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„Extraktion quantifizierbarer Information aus komplexen Systemen"

Minimization of Non-smooth, Non-convex Functionals by Iterative Thresholding

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Minimization of non-smooth, non-convex functionals by iterative thresholding

Kristian Bredies* Dirk A. Lorenz†

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Abstract

Numerical algorithms for a special class of non-smooth and non-convex minimization problems in infinite dimensional Hilbert spaces are considered. The functionals under consideration are the sum of a smooth and non-smooth functional, both possibly non-convex. We propose a generalization of the gradient projection method and analyze its convergence properties. For separable constraints in the sequence space, we show that the algorithm amounts to an iterative thresholding procedure. In this special case we prove strong subsequentional convergence and, moreover show that the limit satisfies necessary conditions for a global minimizer. Eventually, the algorithm is applied to ℓ^p-penalties arising in the recovery of sparse data and numerical tests are presented.

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1 Introduction

In this article we develop an algorithm which aims at minimizing non-smooth and non-convex functionals, covering the important special case of Tikhonov functionals for non-linear operators and non-convex penalty terms. The minimization of non-convex and non-smooth functionals is a delicate matter. On the one hand, there is a class of popular generic algorithms such as simulated annealing, genetic algorithms and other derivative-free methods which can be performed with minimal assumptions on the objective functional. However, they tend to be impractical in higher-dimensional or infinite-dimensional spaces. On the other hand, many minimization algorithms for non-convex problems which are based on derivatives can be applied when the functional is sufficiently smooth. Convergence can, under suitable conditions, be established, also in infinite-dimensional spaces [21, 22]. When it comes to non-convex and non-smooth optimization problems in high (or infinite) dimensions as it is typical for inverse problems in mathematical image processing or partial differential

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equations, there are only a few algorithms available, for instance the graduated non-convexity algorithm which works for finite-dimensional, large-scale problems [1, 19, 20]. In the present paper, a new method for the numerical minimization of such functionals is proposed.

More precisely, we introduce a generalized gradient projection method which involves non-convex proximity mappings and show convergence. Our plan of establishing convergence can roughly be summarized as follows: First we show, that the proposed algorithm reduces the functional value in every step. By coercivity this will give us a weakly convergent subsequence. Then we use arguments for the specific case of separable penalties to show strong convergence of a subsequence and we show that the limit is a critical point of the functional. Under additional assumptions we get strong convergence of the whole sequence.

Our problem in general reads as follows: Let X be a Hilbert space, $S : X \to [0, \infty[$ a differentiable functional and $R : X \to [0, \infty]$ possibly non-smooth. For a positive parameter α we consider the minimization problem

$$\min_{u \in X} \; T_\alpha(u) \quad \text{with} \quad T_\alpha(u) = S(u) + \alpha R(u). \tag{1}$$

Such a T_α typically models the Tikhonov functional associated with the inverse problem $K(u) = g$ where $K : X \to Y$ is a weakly sequentially closed and sufficiently smooth mapping into the Hilbert space Y. Here, S measures, for example, the discrepancy, i.e.

$$S(u) = \frac{\|K(u) - g\|^2}{2}$$

while R serves as a regularization term, which is non-convex and non-smooth in our setting.

As a case of particular importance we will consider separable penalties, i.e. for $X = \ell^2$ and $\phi : [0, \infty[\to [0, \infty[$ we consider

$$R(u) = \sum_{k \in \mathbf{N}} \phi(|u_k|)$$

(while $R(u) = \infty$ whenever the sum does not converge). Note that we do not assume ϕ (and hence R) to be either convex or smooth.

One may be tempted to use the approach via surrogate functionals as proposed in [9] and applied to non-linear problems in [23]. In this approach, one replaces the functional with

$$\Phi(u, a) = T_\alpha(u) + \tfrac{C}{2} \|u - a\|^2 + \tfrac{1}{2} \|K(u) - K(a)\|^2$$

and defining an iteration through $u^{n+1} \in \operatorname{argmin}_u \Phi(u, u^n)$. It is easy to see that this produces decreasing functional values but as shown in [23] one has to solve a "fixed-point problem" in each iteration. In the non-convex case this fixed-point equation becomes an inclusion with a discontinuous operator and hence, there is no guarantee for convergence. Hence, we will not pursue this direction but use the generalized gradient projection algorithm from [5] (see also [3, 7] in which a generalized conditional gradient method was used in the case of convex constraints and [4] for the case of Banach spaces).

The paper is organized as follows: Section 2 presents the generalized gradient projection method for the case of general functionals $S + \alpha R$. Section 3 treats the case $R(u) = \sum \phi(|u_k|)$ and in Section 4 we specialize the results further to the case $\phi(x) = x^p$ with $0 < p \leq 1$ and $S(u) = \|Ku - g\|^2/2$ with a bounded linear operator K. Section 5 presents numerical experiments and Section 6 summarizes and concludes the paper.

2 The generalized gradient projection method

We first consider the general case of minimizing

$$T_\alpha(u) = S(u) + \alpha R(u).$$

The generalized gradient projection algorithm builds on the gradient projection algorithm for constrained minimization problems

$$\min_{u \in C} S(u),$$

where C is a usually a non-empty, convex and closed set incorporating the constraints. In this case, the method calculates iterates according to $u^{n+1} = P_C(u^n - s_n S'(u^n))$ where P_C denotes the projection onto C and s_n is a properly chosen step-size (cf. [10, 11]). The main idea of the generalized gradient projection algorithm for the solution of (1) is to replace the convex constraint C by a general functional R and to replace the projection P_C by the associated proximity operator, i.e.

$$J_s : u \mapsto \operatorname*{argmin}_{v \in X} \frac{\|u - v\|^2}{2} + s\alpha R(v). \tag{2}$$

While proximity operators are well-studied for convex functionals [24, 25], the non-convex case has been of interest to researchers only recently [15]. The motivation for the consideration of this minimization problem is that, in practice and as we will see later, it is much easier to solve than the original problem since it only involves the regularization term R. Hence, the generalized gradient projection algorithm reads as

$$\begin{cases} u^0 \in \operatorname{dom}(S + \alpha R) \quad , \quad u^{n+1} \in G_{\alpha, s_n}(u^n), \\ G_{\alpha, s_n}(u) = \operatorname*{argmin}_{v \in X} \dfrac{\|u - s_n S'(u) - v\|^2}{2} + s_n \alpha R(v). \end{cases} \tag{3}$$

In [5] the convergence of the generalized gradient projection method is worked out for the case of convex R and it is proved that the algorithm converges linearly in certain cases. In the non-convex case the operator J_s may be set-valued because there may be several global minima—moreover local minima may exist. However, if one is able to evaluate J_s, i.e. to calculate global minimizers of $\frac{1}{2}\|u - v\|^2 + s\alpha R(v)$, descent of the functional value is guaranteed as shown in the following proposition.

Proposition 2.1. *Let $S : X \to [0, \infty[$ be weakly lower-semicontinuous and differentiable with S' being Lipschitz continuous with constant $L > 0$. Furthermore, let $R : X \to [0, \infty]$ be weakly lower-semicontinuous and $\alpha > 0$. Then, J_s is non-empty for each $s > 0$ and it holds for every*

$$v \in G_{\alpha,s}(u)$$

that

$$T_\alpha(v) \leq T_\alpha(u) - \tfrac{1}{2}\big(\tfrac{1}{s} - L\big) \|v-u\|^2 . \tag{4}$$

Proof. Due to the assumptions, the functional in (2) is proper, coercive and weakly lower-semicontinuous for each $u \in X$ and hence, minimizers exist, in particular for the minimization problem in (3). Due to the minimizing property of v it holds that

$$\tfrac{1}{2}\|v-u+sS'(u)\|^2 + s\alpha R(v) \leq \tfrac{1}{2}\|sS'(u)\|^2 + s\alpha R(u)$$

which implies

$$\begin{aligned}\alpha R(v) - \alpha R(u) &\leq \frac{1}{2s}(\|sS'(u)\|^2 - \|v-u+sS'(u)\|^2) \\ &= -\langle S'(u)|\, v-u\rangle - \frac{1}{2s}\|v-u\|^2 .\end{aligned}$$

With this, it follows that

$$(S+\alpha R)(v) - (S+\alpha R)(u) \leq S(v) - S(u) - \langle S'(u)|\, v-u\rangle - \frac{1}{2s}\|v-u\|^2 . \tag{5}$$

Finally, we use the Lipschitz-continuity of S' and the inequality of Cauchy-Schwarz to show

$$\begin{aligned}S(v) - S(u) - \langle S'(u)|\, v-u\rangle &= \int_0^1 \langle S'(u-t(v-u)) - S'(u)|\, v-u\rangle \,\mathrm{d}t \\ &\leq \int_0^1 Lt\|v-u\|^2 \,\mathrm{d}t \\ &= \frac{L}{2}\|v-u\|^2\end{aligned}$$

which immediately implies the assertion. □

From this proposition we conclude that a step-size $0 < s < L^{-1}$ reduces the objective functional T_α. But since T_α is bounded from below we get that the sequence $(T_\alpha(u^n))$ converges. As a direct consequence we moreover have the following corollary.

Corollary 2.2. *In the situation of Proposition 2.1 and with a step-size sequence* (s_n) *satisfying* $0 < \underline{s} \leq s_n \leq \overline{s} < L^{-1}$ *for each* n*, the sequence* (u^n) *generated by the iteration* (3) *obeys*

$$\|u^{n+1} - u^n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{6}$$

Proof. The assertion follows from the estimate

$$\tfrac{1}{2}(\tfrac{1}{s_n} - L)\|u^{n+1} - u^n\|^2 \leq (S+\alpha R)(u^n) - (S+\alpha R)(u^{n+1})$$

and the observation that $(S+\alpha R)(u^n)$ is a converging sequence. □

In general, this does not lead to convergence, but if R is, for example, some power of a norm in a space which is compactly embedded in X, i.e.

$$R(u) = \begin{cases} \|u\|_Z^p & u \in Z \\ \infty & u \notin Z \end{cases}, \qquad Z \hookrightarrow X, \quad p > 0, \tag{7}$$

then from the boundedness of the sequence $(T_\alpha(u^n))$ follows that $(R(u^n))$ is bounded and hence (u^n) is precompact in X, admitting a convergent subsequence $u^{n_j} \to u^*$ as $j \to \infty$.

It is however, not clear whether the limit is a global solution (which is rather unlikely in general) or at least a point which is stationary in some sense. At the current stage, however, we can state the following.

Proposition 2.3. *For each global minimizer u^* of T_α and $0 < s < L^{-1}$, we have*

$$G_{\alpha,s}(u^*) = \{u^*\}.$$

In other words: Each minimizer is a fixed point of the generalized gradient projection method.

Moreover, if $0 < \underline{s} \leq s_n \leq \overline{s} < L^{-1}$, each convergent subsequence of (u^n) according to (3) converges, for some $s \in [\underline{s}, \overline{s}]$, to an element of the closed set

$$U_{\alpha,s} = \{u \in X \mid u \in G_{\alpha,s}(u)\}.$$

Proof. Choosing $v \in G_{\alpha,s}(u^*)$ and utilizing (4) implies

$$T_\alpha(u^*) \leq T_\alpha(v) \leq T_\alpha(u^*) - \Big(\frac{1}{2s} - \frac{L}{2}\Big) \|v - u^*\|^2 ,$$

hence $\|v - u^*\|^2 \leq 0$ and consequently, $v = u^*$.

For the remainder, first examine, for $0 < s < L^{-1}$, the functional

$$\Psi_{\alpha,s}(u) = \frac{\|sS'(u)\|^2}{2} + s\alpha R(u) - \Big(\min_{v \in X} \frac{\|u - sS'(u) - v\|^2}{2} + s\alpha R(v)\Big)$$

which satisfies $\Psi_{\alpha,s} \geq 0$ and $\Psi_{\alpha,s}(u) = 0$ if and only if $u \in G_{\alpha,s}(u)$ or, equivalently, $u \in U_{\alpha,s}$.

We will show that $\Psi_{\alpha,s}$ is lower semi-continuous with respect to both u and s which in particular implies that $U_{\alpha,s}$ is closed. For this purpose, choose a (u^n) with $u^n \to u$, $(s_n) \subset \,]0, L^{-1}[$ such that $s_n \to s$ with $0 < s < L^{-1}$ and fix a $v \in X$ with $R(v) < \infty$. Computations lead to

$$\begin{aligned}\Psi_{\alpha,s_n}(u^n) \geq{}& \frac{\|s_n S'(u^n)\|^2 - \|sS'(u)\|^2}{2} + \frac{\|sS'(u)\|^2 - \|u^n - s_n S'(u^n) - v\|^2}{2} \\ &+ s_n\alpha\big(R(u^n) - R(v)\big)\end{aligned}$$

and by continuity of S' as well as lower semi-continuity of R follows

$$\liminf_{n\to\infty} \Psi_{\alpha,s_n}(u^n) \geq \frac{\|sS'(u)\|^2 - \|u - sS'(u) - v\|^2}{2} + s\alpha\big(R(u) - R(v)\big).$$

This holds true for each $v \in X$ such that $R(v) < \infty$, hence

$$\begin{aligned}\liminf_{n\to\infty} \Psi_{\alpha,s}(u^n) &\geq \frac{\|sS'(u)\|^2}{2} + s\alpha R(u) + \sup_{v\in X}\Big(-\frac{\|u - sS'(u) - v\|^2}{2} - s\alpha R(v)\Big) \\ &= \Psi_{\alpha,s}(u).\end{aligned}$$

Next, suppose that a subsequence of the iterates converges, i.e. $u^{n_j} \to u$ as $j \to \infty$ for some $u \in X$. Note that the step-sizes (s_{n_j}) associated with the subsequence (u_{n_j}) are contained in the compact set $[\underline{s}, \overline{s}]$, hence, one can say without loss of generality, that $s_{n_j} \to s$ for some $s \in [\underline{s}, \overline{s}]$. For each j, one easily sees the identity

$$\alpha R(u^{n_j+1}) - \alpha R(u^{n_j}) = -\frac{\Psi_{\alpha,s_{n_j}}(u^{n_j})}{s_{n_j}} - \left\langle S'(u^{n_j}) \middle| \, u^{n_j+1} - u^{n_j} \right\rangle$$
$$- \frac{1}{2s_{n_j}} \left\| u^{n_j+1} - u^{n_j} \right\|^2 .$$

As we already concluded in the proof of Proposition 2.1, this leads to

$$T_\alpha(u^{n_j+1}) - T_\alpha(u^{n_j}) \le -\frac{\Psi_{\alpha,s_{n_j}}(u^{n_j})}{s_{n_j}} - \Big(\frac{1}{2s_{n_j}} - \frac{L}{2}\Big) \left\| u^{n_j+1} - u^{n_j} \right\|^2$$

implying

$$\Psi_{\alpha,s_{n_j}}(u^{n_j}) \le s_{n_j}\big(T_\alpha(u^{n_j}) - T_\alpha(u^{n_j+1})\big).$$

Since the right-hand side tends to zero as $j \to \infty$ it follows, by lower semi-continuity, that $\Psi_{\alpha,s}(u) = 0$ and hence $u \in U_{\alpha,s}$. □

The sets $U_{\alpha,s}$ in the latter proposition describe, in some sense, the fixed points of the iteration, hence one can say that the generalized gradient projection method converges subsequentially, if it converges, to a fixed point. The main objective of this article is to show a framework in which the iteration (3) is computable in the non-convex setting and in which weaker conditions than the compactness stated in (7) lead to convergence to such a fixed point. We therefore examine, in the following, separable non-convex regularization terms in the sequence space ℓ^2. Due to the special properties these constraints have, it is possible to obtain convergence for ℓ^p-regularization with $0 < p < 1$, for example, without a (strong) compactness assumption.

3 Application to separable constraints

Before turning directly to sparsity constraints in terms of ℓ^p (quasi-)norms we study the application of (3) in $X = \ell^2$ for separable constraints of the form:

$$R(u) = \sum_{k=1}^{\infty} \phi(|u_k|) \tag{8}$$

Throughout this section, we assume the following on ϕ.

Assumption 3.1. Let $\phi : [0, \infty[\, \to [0, \infty[$ be continuous with $\phi(0) = 0$ and $\phi(x) \to \infty$ whenever $x \to \infty$. It is moreover assumed that for each $b > 0$ there exists an $a > 0$ such that $\phi(x) \ge ax^2$ for $x \in [0, b]$.

Note that this ensures coercivity and weak sequential lower-semicontinuity in ℓ^2, see Lemmas 3.2 and 3.4, respectively, in [6]. Furthermore, we usually assume that $S : \ell^2 \to [0, \infty[$ is weakly sequentially lower semi-continuous and differentiable with Lipschitz continuous derivative whose constant is $L > 0$ (this only excludes trivial problems).

Let us take a closer look at the generalized gradient projection method (3). Assuming that $S'(u)$ is computationally accessible, the main problem is the evaluation of the proximity operator (as it is also pointed out in [15], where similar, more general considerations can be found). But fortunately, as one can easily see, in the case of separable constraints, computing J_s is reduced to the solution of the one-dimensional minimization problem

$$j_s(x) \in \operatorname*{argmin}_y \tfrac{1}{2}(y-x)^2 + s\alpha\phi(|y|)$$

for $x \in \mathbf{R}$. In fact, by symmetry, we can moreover restrict ourselves to the case $x \geq 0$. Knowing the (generally multi-valued) mapping j_s, iteration (3) amounts to

$$\begin{cases} u^0 \text{ such that } \sum_{k=1}^{\infty} \phi(|u_k^0|) < \infty, \\ u_k^{n+1} \in J_s\big(u^n - s_n S'(u^n)\big)_k = j_{s_n}\big(u_k^n - s_n S'(u^n)_k\big). \end{cases} \tag{9}$$

The following is concerned with the convergence analysis of this iteration. As it will turn out, subsequential convergence to a stationary point can be ensured under sufficiently general conditions. In addition to that, some necessary properties of the sought global minimizers of (1) are derived and the algorithm is adapted such that it converges to points where these conditions are met.

We start with an observation on j_s which resembles a result from [20].

Lemma 3.2. *Let ϕ be non-decreasing on $[0,\infty[$ as well as differentiable on $]0,\infty[$ with $\phi'(x) \to \infty$ for $x \to 0$. Then there exists a $\kappa_s > 0$ which depends monotonically increasing on s such that*

$$|x| < \kappa_s \implies j_s(x) = 0.$$

Proof. By definition of j_s we have, for $j_s(x) \neq 0$,

$$x = j_s(x) + s\alpha\phi'\big(j_s(x)\big) = (I + s\alpha\phi')\big(j_s(x)\big).$$

Without loss of generality, consider $x > 0$. Since $\phi' \geq 0$ and $\phi'(y)$ tends to infinity for $y \to 0$ the value $\kappa_s = \inf_{y>0}\ y + s\alpha\phi'(y)$ is positive meaning that for all $y > 0$ we have $(I + s\alpha\phi')(y) \geq \kappa_s$ which proves the claim. Moreover, from the definition of κ_s that it depends monotonically increasing on s. ∎

We give properties of the iterates of the generalized gradient projection method. The crucial observation is that due to the fact that j_s maps either to 0 or to a value with modulus greater or equal to κ_s, so a change of support always implies a "jump" of size κ_s. We are interested in examining what a change of support implies for the functional descent. This is closely connected to the following property of S.

Assumption 3.3. Let $S : \ell^2 \to [0,\infty[$ be continuously differentiable with Lipschitz constant L. Assume there is an orthogonal projection P onto a finite-dimensional space and an $L^* < L$ such that with $Q = I - P$ the estimate

$$\|S'(u) - S'(v)\|^2 \leq L^2 \|P(u-v)\|^2 + (L^*)^2 \|Q(u-v)\|^2 \tag{10}$$

is satisfied for each $u, v \in \ell^2$.

Such an assumption leads to the following refinement of Proposition 2.1 which also gives functional descent for a step-size $s = L^{-1}$.

Proposition 3.4. *Suppose that S satisfies Assumption 3.3 and let $0 < s \le L^{-1}$.*

Then, for each $u \in \ell^2$ and $v \in G_{\alpha,s}(u)$, the functional descent of T_α can be estimated by

$$T_\alpha(v) \le T_\alpha(u) - \frac{1}{2}\Big(\frac{1}{s} - L\Big) \|P(v-u)\|^2 - \frac{1}{2}\frac{sL^* + 1}{sL + 1}\Big(\frac{1}{s} - L^*\Big) \|Q(v-u)\|^2 . \quad (11)$$

Proof. For the verification of the claimed estimate, we can restrict ourselves to the case $u \neq v$. In analogy to the proof of Proposition 2.1, one gets to the intermediate step (5) which can be further estimated, using Cauchy-Schwarz as well as (10), according to

$$\begin{aligned} T_\alpha(v) - T_\alpha(u) &\le \int_0^1 \big\langle S'\big(u + t(v-u)\big) - S'(u) \big|\, v - u\big\rangle \,\mathrm{d}t - \frac{\|v-u\|^2}{2s} \\ &\le \int_0^1 t\big(L^2 \|P(v-u)\|^2 + (L^*)^2 \|Q(v-u)\|^2\big)^{1/2} \,\mathrm{d}t\, \|v-u\| - \frac{\|v-u\|^2}{2s} \\ &\le \frac{1}{2}\Big(\big(L^2 \|P(v-u)\|^2 + (L^*)^2 \big\|Q(v-u)^2\big\|\big)^{1/2} - \frac{\|v-u\|}{s}\Big) \|v-u\| . \end{aligned} \quad (12)$$

Note that $L^* \le L \le 1/s$, so the difference on the right hand side is actually non-positive. By writing $\sqrt{a} - \sqrt{b} = (a-b)/(\sqrt{a} + \sqrt{b})$ and estimating it becomes

$$\begin{aligned} &\big(L^2 \|P(v-u)\|^2 + (L^*)^2 \|Q(v-u)\|^2\big)^{1/2} - \frac{\|v-u\|}{s} \\ &\qquad \le \Big(L + \frac{1}{s}\Big)^{-1} \frac{\big(L^2 - s^{-2}\big) \|P(v-u)\|^2 + \big((L^*)^2 - s^{-2}\big) \|Q(v-u)\|^2}{\|v-u\|} . \end{aligned} \quad (13)$$

Combining (12) and (13), rewriting $(L^*)^2 - s^{-2} = (L^* + s^{-1})(L^* - s^{-1})$ and expanding finally gives (11). □

In the context of Assumption 3.3, we also observe the following.

Lemma 3.5. *Let $P : \ell^2 \to \ell^2$ be an orthogonal projection onto a finite-dimensional space and $Q = I - P$. Then, for each $0 < \varepsilon < 1$ there exists a k_0 such that for the truncation operator $(M_{k_0}u)_k = 0$ if $k \le k_0$ and $(M_{k_0}u)_k = u_k$ otherwise holds*

$$\|M_{k_0}Pu\| \le \varepsilon \|Pu\| \qquad , \qquad (1-\varepsilon)\|M_{k_0}u\| \le \|QM_{k_0}u\|$$

for all $u \in \ell^2$.

Proof. With an orthonormal basis $z_1, \ldots, z_m$ of range P, we can write

$$M_{k_0}Pu = \sum_{j=1}^m \langle z_j |\, Pu\rangle\, M_{k_0}z_j \quad \Rightarrow \quad \|M_{k_0}Pu\| \le \Big(\sum_{j=1}^m \|M_{k_0}z_j\|\Big) \|Pu\| ,$$

so choosing k_0 large enough that the sum on the right-hand side does not exceed ε yields $\|M_{k_0}Pu\| \le \varepsilon \|Pu\|$. Likewise,

$$PM_{k_0}u = \sum_{j=1}^m \langle M_{k_0}z_j |\, M_{k_0}u\rangle\, z_j \quad \Rightarrow \quad \|PM_{k_0}u\| \le \Big(\sum_{j=1}^m \|M_{k_0}z_j\|\Big) \|M_{k_0}u\|$$

which implies

$$\|QM_{k_0}u\| \geq \|M_{k_0}u\| - \|PM_{k_0}u\| \geq (1-\varepsilon)\,\|M_{k_0}u\|\,.$$

□

Lemma 3.2 together with Proposition 3.4 implies that the sequence generated by (9) eventually does not change its support from some iterate on.

Lemma 3.6. *Let ϕ fulfill the assumption of Lemma 3.2 and let R be according to* (8). *Let furthermore (u^n) be generated by iteration* (9) *and the step-size sequence (s_n) satisfy*

$$0 < \underline{s} \leq s_n \leq \overline{s} < L^{-1}.$$

Then all iterates u^n $(n \geq 1)$ have a finite support and the support only changes finitely many times. In case Assumption 3.3 is satisfied and $\overline{s} = L^{-1}$, the supports of all u^n for $n \geq 1$ are still contained in a finite set.

Proof. First assume that $\overline{s} < L^{-1}$. Due to Lemma 3.2 there exists a $\kappa_{\underline{s}} > 0$ such that $|x| < \kappa_{\underline{s}}$ implies $j_{s_n}(x) = 0$ for all n, the latter since $s_n \geq \underline{s} > 0$ and κ_s depends monotonically increasing on s. Hence, each non-zero component of u^n has magnitude greater or equal to $\kappa_{\underline{s}}$. We conclude that if the support of u^{n+1} differs from u^n, we have $\|u^{n+1} - u^n\| \geq \kappa_{\underline{s}}$. But from Proposition 2.1 we deduce that also

$$\Big(\frac{1}{s_n} - L\Big)\frac{\kappa_{\underline{s}}^2}{2} \leq T_\alpha(u^n) - T_\alpha(u^{n+1})$$

with the right-hand side going to zero as $n \to \infty$. Hence, a change of the support can only occur a finite number of times.

In case $\overline{s} = L^{-1}$ and when Assumption 3.3 is satisfied, first assume that $\|Q(u^{n+1} - u^n)\|$ does not converge to 0, meaning that there exists a $c > 0$ such that $\|Q(u^{n+1} - u^n)\| \geq c$ for infinitely many n. For these n, Proposition 3.4 yields

$$\frac{c}{2}\frac{s_nL^* + 1}{s_nL + 1}\Big(\frac{1}{s_n} - L^*\Big) \leq T_\alpha(u^n) - T_\alpha(u^{n+1})$$

which is a contradiction since the right-hand side tends to zero as $n \to 0$ while the left-hand side is bounded away from zero. Hence, $\|Q(u^{n+1} - u^n)\| \to 0$.

Next, we choose k_0 according to Lemma 3.5 (with $\varepsilon = 1/2$, for instance) and consider the mappings QM_{k_0} as well as $QM_{k_0}^\perp = Q(I - M_{k_0})$. Note that $\operatorname{range} QM_{k_0} \cap \operatorname{range} QM_{k_0}^\perp$ is a finite-dimensional space on which the pseudo-inverse $(QM_{k_0})^\dagger$ is linear and continuous. Denoting by Z the projection on this space and by $v^n = u^{n+1} - u^n$ we have

$$Qv^n = \underbrace{QM_{k_0}v^n + ZQM_{k_0}^\perp v^n}_{=w^n} + (I - Z)QM_{k_0}^\perp v^n.$$

It follows that $w^n \to 0$ as $n \to \infty$ since

$$\operatorname{range} Q = (\operatorname{range} QM_{k_0} + \operatorname{range} ZQM_{k_0}^\perp) \oplus \operatorname{range}(I - Z)QM_{k_0}^\perp$$

by the construction of Z. We conclude, by Lemma 3.5, that

$$M_{k_0}v^n + \underbrace{M_{k_0}(QM_{k_0})^\dagger ZQM_{k_0}^\perp v^n}_{=x^n} \to 0$$

as $n \to \infty$. Since (u^n) is bounded, (x^n) is bounded and, moreover, contained in a finite-dimensional subspace of ℓ^2, hence according to Lemma 3.5 we can achieve for arbitrary $\varepsilon > 0$ that $\|M_{k_1} x^n\| \leq \varepsilon/2$ for each n, by simply choosing $k_1 \geq k_0$ large enough. Also, $\|M_{k_1} v^n\| \leq \|M_{k_0} v^n + x^n\| + \|M_{k_1} x^n\|$ so by choosing n_0 suitably it follows $\|M_{k_0} v^n + x^n\| \leq \varepsilon/2$ for $n \geq n_0$ and hence $\|M_{k_1} v^n\| \leq \varepsilon$ for these n.

Letting $0 < \varepsilon < \kappa_{\underline{s}}$ eventually allows us to conclude that $M_{k_1} u^n$ has fixed support for $n \geq n_0$ since the opposite would imply that $\|M_{k_1} v^n\| \geq \kappa_{\underline{s}}$ for some $n \geq n_0$ which contradicts the above. Consequently, all supports of u^n for $n \geq 1$ are contained in a finite set. □

By the above lemma we have the existence of a strong accumulation point of the sequence (u^n):

Corollary 3.7. *Every subsequence of (u^n) has a strong accumulation point u^*. In the case $\overline{s} < L^{-1}$, this accumulation point is a fixed point in the sense $u^* \in G_{\alpha,s}(u^*)$ for some $s \in [\underline{s}, \overline{s}]$.*

Proof. By assumption, T_α is coercive in ℓ^2 and hence, there is a subsequence (u^{n_j}) which has a weak accumulation point u^* in ℓ^2. By Lemma 3.6 there is an iteration index j_0 and a finite set $J \subset \mathbf{N}$ such that $u_k^{n_j} = 0$ for $j \geq j_0$ and $k \in J$. Hence, we have for the finitely many $k \in J$ that $u_k^{n_j} \to u_k^*$ as $j \to \infty$ and infinitely often $u_k^{n_j} = u_k^* = 0$ (for $j > j_0$). Finally, we conclude that u^{n_j} converges strongly to u^*. The above argumentation holds true for every subsequence of (u^n).

Furthermore, if $\overline{s} < L^{-1}$, one can apply Proposition 2.3 to get that $u^* \in G_{\alpha,s}(u^*)$ for some $s \in [\underline{s}, \overline{s}]$. □

Note that similar arguments have been used in [2] for $\phi = \chi_{\mathbf{R}\setminus\{0\}}$ in the finite dimensional case. Hence, the generalized gradient projection method converges, subsequentially, to a fixed point. In general, we do not know whether this fixed point is a global minimizer, it is no even clear if it still is a local minimizer. One can, however, derive necessary conditions for the global minimizer and make sure that the (subsequential) limits of the algorithm converge to points where these conditions are met. Such an approach is carried out in the following. It will turn out that one actually has to take care of the step-size sequence (s_n): As we will see, it is essential that they converge to L^{-1} on the one hand. On the other hand, one still has to ensure convergence of the algorithm. These requirements call for a further analysis of the situation. But first, let us summarize the result on convergence for step-sizes not approaching L^{-1}.

Theorem 3.8. *Let $S : X \to [0, \infty[$ be continuously differentiable with Lipschitz continuous derivative (with constant L) and $0 < \underline{s} \leq s_n \leq \overline{s} < L^{-1}$. Furthermore, let R be according to (8) with a ϕ satisfying Assumption 3.1 as well as ϕ being non-decreasing and continuously differentiable on $]0, \infty[$ with $\phi'(x) \to \infty$ for $x \to 0$.*

Then, the sequence (u^n) according to (3) has a strong accumulation point. Each accumulation point is a fixed-point of $G_{\alpha,s}$ for some $s \in [\underline{s}, \overline{s}]$. If, additionally, there exists an isolated accumulation point u^*, then the whole sequence converges to the fixed point u^*.

Proof. One can easily convince oneself of the validity of the prerequisites for Lemma 3.6 and Corollary 3.7 which gives a strong accumulation point u for each subsequence of (u^n) which is moreover a fixed point of $G_{\alpha,s}$ for some $s \in [\underline{s}, \overline{s}]$.

Now suppose that u^* is an isolated accumulation point for (u^n), i.e. there is a $\delta > 0$ such that $\|u^* - u\| \geq \delta$ for each accumulation point $u \neq u^*$. Assume that not the whole sequence converges to u^*, i.e. there exists a $0 < \varepsilon < \delta$ and infinitely many n with $\|u^n - u^*\| \geq \varepsilon$. Denote by N the set of these n. Since there exists a subsequence of (u^n) with limit u^* we can find infinitely many n with $n \in N$ and $n+1 \notin N$. Denote by (u^{n_j}) the subsequence associated with these n. By construction $\|u^{n_j+1} - u^*\| \leq \varepsilon < \delta$, so $u^{n_j+1} \to u^*$ and since $\|u^{n_j} - u^{n_j+1}\| \to 0$, also $u^{n_j} \to u^*$. Because of $n_j \in N$, this is a contradiction, hence the assumption that not the whole sequence converged must have been wrong. □

We now proceed with the case where the step-size sequence (s_n) approaches L^{-1}. In order to state necessary conditions for global minimizers, let us first derive sufficient conditions for the continuity of the j_s outside of the "dead zone" $\{x \in \mathbf{R} \mid j_s(x) = 0\}$. For that purpose, we introduce some additional assumptions on ϕ, which are supposed to the fulfilled (in addition to Assumption 3.1) in the remainder of this section.

Assumption 3.9. Let $\phi : [0, \infty[\to [0, \infty[$ continuous with $\phi(0) = 0$ satisfy the conditions

(a) the derivative ϕ' is strictly convex on $]0, \infty[$ with $\phi'(x) \to \infty$ for $x \to 0$ and $\phi'(x)/x \to 0$ for $x \to \infty$,

(b) some single-valued selection of $x \mapsto \partial\phi'(x)x$ is locally integrable on $[0, \infty[$.

We need a preparatory lemma for proving continuity of the j_s.

Lemma 3.10. *Assumption 3.9 implies the following properties:*

(a) for each $s > 0$, the function $\rho_s : y \mapsto y + s\alpha\phi'(y)$ exists on $\mathbf{R}_+$, is strictly convex and attains a minimum at $y_s > 0$,

(b) the function $\psi : y \mapsto 2\big(\phi(y) - y\phi'(y)\big)/y^2$ is strictly decreasing and one-to-one on $]0, \infty[\to]0, \infty[$,

(c) we have, for $y > 0$ and any $z \in \partial\phi'(y)$, that $\psi(y) > -z$.

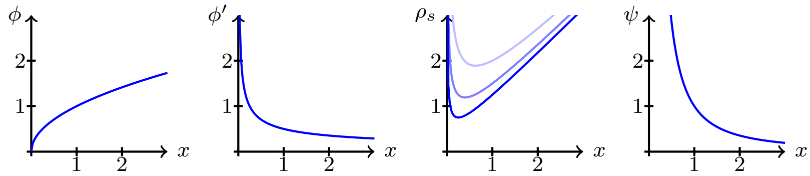
Proof. Since it is convex, the function ϕ' is continuous. Moreover, ϕ' has to satisfy $\lim_{y\to\infty} \phi'(y) \geq 0$ (with ∞ allowed) since otherwise

$$\lim_{y\to\infty} \phi(y) = \lim_{y\to\infty} \int_0^y \phi'(t) \, \mathrm{d}t \to -\infty$$

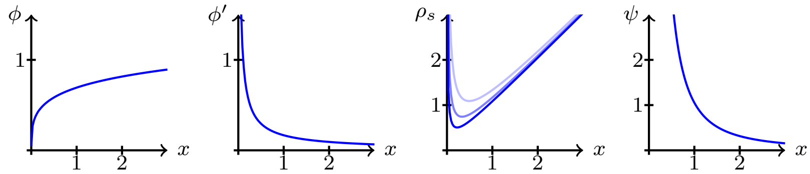
which contradicts $\phi \geq 0$. Thus, each ρ_s is a strictly convex function with $\rho_s(x) \to \infty$ for $y \to 0$ and $y \to \infty$ which implies that a unique minimizer $y_s > 0$ exists.

Next, note that $\phi''(y) = \partial\phi'(y)$ for almost every $y > 0$. So, we can use Taylor expansion of ϕ in y to get

$$\phi(y) - y\phi'(y) = -\int_0^y t\partial\phi'(t) \, \mathrm{d}t.$$



$\phi(x) = \sqrt{x}$



$\phi(x) = \log(x^{1/3} + 1)$

Figure 1: Illustration of some ϕ satisfying Assumption 3.9, ϕ', ρ_s ($\alpha = 1$, $s \in \{1/2, 1, 2\}$) and ψ.

Since $\partial\phi'(t)$ is strictly monotonically increasing, for $y > 0$ and any $z \in \partial\phi'(y)$,

$$-\int_0^y t\partial\phi'(t)\,\mathrm{d}t > -z\int_0^y t\,\mathrm{d}t = -\frac{zy^2}{2}.$$

This already proves $\psi(y) > -z$. For ψ being strictly decreasing, consider $y > 0$ where $\partial\phi'(y) = \phi''(y)$ and deduce

$$\psi'(y) = \frac{2}{y}\Big(\frac{2}{y^2}\int_0^y t\partial\phi'(t)\,\mathrm{d}t - \phi''(y)\Big) < 0 \quad \Leftrightarrow \quad \psi(y) > -\phi''(y),$$

where the latter has already been established. Hence, $\psi'(y) < 0$ almost everywhere and, consequently, ψ is strictly monotonically decreasing. Moreover, $\psi(y) > -\phi''(y)$ also implies $\psi(y) \to \infty$ as $y \to 0$ since $\phi'(y)$ is bounded around 0 whenever $\phi''(y)$ is bounded around 0. Finally, from the assumption $\phi'(y)/y \to 0$ as $y \to \infty$ follows that $\phi(y)/y^2 \to 0$: For each $\varepsilon > 0$ there is a y_0 such that for $y \geq y_0$ we have $|\phi'(y)/y| \leq \varepsilon$ and hence

$$\phi(y) - \phi(y_0) = \int_{y_0}^y t\phi'(t)/t\,\mathrm{d}t \leq \frac{\varepsilon}{2}(y^2 - y_0^2) \quad \Rightarrow \quad \frac{\phi(y)}{y^2} \leq \frac{\phi(y_0)}{y^2} + \frac{\varepsilon}{2}.$$

One can choose a y_1 such that $\phi(y_0)/y^2 < \varepsilon/2$ for $y \geq y_1$, so for y large enough we have $\phi(y)/y^2 < \varepsilon$. Consequently, $\psi(y) \to 0$ as $y \to \infty$ and, together with the above, $\psi : [0, \infty[\to [0, \infty[$ is one-to-one. $\square$

Example 3.11. For $p \in]0, 1[$, it can be verified that the functions $\phi(x) = x^p$ as well as $\phi(x) = \log(x^p + 1)$ satisfy Assumption 3.9. In Figure 1, you can see an illustration of the constructions in Lemma 3.10.

The continuity properties can now easily be deduced.

Lemma 3.12. *The functions j_s obey*

$$j_s(x) = \begin{cases} 0 & \text{for } |x| \leq \tau_s \\ \operatorname{sign}(x)\rho_s^{-1}(|x|) & \text{for } |x| \geq \tau_s, \end{cases}$$
$$|j_s(x)| \in \{0\} \cup \{x \geq \lambda_s\}$$

where

$$\lambda_s = \psi^{-1}\big((s\alpha)^{-1}\big), \quad \tau_s = \rho_s \circ \psi^{-1}\big((s\alpha)^{-1}\big). \tag{14}$$

In particular, j_s takes two values if $|x| = \tau_s$, is left- and right continuous there (in the sense that $j_s(\pm\tau)$ are exactly the left- and right limits) as well as continuous on $\{|x| \neq \tau_s\}$.

Proof. Let $s > 0$ be fixed and denote by $F_x(y) = \frac{1}{2}(y-x)^2 + s\alpha\phi(|y|)$. Furthermore, assume, without loss of generality, $x \geq 0$ such that we only need to minimize F_x over $[0, \infty[$. Note the identities $F_x' = \rho_s - x$ and $F'' = \rho_s'$. From Lemma 3.10 we know that there is a unique minimizer $y_s > 0$ of ρ_s. Consequently, $F_x''(y) = \rho_s'(y) < 0$ for $y < y_s$ meaning that local minimizers $y^* \neq 0$ of F_x obey $y^* \geq y_s$.

Thus, we can conclude that $j_s(x) = 0$ whenever $x < \rho_s(y_s)$ since no $y^* > 0$ exists for which these necessary conditions are satisfied. On the other hand, if $x \geq \rho_s(y_s)$, a unique $y^* \geq y_s$ with $\rho_s(y^*) = x \Leftrightarrow F_x'(y^*) = 0$ exists and we just have to compare the values of $F_x(0)$ and $F_x(y^*)$ in order to determine the minimizers. It turns out that

$$\begin{aligned} F_x(y^*) - F_x(0) &= \frac{(s\alpha\phi'(y^*))^2}{2} + s\alpha\phi(y^*) - \frac{(y^* + s\alpha\phi'(y^*))^2}{2} \\ &= s\alpha\big(\phi(y^*) - y^*\phi'(y^*)\big) - \frac{(y^*)^2}{2} \end{aligned}$$

which leads to

$$\begin{aligned} \operatorname{sign}\big(F_x(y^*) - F_x(0)\big) &= \operatorname{sign}\big(\psi(y^*) - (s\alpha)^{-1}\big) = \operatorname{sign}\big(\psi^{-1}((s\alpha)^{-1}) - y^*\big) \\ &= \operatorname{sign}(\lambda_s - y^*) \end{aligned}$$

the latter since ψ is invertible and strictly monotonically decreasing, see Lemma 3.10 (b). Finally, ρ_s is also strictly monotonically increasing on $\{y \geq y_s\}$ and $x = \rho_s(y^*)$, hence

$$\operatorname{sign}\big(F_x(y^*) - F_x(0)\big) = \operatorname{sign}\big(\rho_s(\lambda_s) - x\big) = \operatorname{sign}(\tau_s - x).$$

Note that necessarily $\tau_s \geq \rho_s(y_s)$, hence $j_s(x) = 0$ for all $x < \tau_s$, $j_s(x) = \rho_s^{-1}(x)$ for $x > \tau_s$. For $x = \tau_s$, there are the two minimizers 0 and λ_s.

Finally observe that ρ_s can be continuously inverted on $[y_0, \infty[$, hence j_s is continuous on $[\tau_s, \infty[$ with the left limit in τ_s being λ_s. The claimed continuity on $[0, \tau_s]$ is trivial and it is easy to see that $\operatorname{range}(j_s) = \{0\} \cup \{|x| \geq \lambda_s\}$. □

Remark 3.13. We remark that the threshold τ_s is always greater than the minimum of ρ_s: Since ρ_s is strictly convex, the minimizer y_s satisfies $z = -(s\alpha)^{-1} \in \partial\phi'(y_s)$. According to the definition, $(s\alpha)^{-1} = \psi(\lambda_s)$, so due to Lemma 3.10 (c) we have $\psi(y_s) > -z = \psi(\lambda_s)$ and hence $y_s < \lambda_s$. Since ρ_s is strictly monotonically increasing on $[y_s, \infty[$, $\min_{y>0} \rho_s(y) = \rho_s(y_s) = \kappa_s < \tau_s$ follows. Moreover, note that in particular we have that $\partial\rho_s(\tau_s)$ does not contain 0.

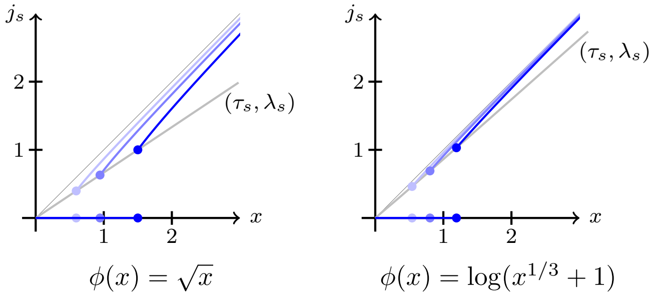


Figure 2: Illustration of j_s for some particular ϕ and $s \in \{1/4, 1/2, 1\}$ ($\alpha = 1$). Additionally, the curve of breaking points (τ_s, λ_s) is depicted.

You can find illustrations of the functions j_s for some particular ϕ in Figure 2. The thresholds (14) for $s = L^{-1}$ play an important role for global optimality as they occur in the necessary conditions.

Proposition 3.14. *For $\lambda_{L^{-1}}$ and $\tau_{L^{-1}}$ according to* (14) *each global minimizer u^* of T_α possesses the following properties:*

(a) The set $J = \{k \in \mathbf{N} \mid u_k^* \neq 0\}$ is finite,

(b) for all $k \in J$ it holds that $-S'(u^*)_k = \alpha \operatorname{sgn}(u_k^*)\phi'(|u_k^*|)$ and $|u_k^*| \geq \lambda_{L^{-1}}$,

(c) For all $k \notin J$ it holds that $|S'(u^*)_k| \leq L\tau_{L^{-1}}$.

Proof. Suppose $u^* \in \ell^2$, $u^* \neq 0$ is a global minimizer of T_α. Obviously, $-S'(u^*) \in \ell^2$. Exploiting the condition that $\lim_{x\downarrow 0} \phi'(x) = \infty$, we are able to find an $\varepsilon > 0$ such that $|\phi'(x)| \geq 1$ for $|x| \leq \varepsilon$ and $x \neq 0$. Take a k for which $|u_k^*| \leq \varepsilon$ and $u_k^* \neq 0$, differentiate with respect to that component to get

$$S'(u^*)_k + \alpha \operatorname{sgn}(u_k^*)\phi'(|u_k^*|) = 0 \quad \Rightarrow \quad |S'(u^*)|_k = \alpha\phi'(|u_k^*|) \geq \alpha.$$

Consequently, $\alpha^{-1} \|S'(u^*)\| \geq \#\{|u_k^*| \leq \varepsilon \wedge u_k^* \neq 0\}$ and since the norm is finite as well as only finitely many u_k^* can satisfy $|u_k^*| > \varepsilon$, $J = \{u_k^* \neq 0\}$ has to be finite. This proves the first assertion as well as the first part of the second.

Next, we will show that if $u_k^* \neq 0$ and $|u_k^*| < \lambda_{L^{-1}}$ for some k, setting u_k^* to zero strictly decreases the functional value of T_α and hence, u^* was not a global minimizer. Let $v_k^* = 0$ and $v_l^* = u_l^*$ for $l \neq k$ such that $v^* - u^* = -u_k^* e_k$. Expand S at u^* and compare $T_\alpha(v^*)$ with $T_\alpha(u^*)$:

$$\begin{aligned} T_\alpha(u^*) - T_\alpha(v^*) = &- \langle S'(u^*) | \, v^* - u^* \rangle \\ &- \int_0^1 \big\langle S'\big(u^* + t(v^* - u^*)\big) - S'(u^*) \big| \, v^* - u^* \big\rangle \, \mathrm{d}t \\ &+ \alpha \sum_{l=1}^\infty \phi(|u_l^*|) - \alpha \sum_{l \neq k} \phi(|u_l^*|) \\ \geq{}& u_k^* S'(u^*)_k - L\frac{(u_k^*)^2}{2} + \alpha\phi(|u_k^*|) \\ ={}& \frac{\alpha(u_k^*)^2}{2}\Big(\frac{2\big(\phi(|u_k^*|) - |u_k^*|\phi'(|u_k^*|)\big)}{(u_k^*)^2} - (L^{-1}\alpha)^{-1}\Big) \end{aligned}$$

remembering that $u_k^* \neq 0$ implies $S'(u^*)_k = -\alpha \operatorname{sgn}(u_k^*)\phi'(|u_k^*|)$. The term on the right-hand side is positive if and only if $|u_k^*| < \psi^{-1}\big((L^{-1}\alpha)^{-1}\big) = \lambda_{L^{-1}}$ (also confer Lemma 3.10), thus $|u_k^*| < \lambda_{L^{-1}}$ implies $T_\alpha(v^*) < T_\alpha(u^*)$. Hence, for u^* being a global minimizer, it is necessary that $|u_k^*| \geq \lambda_{L^{-1}}$ for each $k \in J$, concluding the proof of the second assertion.

For the remainder, we will utilize a similar line of argumentation. Take a $k \notin J$, i.e. $u_k^* = 0$ and denote by $w_k^* = S'(u^*)_k$. Our interest is in finding sufficient criteria which lead to a functional descent by just varying the k-th component. Let $v_l^* = u_l^*$ for $l \neq k$ and $v_k^* \in \mathbf{R}$. With this, we get, similarly to the above,

$$\begin{aligned} T_\alpha(u^*) - T_\alpha(v^*) &\geq w_k^* v_k^* - L\frac{(v_k^*)^2}{2} - \alpha\phi(|v_k^*|) \\ &= -L\Big(\frac{1}{2}\Big(v_k^* - \frac{w_k^*}{L}\Big)^2 + \frac{\alpha}{L}\phi(|v_k^*|)\Big) + \frac{(w_k^*)^2}{2L}. \end{aligned} \tag{15}$$

Maximizing the term on the right hand side with respect to v_k^* gives $v_k^* = j_{L^{-1}}\big(w_k^* L^{-1}\big)$ which is only non-zero if $|w_k^*| \geq L\tau_{L^{-1}}$, see Lemma 3.12. From $j_{L^{-1}}$ being the solution of a minimization problem we also know that

$$|w_k^*| > L\tau_{L^{-1}} \;\Rightarrow\; \frac{1}{2}\Big(v_k^* - \frac{w_k^*}{L}\Big)^2 + \frac{\alpha}{L}\phi'(|v_k^*|) < \frac{(w_k^*)^2}{2L^2}$$

and plugging this into (15) yields $T_\alpha(u^*) - T_\alpha(v^*) > 0$. Consequently, if $|w_k^*| > L\tau_{L^{-1}}$ for a $u_k^* = 0$ then u^* was not a global minimizer what was to show. □

The properties (a)–(c) tell us what is necessary for the generalized gradient projection method to be able to detect global minimizers.

Remark 3.15. In order to get consistent results, we use, in the sequel, a single-valued selection of $G_{\alpha,s}$:

$$G^+_{\alpha,s}(u)_k = \begin{cases} 0 & \text{if } u_k \neq 0, |u_k - sS'(u)_k| < \tau_s \\ 0 & \text{if } u_k = 0, |u_k - sS'(u)_k| \leq \tau_s \\ \rho_s^{-1}(u_k - sS'(u)_k) & \text{if } u_k \neq 0, |u_k - sS'(u)_k| \geq \tau_s \\ \rho_s^{-1}(u_k - sS'(u)_k) & \text{if } u_k = 0, |u_k - sS'(u)_k| > \tau_s \end{cases} \tag{16}$$

This rule basically tells in the ambiguous cases to select 0 whenever $u_k = 0$ and the non-zero value of j_s otherwise.

Proposition 3.16. *A global minimizer u^* for the Tikhonov functional T_α is a fixed point of $G^+_{\alpha,L^{-1}}$. The other way around, each fixed point of this mapping satisfies the properties (a)–(c) of Proposition 3.14.*

Proof. Let u^* be a minimizer and denote again by J the set of indices where $u_k^* \neq 0$. If $k \in J$, then $-L^{-1}S'(u^*)_k = L^{-1}\alpha \operatorname{sgn}(u_k^*)\phi'(|u_k^*|)$ as well as $|u_k^*| \geq \lambda_{L^{-1}}$, hence

$$\begin{aligned} \big|u_k^* - L^{-1}S'(u^*)_k\big| &= \big|u_k^* + L^{-1}\alpha \operatorname{sgn}(u_k^*)\phi'(|u_k^*|)\big| = \rho_{L^{-1}}\big(|u_k^*|\big) \\ &\geq \rho_{L^{-1}}(\lambda_{L^{-1}}) = \tau_{L^{-1}} \end{aligned}$$

by Proposition 3.14. Consequently, by the particular single-valued selection $v^* = G^+_{\alpha,L^{-1}}(u^*)$ according to (16), the corresponding $j_{L^{-1}}$ always yields the non-zero value, i.e.

$$v^*_k = \rho^{-1}_{L^{-1}}\big(u^*_k - L^{-1}S'(u^*)_k\big) = \rho^{-1}_{L^{-1}}\big(u^*_k + L^{-1}\alpha\,\mathrm{sign}(u^*_k)\phi'(|u^*_k|)\big) = u^*_k$$

by the definition of $\rho_{L^{-1}}$.

Take a $k \notin J$ and observe that

$$|L^{-1}S'(u^*)_k| \leq \tau_{L^{-1}} \quad \Rightarrow \quad G^+_{\alpha,L^{-1}}\big(u^* - L^{-1}\alpha S'(u^*)\big)_k = 0$$

again by the single-valued selection (16). Hence $v^*_k = u^*_k = 0$ and consequently, u^* is a fixed point.

Now suppose that u^* obeys $u^* = G^+_{\alpha,L^{-1}}(u^*)$. Obviously, u^* has only finitely many non-zero coefficients, meaning that property (a) of Proposition 3.14 is satisfied. For $u^*_k \neq 0$ we have

$$u^*_k - L^{-1}S'(u^*)_k = u^*_k + L^{-1}\alpha\,\mathrm{sign}(u^*_k)\phi'(|u^*_k|)$$

thus $S'(u^*)_k + \alpha\,\mathrm{sign}(u^*_k)\phi'(|u^*_k|) = 0$ and since $\mathrm{range}(j_{L^{-1}}) = \{0\} \cup \{|x| \geq \lambda_{L^{-1}}\}$, see Proposition 3.12, we also have property (b). Finally, $u^*_k = 0$ means that $L^{-1}|S'(u^*)_k| \leq \tau_{L^{-1}}$, so $|S'(u^*)_k| \leq L\tau_{L^{-1}}$, implying that u^* also obeys property (c). □

These results suggest that the generalized gradient projection method should actually look for fixed points of $G_{\alpha,L^{-1}}$. There are, however, some difficulties with taking L^{-1} as a step-size since no strict descent of T_α can be guaranteed, see Proposition 3.4. Thus, one can only try to have subsequential convergence to a fixed point of $G^+_{\alpha,L^{-1}}$, which is indeed possible.

Lemma 3.17. *If $s_n \to s^*$ monotonically increasing and $x_n \to x^*$ with $|x| \neq \tau_{s^*}$ then $j_{s_n}(x_n) \to j_{s^*}(x^*)$. For $|x^*| = \tau_{s^*}$, the following implications hold*

$$|x_n| \geq \tau_{s_n} \;\Rightarrow\; j_{s_n}(x_n) \to \mathrm{sign}(x^*)\lambda_{s^*} \quad , \quad |x_n| \leq \tau_{s_n} \;\Rightarrow\; j_{s_n}(x_n) \to 0.$$

Proof. First suppose $|x^*| < \tau_{s^*}$ meaning that from some index on, $|x_n| < \tau_{s^*} \leq \tau_{s_n}$ since τ_s depends monotonically decreasing on s. Consequently, $j_{s_n}(x_n) = 0$ from some index on and $\lim_{n\to\infty} j_{s_n}(x_n) = 0 = j_{s^*}(x^*)$. Note that this also gives $j_{s_n}(x_n) \to 0$ whenever $|x_n| \leq \tau_{s_n}$, in particular for $|x^*| = \tau_{s^*}$.

Now assume $|x^*| > \tau_{s^*}$ implying that from some index on, $|x_n| > \tau_{s_n}$ since $\tau_{s_n} \to \tau_{s^*}$ from above. Hence $\big(x^*, s^*, j_{s^*}(x^*)\big)$ is in

$$M = \{(x,s,y) \in \mathbf{R}^3 \;\big|\; s \in \,]0,\infty[\;,\; |x| \geq \kappa_s \;,\; |y| \geq y_s\},$$

denoting by y_s and κ_s again the minimizing argument and minimum of ρ_s, respectively. It is easily seen that $F : M \to \mathbf{R}$ defined by

$$F(x,s,y) = y - x + s\alpha\phi'(y) = \rho_s(y) - x$$

is locally Lipschitz continuous in M. Moreover, for $|x^*| \geq \tau_{s^*}$ we have that the generalized partial derivative satisfies $\frac{\partial F}{\partial y}\big(x^*, s^*, j_{s^*}(x^*)\big) > 0$, the latter since $\tau_{s^*} > \rho_{s^*}(y_{s^*})$, see Remark 3.13. The implicit function theorem for Lipschitz functions [8] implies the Lipschitz continuity of the mapping locally parametrizing the fiber $F^{-1}(0)$, i.e. $(s,x) \mapsto \rho_s^{-1}(x) = j_s(x)$. Due to that, $j_{s_n}(x_n) \to j_{s^*}(x^*)$. Finally, the latter also applies to the case where $|x_n| \geq \tau_{s_n}$ and $|x^*| = \tau_{s^*}$, yielding $j_{s_n}(x_n) \to \mathrm{sign}(x^*)\lambda_{s^*}$. □

Lemma 3.18. *Let $s_n \to s$ monotonically increasing and (u^{n_j}) be a subsequence with $u^{n_j} \to u$. Then it holds that $G^+_{\alpha,s_{n_j}}(u^{n_j}) \to G^+_{\alpha,s}(u)$.*

Proof. Since S' is Lipschitz continuous, it is easy to see that $u^{n_j} - s_{n_j} S'(u^{n_j}) \to u - sS'(u)$ and in particular $u^{n_j}_k - s_{n_j} S'(u^{n_j})_k \to u_k - sS'(u)_k$ for each k. Now, since u^{n_j} are iterates, the range property of j_{s_n} (see Lemma 3.12) applies, i.e. $u^{n_j}_k = 0$ or $|u^{n_j}_k| \geq \lambda_{s_{n_j}}$. One can easily derive from (14) that $\lambda_{s_{n_j}} \to \lambda_s$ monotonically increasing, thus either $u_k = 0$ or $|u_k| \geq \lambda_s$. Consequently, $u_k = 0$ for almost every k and $u^{n_j}_k = 0$ for all of these k and from some index on. It follows $|u^{n_j}_k - sS'(u^{n_j})_k| \leq \tau_{s_{n_j}}$ and $|u_k - sS'(u)_k| \leq \tau_s$ since $\tau_{s_{n_j}} \to \tau_s$ monotonically decreasing (again, see (14)). Hence, $G^+_{\alpha,s}(u)_k = 0 = \lim_{j\to\infty} G^+_{\alpha,s_{n_j}}(u_{n_j})_k$ by the single-valued selection (16).

Analogously, from some index on we have $|u^{n_j}_k| \geq \tau_{s_{n_j}}$ for all k for which $u_k \neq 0$. Of course, this also implies for u_k that $|u_k| \geq \tau_s$, thus, by the continuity statement of Lemma 3.17 and (16) we get

$$G^+_{\alpha,s_{n_j}}(u_{n_j})_k = \rho^{-1}_{s_{n_j}}\big(u^{n_j}_k - s_{n_j} S'(u^{n_j})_k\big) \to \rho^{-1}_s\big(u_k - sS'(u)_k\big) = G^+_{\alpha,s}(u)_k$$

and consequently, the desired statement. □

The previous lemmas and propositions are the essential ingredients for proving subsequential convergence of the generalized gradient projection method when s_n is increasing monotonically with limit L^{-1} in a suitable manner. This will be done in the following theorem.

Theorem 3.19. *If, under the same prerequisites as for Theorem 3.8, the Assumptions 3.3 and 3.9 hold true and the iteration is performed according to* (16) *with step-size choice $s_n = n/(Ln+1)$, then there exists a strong accumulation u point of (u^n) which is a fixed point of $G^+_{\alpha,L^{-1}}$, i.e. it obeys (a)–(c) of Proposition 3.14. Furthermore, the whole sequence converges to u if*

1. *$u_k - L^{-1}S'(u)_k \neq \pm\tau_{L^{-1}}$ for each k and*
2. *u is a strong local minimizer for T_α in the sense that there exists a continuous, strictly monotonically increasing $\xi : [0,\delta] \to [0,\infty[$ for some $\delta > 0$ with $\xi(0) = 0$ such that*

   $$T_\alpha(u) + \xi(\|v - u\|) \leq T_\alpha(v)$$

   whenever $\|v - u\| \leq \delta$ and $\operatorname{supp} v \subset \operatorname{supp} u$.

Proof. We first establish the existence of a subsequence (u^{n_j}) which satisfies $\|u^{n_j+1} - u^{n_j}\| \to 0$. Assume the opposite which means that for each $\varepsilon > 0$ there exists a n_0 such that $\|u^{n+1} - u^n\|^2 \geq \varepsilon^2$ for $n \geq n_0$. According to Proposition 2.1,

$$\frac{\varepsilon^2}{2}\Big(\frac{1}{s_n} - L\Big) \leq T_\alpha(u^n) - T_\alpha(u^{n+1})$$

which yields, by the particular choice of s_n, after summing up,

$$\sum_{n=n_0}^{n_1-1} \frac{1}{n} \leq \frac{2}{\varepsilon^2}\big(T_\alpha(u^{n_0}) - T_\alpha(u^{n_1})\big)$$

for any $n_1 > n_0$. Since $(T_\alpha(u^n))$ is a decreasing sequence, the right-hand side is bounded as $n_1 \to \infty$ while the right-hand is not, a contradiction.

Hence, there has to be a subsequence for which $\|u^{n_j+1} - u^{n_j}\| \to 0$ as $j \to 0$. Moreover, Corollary 3.7 implies that by further restriction to a subsequence, also denoted by (u^{n_j}), we can achieve that $u^{n_j} \to u$ for some $u \in \ell^2$. From Lemma 3.18 we know that $u^{n_j+1} = G^+_{\alpha,s_{n_j}}(u^{n_j}) \to G^+_{\alpha,L^{-1}}(u)$ and, consequently, $u = G^+_{\alpha,L^{-1}}(u)$ from which Proposition 3.16 gives the desired properties.

For the convergence statement regarding the whole sequence, note that since $u^{n_j} - s_{n_j} S'(u^{n_j}) \to u - L^{-1}S'(u)$, we can assume that for some j_0 it holds that $|u^{n_j}_k - s_{n_j} S'(u^{n_j})_k| \neq \tau_{L^{-1}}$ for each k and for $j \geq j_0$. In Lemma 3.17 it is also shown that $(s, x) \mapsto j_s(x)$ is locally Lipschitz continuous as long as the values do not cross the threshold τ_s (which is the case for (u^{n_j})), hence there is a $C \geq 1$ such that

$$\|u^{n_j+1} - u\| \leq C \|u^{n_j} - u\| + \frac{C}{L(Ln_j + 1)}$$

for $j \geq j_1$ where $j_1 \geq j_0$ is suitably chosen. By choosing j_2 such that for $j \geq j_2 \geq j_1$ it moreover follows that $n_j \geq L^{-1}\bigl(2C/(\delta L) - 1\bigr)$ and $T_\alpha(u^{n_j}) - T_\alpha(u) \leq \xi\bigl(\delta/(2C)\bigr)$ we get

$$\xi(\|u^{n_j} - u\|) \leq \xi\bigl(\delta/(2C)\bigr) \quad \Rightarrow \quad \|u^{n_j} - u\| \leq \frac{\delta}{2C} \quad \Rightarrow \quad \|u^{n_j+1} - u\| \leq \delta$$

as well as

$$T_\alpha(u^{n_j+1}) - T_\alpha(u) \leq T_\alpha(u^{n_j}) - T_\alpha(u) \leq \xi\bigl(\delta/(2C)\bigr).$$

By induction, it follows that there is n_0 such that for $n \geq n_0$ it holds that $\|u^n - u\| \leq \delta$ and $T_\alpha(u^n) \to T_\alpha(u)$. This implies $\xi(\|u^n - u\|) \to 0$ and thus $\|u^n - u\| \to 0$ what was to show. □

To summarize, this means that one can design an algorithm for which at least one subsequential limit u shares the same necessary conditions as the global minimizer u^*. Under some additional assumptions, the convergence of the whole sequence may be established. Such an observation does not prove that the algorithm indeed runs into a global minimizer but makes sure that there is a chance and helps to avoid stationary points which are certainly not a global minimizer. Hence, if Theorem 3.19 is applicable, one can argue that the generalized gradient projection method produces reasonable approximations to a minimizer of T_α.

4 Application to ℓ^p penalties

This section deals with the special case $\Phi(x) = |x|^p$ with $0 < p < 1$ which is clearly a non-convex separable penalty. First, it is analyzed how the proximity operator can be calculated (cf. [16, 18]). This result can be easily derived from the general statements of Lemma 3.12.

Lemma 4.1. *Let $0 < p < 1$ and $\alpha > 0$. The function $\phi(x) = |x|^p$ satisfies Assumptions 3.1 and 3.9. The corresponding j_s and thresholds according to* (14) *can be expressed by*

$$j_s(x) = \begin{cases} 0 & \text{for } |x| \leq \tau_s \\ \big(\,\cdot\, + s\alpha p \operatorname{sign}(\,\cdot\,)|\,\cdot\,|^{p-1}\big)^{-1} & \text{for } |x| \geq \tau_s \end{cases},$$
$$\tau_s = \frac{2-p}{2-2p}\big(2s\alpha(1-p)\big)^{\frac{1}{2-p}} \quad , \quad \lambda_s = \big(2s\alpha(1-p)\big)^{\frac{1}{2-p}}.$$

Proof. All the statements can easily be verified by computation. Assumption 3.1 is trivially fulfilled. Regarding Assumption 3.9, note that ϕ is arbitrarily differentiable on $]0, \infty[$. The function ϕ' is strictly convex because of $\phi''' > 0$ for positive arguments. We have $\phi'(x) \to \infty$ as $x \to 0$ as well as $\phi'(x)/x = px^{p-2} \to 0$ as $x \to \infty$. Moreover, $x\phi''(x) = p(p-1)x^{p-1}$ is easily seen to be locally integrable on $[0, \infty[$.

Clearly, $\psi(y) = 2(1-p)y^{p-2}$, so one computes $\lambda_s = \big(2s\alpha(1-p)\big)^{1/(2-p)}$ and consequently

$$\tau_s = \rho_s(\lambda_s) = \lambda_s + s\alpha p \lambda_s^{p-2}\lambda_s = \Big(1 + \frac{p}{2-2p}\Big)\lambda_s = \frac{2-p}{2-2p}\big(2s\alpha(1-p)\big)^{\frac{1}{2-p}}.$$

□

Knowing the thresholds τ_s and λ_s is crucial for performing the generalized gradient projection method, see Lemma 3.12. With Proposition 3.14 in mind, we also get the following.

Remark 4.2. If $S : \ell^2 \to [0, \infty[$, one can immediately estimate the number of non-zero coefficients for a minimizer u^* of T_α:

$$S(0) = T_\alpha(0) \geq T_\alpha(u^*) \geq \alpha \left\|u^*\right\|_p^p \geq \#\{u_k^* \neq 0\}\alpha\big(2L^{-1}\alpha(1-p)\big)^{p/(2-p)}$$

implies

$$\#\{u_k^* \neq 0\} \leq 2S(0)\big((2\alpha)^{2/p}L^{-1}(1-p)\big)^{p/(p-2)} .$$

It is remarkable that this quantity stays bounded as $p \to 0$.

The above shows that the generalized gradient projection method is applicable for ℓ^p-penalties and leads to convergence whenever one of the Theorems 3.8 or 3.19 is applicable. Roughly speaking, without additional assumptions we only get subsequential convergence to stationary points. In some situations, where more about the structure of the problem is known, one can actually see that the additional assumptions introduced in Section 3 are satisfied and convergence can be established. One class for which this is possible is the usual setting of linear inverse problems. We will deal, in the following, with the problem of minimizing

$$\min_{u \in \ell^2} \frac{\|Ku - f\|_H^2}{2} + \alpha \sum_{k=1}^{\infty} |u_k|^p \tag{17}$$

with a Hilbert space H, $K : \ell^2 \to H$ being a linear and continuous operator, $f \in H$ some data and $\alpha > 0$, $0 < p < 1$. Hence, $S(u) = \frac{1}{2}\left\|Ku - f\right\|^2$ and $R(u) = \|u\|_p^p$.

Note that $S'(u) = K^*(Ku - f)$ is Lipschitz continuous. It is easy to see that the Lipschitz constant is given by $\|K^*K\|$ and Theorem 3.8 is applicable with $0 < \underline{s} \leq s_n \leq \overline{s} < \|K^*K\|$ yielding subsequential convergence to stationary points. We observe that Assumption 3.3 is also satisfied in many cases:

Lemma 4.3. *In the situation of* (17)*, if the space for which* $K^*Ku = \|K^*K\| u$ *holds is finite-dimensional and the eigenvalues of* K^*K *do not accumulate at* $\|K^*K\|$*, then Assumption 3.3 is fulfilled.*

Proof. The statement follows immediately by setting P as the projection onto the eigenspace for $\|K^*K\|$ and noting that $K^*K = \|K^*K\| P + QK^*KQ$ where $\|QK^*KQ\| < \|K^*K\|$. □

In particular, Theorem 3.19 is applicable for compact operators and hence, we have strong subsequential convergence to a fixed point even if $s_n = n/(Ln + 1)$. In the remainder of this section, we show that there are conditions on K under which the choice of a sufficiently small p and a special parameter choice almost certainly lead to convergence of the whole sequence. We start with establishing under which conditions one can guarantee that stationary points are strong local minimizer in the sense of Theorem 3.19.

Proposition 4.4. *Let* $f \in H$ *and* $K : \ell^2 \to H$ *be a fixed linear, continuous and compact operator satisfying the finite basis injectivity property, i.e.* K *is injective whenever restricted to finitely many coefficients.*

Consider the problem of minimizing the Tikhonov functional T_α *in dependence of* p *and the parameter choice* $\alpha(p) = \gamma\alpha_0^p$ *with* $\alpha_0, \gamma > 0$ *and* $\gamma \geq \frac{\|f\|^2}{2}$*, i.e.*

$$T_{\alpha(p)}(u) = \frac{\|Ku - f\|^2}{2} + \gamma\alpha_0^p \|u\|_p^p. \tag{18}$$

There exists a $p^* \in]0, 1[$ *such that for each* $0 < p < p^*$ *every stationary point for the corresponding* $(u(p)^n)$ *with step-size choice* $s_n = n/(Ln + 1)$ *and* $u^0 = 0$ *is a strong local minimizer.*

Proof. We first derive the bound $\|u\| \leq \alpha_0^{-1}$ for u with $T_{\alpha(p)}(u) \leq \frac{\|f\|^2}{2}$ which is independent of K and $p \in]0, 1[$. For the non-trivial case $u \neq 0$, this can be achieved by

$$\gamma\alpha_0^p \|u\|_p^p \leq T_{\alpha(p)}(u) \leq \frac{\|f\|^2}{2} \quad \Rightarrow \quad \|u\|_p \leq \Big(\frac{\|f\|^2}{2\gamma}\Big)^{1/p} \frac{1}{\alpha_0} \leq \frac{1}{\alpha_0}$$

and

$$\frac{|u_k|^2}{\|u\|^2} \leq \frac{|u_k|^p}{\|u\|^p} \quad \Rightarrow \quad 1 = \sum_{k=1}^{\infty} \frac{|u_k|^2}{\|u\|^2} \leq \sum_{k=1}^{\infty} \frac{|u_k|^p}{\|u\|^p} = \frac{\|u\|_p^p}{\|u\|^p} \quad \Rightarrow \quad \|u\| \leq \|u\|_p.$$

Next, note we have for $p \in [0, 1[$ that

$$\tau_{L^{-1}}(p) = \frac{2-p}{2-2p}\big(2L^{-1}\gamma\alpha_0^p(1-p)\big)^{1/(2-p)} > 0$$

hence there is a $\tau^* > 0$ such that $\tau_{L^{-1}}(p) \geq \tau^*$ on each interval $[0, p_0]$ where $p_0 \in]0, 1[$. Consequently, due to the compactness of K and f being in H, we can find a k_0 such that $|K^*(Ku - f)|_k \leq L\tau^*$ for each $k \geq k_0$ and $\|u\| \leq \alpha_0^{-1}$.

Since we start the algorithm with $u^0 = 0$, all $u(p)^n$ satisfy $T_{\alpha(p)}\big(u(p)^n\big) \leq T_{\alpha(p)}(u^0) = \|f\|^2/2$, hence one can say that for each $p \in [0, p_0]$ and $k \geq k_0$ the estimate

$$\big|s_nK^*\big(Ku(p)^n - f\big)\big|_k \leq L^{-1}\big|K^*\big(Ku(p)^n - f\big)\big|_k \leq \tau^* \leq \tau_{L^{-1}}(p) \leq \tau_{s_n}(p)$$

holds. By induction, $u(p)^n_k = 0$ for all $k \geq k_0$ and $p \in [0, p_0]$. Thus, it suffices to consider the iteration on the coefficients $1, \ldots, k_0 - 1$ and the restriction of K^*K to these coefficients which we will denote by $K_0^*K_0$ in the following. Let $c > 0$ be the smallest eigenvalue of $K_0^*K_0$ (which is positive due to finite basis injectivity) and set $p^* = \min\{p_0, 2cL\}$. In the following, we consider $0 < p < p^*$ as fixed and omit the dependence on p. We will see that such p always implies that T_α has stationary points being strong local minimizers.

Suppose that u^* is a stationary point for (u^n), i.e. $u^{n_j} \to u^*$ with $u^* = G^+_{\alpha,L^{-1}}(u^*)$. As it is argued in Proposition 3.16, we get $K_0^*(K_0u^* - f)_k + \alpha p|u_k^*|^{p-2}u_k^* = 0$ and $|u_k^*| \geq \lambda_{L^{-1}}$ if $u_k^* \neq 0$. Restrict, T_α to the space corresponding to $\operatorname{supp} u^*$, denoted by $\bar{T}_\alpha$, as well as K, denoted by K_1, and compute

$$F(u) = \nabla \bar{T}_\alpha(u) = K_1^*(K_1u - f) + \alpha p|u|^{p-2}u$$

for which holds $F(u^*) = 0$ (taking the restriction into account). This is a smooth mapping whenever each $u_k \neq 0$, so check that the Jacobian matrix

$$\nabla F(u^*) = K_1^*K_1 - \alpha p(1-p)\operatorname{diag}(|u^*|^{p-2})$$

is positive definite. For v with $\operatorname{supp} v \subset \operatorname{supp} u$ we can estimate, according to the definition of c,

$$\begin{aligned}\langle v|\, \nabla F(u^*)v\rangle &= \|K_1v\|^2 - \alpha p(1-p)\sum_k |u_k^*|^{p-2}v_k^2 \\ &\geq \big(c - \alpha p(1-p)\lambda_{L^{-1}}^{p-2}\big)\|v\|^2 = \Big(c - \frac{p}{2L}\Big)\|v\|^2 \ .\end{aligned}$$

By the choice of p, $\nabla F(u^*) = \nabla^2 \bar{T}_\alpha(u^*)$ is positive definite and hence, u^* is a strong local minimizer for some $\delta > 0$ and $\xi(t) = c^*t^2$ where $c^* > 0$. □

Remark 4.5. The "finite basis injectivity" (FBI) property also plays an important role in the context of linear (as well as non-linear) inverse problems with sparsity constraints. One the one hand, it is one of the crucial ingredients under which order-optimal convergence rates can be established [6, 12, 13, 17]. On the other hand, under the assumption that the FBI property holds, one can achieve good convergence rates for numerical algorithms for minimization problems with sparsity constraints [5, 14]

In order to apply Theorem 3.19 concerning the convergence of the whole sequence, each stationary point u of (u^n) must not "hit" the points of discontinuity, i.e.

$$u_k - L^{-1}\big(K^*(Ku - f)\big)_k \neq \pm\tau_{L^{-1}}$$

for each k. In fact, one can easily construct cases in which this condition is violated, for instance where K is the identity on ℓ^2 and $f_k = \pm\tau_k$ for some k. In this example, however, one can see that varying p slightly changes $\tau_{L^{-1}}$ and therefore, for a fixed f the above condition is fulfilled generically.

Likewise, one can proceed and try to establish an analogous statement for general K. However, this would result in a large additional amount of technical considerations which do not add too much to this article since the result only covers a special situation. Moreover, in practice it seems like the stationary points never violate this assumption. For these reasons, we omit a thorough examination of this issue and leave it to the reader who is interested in it. To this point, we summarize in the following convergence theorem:

Theorem 4.6. *For f, K and $\alpha(p)$ according to Proposition 4.4, there is a $p > 0$ such that the generalized gradient projection method for* (18) *converges, subsequentially, to a stationary point u which satisfies the (a)–(c) of Proposition 3.14. If, additionally,*

$$\left|u_k - L^{-1}\big(K^*(Ku - f)\big)_k\right| \neq \tau_{L^{-1}}$$

for each k, then the whole sequence converges to u.

Proof. This follows immediately from Theorem 3.19 for which the prerequisites are given by the Lemmas 4.1 and 4.3. The additional part can be obtained from Proposition 4.4 and, of course, the assumption. □

We conclude this section with some remarks about this result and its connection to minimization with an ℓ^0-penalty term.

Remark 4.7. Let us comment on the parameter choice $\alpha(p) = \gamma\alpha_0^p$ with $\alpha_0, \gamma > 0$ with $\gamma \geq \frac{\|f\|^2}{2}$. As it is shown in the proof of Proposition 4.4, this leads to a bound on the norm of solutions which is uniform in p. However, as $p \to 0$, $\alpha(p)$ tends to γ which cannot be made arbitrarily small. The necessity for this can be explained, to a certain extend, in the fact that the "limit problem" for $p = 0$, i.e.

$$\min_{u\in\ell^2} \frac{\|Ku - f\|^2}{2} + \bar{\alpha}\#\{u_k \neq 0\} \tag{19}$$

is not well-posed for general $\bar{\alpha} > 0$ due to the lack of coercivity of the functional $\#\{u_k \neq 0\}$. For $\bar{\alpha} = \gamma \geq \frac{\|f\|^2}{2}$, however, it is easy to see that the above functional is globally minimized by the trivial solution $u = 0$.

On the other hand, letting $f \neq 0$ and $0 < \bar{\alpha} < \frac{\|f\|^2}{2}$, one can construct a compact linear operator such that the minimization problem admits no solution: Set $Ke_n = \frac{1}{n}(f + \frac{1}{n}e_n)$ and deduce, for example with $u_n = ne_n$, that

$$\inf_{u\in\ell^2} \frac{\|Ku - f\|^2}{2} + \bar{\alpha}\#\{u_k \neq 0\} \leq \lim_{n\to\infty} \frac{\|Ku^n - f\|^2}{2} + \bar{\alpha} = \bar{\alpha} \ .$$

But each $u = ve_k$ for some $v \neq 0$ and $k \in \mathbf{N}$ results in a $\|Ku - f\|^2/2 + \bar{\alpha} > \bar{\alpha}$, the same holds true for $u = 0$ and and u with $\#\{u_k \neq 0\} \geq 2$. Hence, no minimizer exists and the choice of $\gamma \geq \frac{\|f\|^2}{2}$ is sharp for compact linear operators K.

Remark 4.8. As the previous remark showed, problem (19) generally admits no solution. One can, however, also try to apply the generalized gradient projection method with quadratic fidelity and the separable penalty term

$$S(u) = \frac{\|Ku - f\|^2}{2} \quad , \quad R(u) = \alpha\sum_{k=1}^{\infty}\phi(|u_k|) \quad , \quad \phi(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \ . \end{cases}$$

One easily sees that the minimization problem for J_s, i.e.

$$\min_{v\in\ell^2} \sum_{k=1}^{\infty}\big(u_k - s\big(K^*(Ku - f)\big)_k\big)^2 + \alpha\phi(|u_k|),$$

can be solved by doing a hard-thresholding on $w = u - sK^*(Ku - f)$:

$$J_s\big(w\big)_k = \begin{cases} 0 & \text{for } |w_k| \le \sqrt{2s\alpha} \\ w_k & \text{for } |w_k| \ge \sqrt{2s\alpha} \end{cases}$$

This iteration has been studied in finite dimensions in [2].

Note that the structure of the iteration is the same as for the ℓ^p-penalties with $0 < p < 1$. Furthermore, with $0 < s < L^{-1}$, Proposition 2.1 is still applicable and since hard-thresholding also induces a jump, the iterates (u^n) do not change the sign pattern from some index on, reducing the iteration to

$$u_k^{n+1} = u_k^n - s\big(K^*(Ku^n - f)\big)_k$$

for k being in some finite set J. This amounts to a Landweber iteration in finite dimensions which always converges.

Hence, the hard-thresholding operation of [2] still converges in the infinite-dimensional setting even though a global minimizer does not necessarily exist. The ℓ^p-setting where $0 < p < 1$ has the advantage that we can get both: well-posedness as well as convergence of the algorithm.

5 Numerical examples

We illustrate the behavior of the proposed algorithm with two examples. The purpose of this section is to give an impression of the characteristics of the algorithm, especially since it behaves somehow discontinuous.

5.1 Reconstruction with a partial DCT-matrix

The purpose of this example is, to demonstrate the typical behavior of the iterated thresholding algorithm on a simple example. We considered the finite dimensional problem of reconstruction of a spiky signal from partial discrete cosine transformed (DCT) measurements. We generated an operator by taking 64 random rows of a DCT matrix of size 256×256. We generated spiky data by randomly choosing ten entries to have normally distributed values, added 5% noise and chose $\alpha = 5 \cdot 10^{-4}$. Moreover, we chose to use the increasing step-size rule $s_n = n/(nL + 1)$. Figure 3 illustrates the behavior of the iteration. We plotted the behavior of the functional value T_α and also the norm difference between two iterates

$$r^n = \big\|u^n - J_{s_n}\big(u^n - s_n K^*(Ku^n - g^\delta)\big)\big\| . \tag{20}$$

We observe that the functional value is monotonically decreasing and from time to time it jumps down. This effect is due to the jump in the iteration mapping G_{α,s_n} and happens when one coefficient jumps out of or into the dead-zone of G_{α,s_n}. These jumps are also present in the plot of the residual r and of course they are smaller for larger p since the jump in G_{α,s_n} is smaller. Finally one observes that, from some point on, the residual decreases monotonically and this may be due to the fact that the support of the minimizer is identified and hence, is not changing anymore and the algorithm behaves like a usual gradient descent.

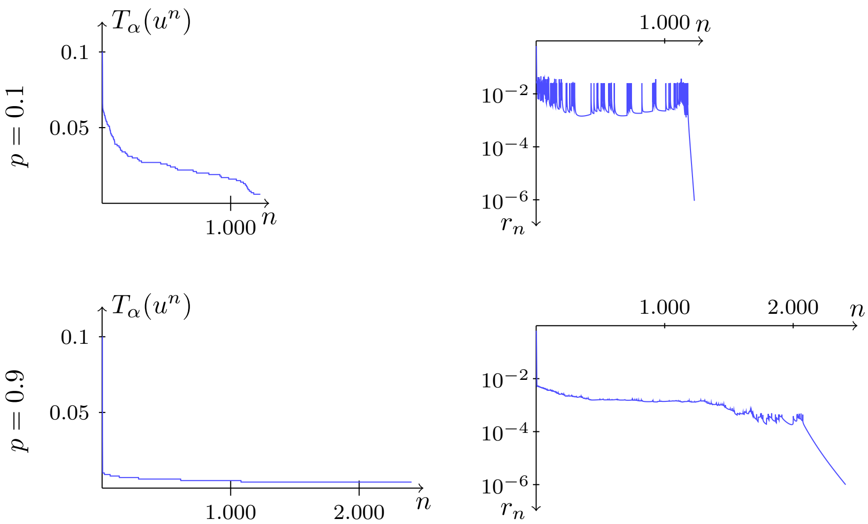


Figure 3: Two typical runs for the partial-DCT example. Top row: $p = 0.1$, bottom row: $p = 0.9$. Left column: development of the objective value $T_\alpha(u^n)$, right column: development of the residual from (20).

5.2 Deblurring of spikes

This purpose of the next example is to show the difference of the minimizers for different values of p. In this numerical example we considered a discretized linear blurring operator F combined with a synthesis operator B associated with simple hat-functions. We generated data which just consists of a few spikes and hence, has a sparse representation in hat functions. We generated noisy data with 5% noise, see Figure 4. Then we chose $\alpha = 5 \cdot 10^{-5}$ and $p = 0.1$. Motivated by the previous example we applied our algorithm until the norm of the residual r_n from (20) fell below the threshold $1 \cdot 10^{-9}$, leading to a reconstruction $u_p^{\alpha,\delta}$. To make a comparison with different values for p we calculated the discrepancy for $\left\| FBu_p^{\alpha,\delta} - g^\delta \right\|$ and chose α such that we obtained the same discrepancy for different values of p. The result is depicted in Figure 5. Concerning the properties of the solutions one may note the following things:

- Smaller values of p lead to higher sparsity for the same discrepancy.
- Smaller values of p lead to a more accurate reconstruction of the height of the peaks.

6 Conclusion

We considered special instances of non-smooth and non-convex minimization problems and proposed a generalization of the well-known gradient projection method. Our analysis shows, that even in the general case of functionals $S + \alpha R$ the proposed algorithm has convenient convergence properties. In the special case of separable constraints our method amounts to an iterative thresholding

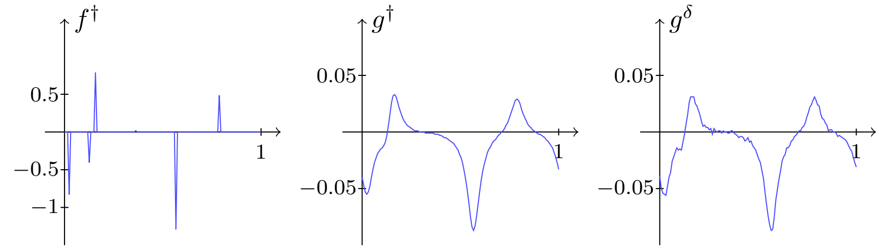


Figure 4: Example with a smoothing operator. Left original solution $f^\dagger$, middle: original data $g^\dagger$, right: noisy data g^δ (5% noise).

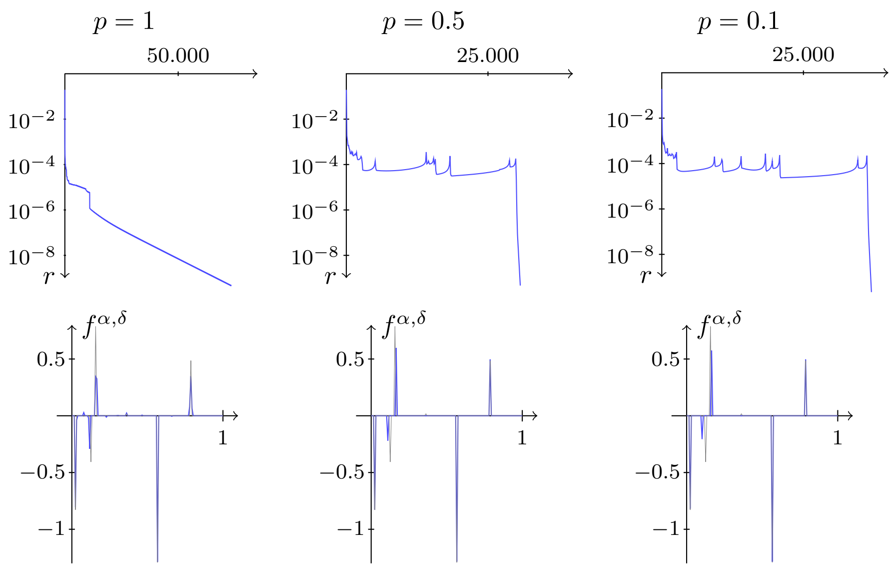


Figure 5: Reconstructions for different values of p. The first row shows the norm of the residual, i.e. $\left\|u^n - J_s(u^n - K^*(Ku^n - g^\delta)\right\|$. The second row shows the reconstructed $f^{\alpha,\delta}$ (and with slim lines the original $f^\dagger$).

procedure and is rather easy to implement: One only needs to calculate the gradient of S and the proximal mapping for the one-dimensional function ϕ which can even be done analytically in some examples.

We remark that non-smooth and non-convex optimization problems are fairly hard to solve. Our algorithm gives strong subsequential convergence and there is good reason to hope that it may reach a global minimizer in the special case of separable constraints.

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