\|) + \frac{1}{q}\left\|Ku^{\alpha,\delta} - g^\delta\right\|^q. \quad (14)\end{aligned}$$

From (9) with $\lambda = 0$ we conclude that ϕ fulfills the quasi-triangle inequality $\phi(x+y) \le C_1(\phi(x) + \phi(y))$. With this and the assumption that $\phi(x) \le C_5x^p$ we estimate

$$\begin{aligned}\phi(\left\|K(u^{\alpha,\delta} - u^\dagger)\right\|) &\le C_1(\phi(\left\|K^{\alpha,\delta} - g^\delta\right\|) + \phi(\delta)) \\ &\le C_1C_5(\left\|K^{\alpha,\delta} - g^\delta\right\|^p + \delta^p) \quad (15)\end{aligned}$$

For $p = q$ the combination of (14) and (15) proves the claim. For $p < q$ we use Young's inequality with $r = q/p$ and $r' = q/(q-p)$ to obtain

$$\alpha CC_1C_5\left\|K^{\alpha,\delta} - g^\delta\right\|^p \le (\alpha CC_1C_5)^{\frac{q}{q-p}} p^{\frac{p}{q-p}}\frac{q-p}{q} + \frac{1}{q}\left\|Ku^{\alpha,\delta} - g^\delta\right\|^q. \quad (16)$$

Combining the inequalities (15) and (16) with (14) proves the claim. □

Again, we may notice that the case $p = q$ provides exact penalization, i.e. the regularization is exact for $\delta = 0$. We state a few remarks on Theorems 4.7 and 4.8.

Remark 4.9 (Error estimates for nonlinear operators). According to Remark 4.6 both Theorem 4.7 and Theorem 4.8 can be generalized for nonlinear operators as soon as the condition (11), resp. (12) is fulfilled.

Example 4.10 (Rates for ℓ^p penalties). Let us take $\phi(x) = x^p$ for $0 < p \le 1$. Then we see that the assumptions of Theorem 4.7 are fulfilled and hence, with the parameter choice $\alpha = \delta^{q-1}$ for $q > 1$, we reproduced the error estimates and convergence rate from [20], i.e. for sparse $u^\dagger$ it holds

$$\left\|u^{\alpha,\delta} - u^\dagger\right\|_1 = \mathcal{O}(\delta) \quad \text{for} \quad \delta \to 0.$$

For $q = 1$ we see that we also get the same rate if α is small enough for any choice of δ.

The assumptions of Theorem 4.8 are also fulfilled. For $p < q$ and sparse $u^\dagger$ we get with the parameter choice $\alpha = \delta^{q-p}$ the convergence rate

$$\left\|u^{\alpha,\delta} - u^\dagger\right\|_p^p = \mathcal{O}(\delta^p).$$

This shows that we get the rate $\mathcal{O}(\delta)$ not only for the 1-norm but also for the corresponding stronger p-quasi-norm. For $p < 1$ this is an improvement over the main result from [20] (Theorem 5). We may also apply Theorem 4.8 in the case $p = q$ (i.e. we use $q \leq 1$) and deduce for sparse $u^\dagger$ that it holds

$$\left\| u^{\alpha,\delta} - u^\dagger \right\|_p = \mathcal{O}(\delta).$$

for α small enough. Note that Theorem 4.8 is not applicable for $p > 1$ since in this case the condition (9) from Lemma 4.3 is not fulfilled.

Example 4.11. Due to Remark 4.2 and 4.4 we may apply Theorem 4.7 also to the slowly growing function and the sharp cusp from Example 3.9 and 3.10 respectively. Similar to the case of ℓ^p penalties, Theorem 4.7 gives a rate of δ for the 1-norm in this case.

We may also apply Theorem 4.8 to the slowly growing function but not to the sharp cusp since this is not bounded from above by some p-th power.

Remark 4.12 (Another error estimate). Similar to Theorem 4.8, we may derive yet another error estimate by using the following regularization:

$$u^{\alpha,\delta} \in \operatorname*{argmin}_u \phi(\left\| Ku - g^\delta \right\|) + \alpha \sum_k \phi(|u_k|)$$

(which can be seen to be a regularization by following the proof of Theorem 2.3). Under the assumptions of Lemma 4.3 we may arrive at the estimate

$$R(u^{\alpha,\delta} - u^\dagger) \leq \frac{(1 + \alpha C C_1)\phi(\delta)}{\alpha}$$

which holds for $\alpha \leq (CC_1)^{-1}$. Again, we get exact reconstruction for $\delta = 0$.

5 Conclusion

In this paper we tried to give further insight into the regularization with non-convex penalty terms. By investigating separable penalties we were able to derive sufficient conditions for regularization. Moreover, we stated additional conditions that enabled us to derive several error estimates in Theorems 4.7 and 4.8 and Remark 4.12. We showed that the results are not only valid for ℓ^p penalties but also for a broader class of non-convex sparsity enforcing functions. In particular it turned out that the sparsity enforcing function may both grow arbitrarily slow at infinity and decay arbitrarily slow at zero.

Besides these general observations on possible sparsity enforcing functions we derived a convergence rate of $\mathcal{O}(\delta)$ for the error measured in the p-norm when the discrepancy is measured in q-power of the Banach-space norm, the penalty term is the p-th power of the p-norm. This improves on an earlier result from [20] where the same rate was derived under similar assumption but for the 1-norm.

However, to make non-convex penalties applicable in practice one needs algorithms to solve the corresponding non-convex minimization problems. These are derived in a different work [7] where also numerical experiments are presented.

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References

[1] Thomas Bonesky. Morozov's discrepancy principle and Tikhonov-type functionals. *Inverse Problems*, 25(1):015015(11pp), 2009.

[2] Thomas Bonesky, Kristian Bredies, Dirk A. Lorenz, and Peter Maass. A generalized conditional gradient method for nonlinear operator equations with sparsity constraints. *Inverse Problems*, 23:2041–2058, 2007.

[3] Kristian Bredies. A forward-backward splitting algorithm for the minimization of non-smooth convex functionals in Banach space. *Inverse Problems*, 25(1):015005 (20pp), 2009.

[4] Kristian Bredies. An iterative thresholding-like algorithm for inverse problems with sparsity constraints in Banach space. *Journal of Inverse and Ill Posed Problems*, 17(1):19–28, 2009.

[5] Kristian Bredies and Dirk A. Lorenz. Iterated hard shrinkage for minimization problems with sparsity constraints. *SIAM Journal on Scientific Computing*, 30(2):657–683, 2008.

[6] Kristian Bredies and Dirk A. Lorenz. Linear convergence of iterative soft-thresholding. *Journal of Fourier Analysis and Applications*, 14(5–6):813–837, 2008.

[7] Kristian Bredies and Dirk A. Lorenz. Minimization of non-smooth, non-convex functionals by iterative thresholding. Preprint, 2009.

[8] Kristian Bredies, Dirk A. Lorenz, and Peter Maass. A generalized conditional gradient method and its connection to an iterative shrinkage method. *Computational Optimization and Applications*, 42(2):173–193, 2008.

[9] Martin Burger and Stanley J. Osher. Convergence rates of convex variational regularization. *Inverse Problems*, 20(5):1411–1420, 2004.

[10] Martin Burger, Elena Resmerita, and Lin He. Error estimation for Bregman iterations and inverse scale space methods in image restoration. *Computing*, 81:109–135, 2007. Special Issue on Industrial Geometry (Guest editors: B. Jüttler, H. Pottmann, O. Scherzer).

[11] Antonin Chambolle. An algorithm for total variation minimization and applications. *Journal of Mathematical Imaging and Vision*, 20:89–97, 2004.

[12] Rick Chartrand. Exact reconstructions of sparse signals via nonconvex minimization. *IEEE Signal Processing Letters*, 14:707–710, 2007.

[13] Patrick L. Combettes and Valérie R. Wajs. Signal recovery by proximal forward-backward splitting. *Multiscale Model. Simul.*, 4(4):1168–1200, 2005.

[14] Ingrid Daubechies, Michel Defrise, and Christine De Mol. An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. *Communications in Pure and Applied Mathematics*, 57(11):1413–1457, 2004.

[15] Ingrid Daubechies, Massimo Fornasier, and Ignace Loris. Accelerated projected gradient method for linear inverse problems with sparsity constraints. *Journal of Fourier Analysis and Applications*, 14(5–6):764–792, 2008.

[16] David L. Donoho, Iain M. Johnstone, Jeffrey C. Hoch, and Allen S. Stern. Maximal entropy and the nearly black object. *Journal of the Royal Statistical Society. Series B*, 54(1):41–81, 1992.

[17] Heinz W. Engl, Martin Hanke, and Andreas Neubauer. *Regularization of Inverse Problems*, volume 375 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 2000.

[18] Heinz W. Engl and Gerhard Landl. Convergence rates for maximum entropy regularization. *SIAM J. Numer. Anal.*, 30(5):1509–1536, 1993.

[19] Mário A. T. Figueiredo, Robert D. Nowak, and Stephen J. Wright. Gradient projection for sparse reconstruction: Applications to compressed sensing and other inverse problems. *IEEE Journal of Selected Topics in Signal Processing*, 4:586–597, 2007.

[20] Markus Grasmair. Well-posedness and convergence rates for sparse regularization with sublinear l^q penalty term. FSP 092: Joint Research Program of Industrial Geometry, 74, 2008.

[21] Markus Grasmair, Markus Haltmeier, and Otmar Scherzer. Sparse regularization with ℓ^q penalty term. *Inverse Problems*, 24(5):055020 (13pp), 2008.

[22] Roland Griesse and Dirk A. Lorenz. A semismooth Newton method for Tikhonov functionals with sparsity constraints. *Inverse Problems*, 24:035007 (19pp), 2008.

[23] Torsten Hein. Convergence rates for regularization of ill-posed problems in Banach spaces by approximate source conditions. *Inverse Problems*, 24(4):045007 (10pp), August 2008.

[24] Michael Hintermüller and Karl Kunisch. Total bounded variation regularization as a bilaterally constrained optimization problem. *SIAM Journal on Applied Mathematics*, 64(4):1311–1333, 2004.

[25] Bernd Hofmann, Barbara Kaltenbacher, Christiane Poeschl, and Otmar Scherzer. A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators. *Inverse Problems*, 23(3):987–1010, 2007.

[26] Dirk A. Lorenz. Convergence rates and source conditions for Tikhonov regularization with sparsity constraints. *Journal of Inverse and Ill-Posed Problems*, 16(5):463–478, 2008.

[27] Dirk A. Lorenz. On the role of sparsity in inverse problems. *Journal of Inverse and Ill Posed Problems*, 17(1):61–68, 2009.

[28] Dirk A. Lorenz and Dennis Trede. Optimal convergence rates for Tikhonov regularization in Besov scales. *Inverse Problems*, 24:055010 (14pp), 2008.

[29] Christiane Pöschl. An overview on convergence rates for Tikhonov regularization methods for non-linear operators. *Journal of Inverse and Ill Posed Problems*, 17(1):77–84, 2009.

[30] Ronny Ramlau. Regularization properties of Tikhonov regularization with sparsity constraints. *Electronic Transactions on Numerical Analysis*, 30:54–74, 2008.

[31] Ronny Ramlau and Gerd Teschke. A Tikhonov-based projection iteration for nonlinear ill-posed problems with sparsity constraints. *Numerische Mathematik*, 104(2):177–203, 2006.

[32] Elena Resmerita. Regularization of ill-posed problems in Banach spaces: convergence rates. *Inverse Problems*, 21(4):1303–1314, 2005.

[33] Elena Resmerita and Otmar Scherzer. Error estimates for non-quadratic regularization and the relation to enhancement. *Inverse Problems*, 22(3):801–814, 2006.

[34] Winfried Schirotzek. *Nonsmooth Analysis*. Springer, Berlin, 2007.

[35] Frank Schöpfer, Alfred K. Louis, and Thomas Schuster. Nonlinear iterative methods for linear ill-posed problems in Banach spaces. *Inverse Problems*, 22:311–329, 2006.

[36] Thomas Schuster, Peter Maass, Thomas Bonesky, Kamil S. Kazimierski, and Frank Schöpfer. Minimization of Tikhonov functionals in Banach spaces. *Abstract and Applied Analysis*, 2008:Article ID 192679, 19 pages, 2008.

[37] Curtis R. Vogel and Robert Acar. Analysis of bounded variation penalty methods for ill-posed problems. *Inverse Problems*, 10:1217–1229, 1994.

[38] Clemens A. Zarzer. On Tikhonov regularization with non-convex sparsity constraints. *Inverse Problems*, 25(2):025006, 2009.

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[1] R. Ramlau, G. Teschke, and M. Zhariy. A Compressive Landweber Iteration for Solving Ill-Posed Inverse Problems. Preprint 1, DFG-SPP 1324, September 2008.

[2] G. Plonka. The Easy Path Wavelet Transform: A New Adaptive Wavelet Transform for Sparse Representation of Two-dimensional Data. Preprint 2, DFG-SPP 1324, September 2008.

[3] E. Novak and H. Woźniakowski. Optimal Order of Convergence and (In-) Tractability of Multivariate Approximation of Smooth Functions. Preprint 3, DFG-SPP 1324, October 2008.

[4] M. Espig, L. Grasedyck, and W. Hackbusch. Black Box Low Tensor Rank Approximation Using Fibre-Crosses. Preprint 4, DFG-SPP 1324, October 2008.

[5] T. Bonesky, S. Dahlke, P. Maass, and T. Raasch. Adaptive Wavelet Methods and Sparsity Reconstruction for Inverse Heat Conduction Problems. Preprint 5, DFG-SPP 1324, January 2009.

[6] E. Novak and H. Woźniakowski. Approximation of Infinitely Differentiable Multivariate Functions Is Intractable. Preprint 6, DFG-SPP 1324, January 2009.

[7] J. Ma and G. Plonka. A Review of Curvelets and Recent Applications. Preprint 7, DFG-SPP 1324, February 2009.

[8] L. Denis, D. A. Lorenz, and D. Trede. Greedy Solution of Ill-Posed Problems: Error Bounds and Exact Inversion. Preprint 8, DFG-SPP 1324, April 2009.

[9] U. Friedrich. A Two Parameter Generalization of Lions' Nonoverlapping Domain Decomposition Method for Linear Elliptic PDEs. Preprint 9, DFG-SPP 1324, April 2009.

[10] K. Bredies and D. A. Lorenz. Minimization of Non-smooth, Non-convex Functionals by Iterative Thresholding. Preprint 10, DFG-SPP 1324, April 2009.

[11] K. Bredies and D. A. Lorenz. Regularization With Non-convex Separable Constraints. Preprint 11, DFG-SPP 1324, April 2009.