DFG-Schwerpunktprogramm 1324

„Extraktion quantifizierbarer Information aus komplexen Systemen”

A Two Parameter Generalization of Lions' Nonoverlapping Domain Decomposition Method for Linear Elliptic PDEs

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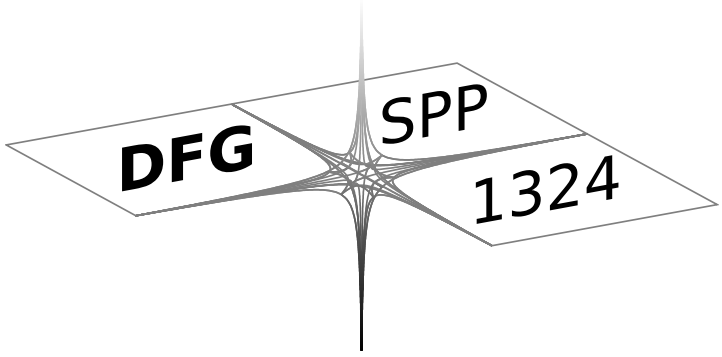

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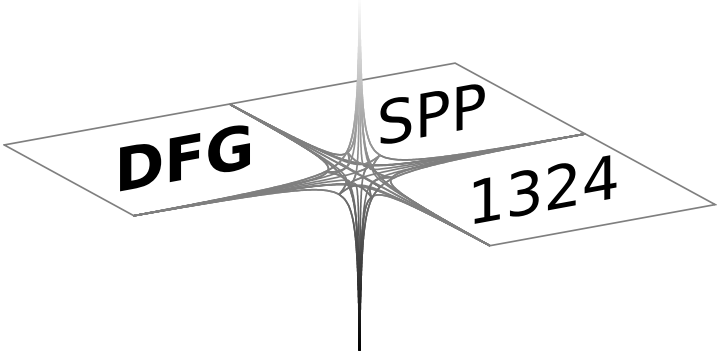

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A TWO PARAMETER GENERALIZATION OF LIONS' NONOVERLAPPING DOMAIN DECOMPOSITION METHOD FOR LINEAR ELLIPTIC PDES

ULRICH FRIEDRICH

Abstract. A two parameter generalization of Lions' nonoverlapping domain decomposition method for linear elliptic PDEs is proposed and studied. The convergence of the iterates is established. A strategy for an adaptive choice of parameters is given and discussed. The new method is numerically compared to Lions' original method and a one parameter generalization using the same strategy for its parameter choice.

1. Introduction

The idea of using nonoverlapping domain decomposition methods (DDM) for solving PDE is to decompose the domain on which the global problem has to be solved into nonoverlapping subdomains. The global problem defines local problems that are coupled by their boundary conditions on the interfaces between adjacent domains. Nonoverlapping DDM start at an initial guess for the unknown boundary data corresponding to the looked for solution, iteratively compute local solutions and use them to update the current approximation of the boundary data. The computation of the local problems can be done in parallel, leading to a possible speed up. Additionally, the domain decomposition itself can be chosen such that it corresponds to the given operator and the geometry of the domain, thus leading to subdomains which allow for much simpler discretizations. In this paper a nonoverlapping DDM for linear elliptic partial differential equations is developed and

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analyzed. Like in the well-known method by Lions (4), Robin boundary conditions are used to model the coupling of the local subproblems. Two subproblems need to be solved per domain at each iteration, one corresponding to the jumps of the function values of the local solutions, i.e., the error in the Dirichlet data, and one to the matching of the normal derivatives, i.e., the error in the Neumann data. Since the unknown global solution has no jumps in its function values across the interfaces and has matching normal derivatives, it is only natural to utilize information about both the error in the Dirichlet and Neumann data of the current solutions for the update.

For Lions' method only one local problem per domain has to be solved per iteration and the method can be interpreted as using a fixed combination of the information gained from the error in the Dirichlet and Neumann data. Those two kinds of error don't need to be equal in size, and a good strategy of weighting the corresponding solutions in the boundary value update will lead to faster convergence of the proposed method than Lions' method or methods that solve the same local problems as in Lions' method, but use them for a Richardson like update of the local boundary data, like in (3).

The weights of the two solutions corresponding to the Dirichlet and Neumann error are chosen such that an interface bias functional is minimized at each iteration. It consists of the jumps in the Dirichlet data and the sum of the normal derivatives along the interfaces, both measured in local L_2 spaces. Using the structure of the Robin boundary data, no normal derivatives have to be computed for the update, and only L_2 inner products occur on the interfaces between subdomains, which are easy to treat numerically.

This paper is organized as follows. In Section 2 the problem setting is described and the methods by Lions (4), Guo and Hou (3) and the new two parameter generalization outlined above are defined. In Section 3 a proof of convergence for a certain range of parameters is given that is independent of the strategy to choose those parameters. In Section 4 the strategy of minimizing the interface bias functional is analyzed more closely and a way to compute the parameters minimizing the functional is given. The proposed method is roughly twice as expensive as Lions' method per iteration, since two subproblems per domain need to be solved, jet these subproblems are independent of one another, leading to a new possibility for further parallelism. Still, even if these subproblems are not parallized and the methods presented are compared to their computational cost, the new method often leads to

faster convergence. To point this out, numerical results using biorthogonal tensor-wavelets for discretization are presented in Section 5. The results are summarized in Section 6.

2. Elliptic setting and method description

2.1. **Scope of problems.** We want to solve a linear elliptic problem with homogeneous Dirichlet boundary conditions of the form

$$Au = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{2.1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain, $A = -\sum_{k,l=1}^{d} \frac{\partial}{\partial x_k}(a_{kl}\frac{\partial}{\partial x_l}) + a_0$ is an elliptic operator, with a_0, $a_{ij} \in L^\infty(\Omega)$, and f is a given right-hand side. For a nonoverlapping DDM we consider subdomains $\Omega_i \subset \Omega$, $i = 1, \ldots, m$, with

$$\Omega_i \cap \Omega_j = \emptyset, \quad i \neq j, \quad \bigcup_{i=1}^{m} \overline{\Omega_i} = \overline{\Omega}, \tag{2.2}$$

and solve local problems

$$\begin{aligned} Au_i &= f \quad \text{in } \Omega_i, \\ u_i &= 0 \quad \text{on } \Gamma_i := \partial\Omega_i \cap \partial\Omega, \\ u_i + \lambda\frac{\partial u_i}{\partial \nu_i} &= g_i \quad \text{on } \gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j,\ j \in N(i), \end{aligned} \tag{2.3}$$

where n_k^i denotes the k-th component of the outward normal vector on Ω_i, $\frac{\partial v}{\partial \nu_i} := \sum_{k,l=1}^{d} a_{kl}\frac{\partial v}{\partial x_l}n_k^i$ the conormal derivative, $\lambda \in \mathbb{R} \setminus \{0\}$ a coupling parameter for the Robin boundary data, and $N(i) := \{j \in \{1, \ldots, m\} | \operatorname{meas}_{d-1}(\gamma_{ij}) > 0\}$ the index set of all relevant neighbors. Here (2.3) is meant to hold in the following variational sense. The solution $u_i \in H^1_{\Gamma_i}(\Omega_i) := \{v \in H^1(\Omega_i) | v_{|\Gamma_i} = 0\}$ fulfills the variational formulation of the local problem given by

$$a_i(u_i, v_i) + \lambda^{-1}\langle u_i, v_i\rangle_{L_2(\partial\Omega_i)} = \langle f, v_i\rangle_{H^1_{\Gamma_i}(\Omega_i)} + \lambda^{-1}\langle g_i, v_i\rangle_{L_2(\partial\Omega_i)}, \tag{2.4}$$

for all $v_i \in H^1_{\Gamma_i}(\Omega_i)$, where

$$a_i(v, w) := \int_{\Omega_i} \sum_{k,l=1}^{d} a_{kl}\frac{\partial v}{\partial x_l}\frac{\partial w}{x_k}, \quad v, w \in H^1_{\Gamma_i}(\Omega_i) \tag{2.5}$$

is the bilinearform induced by A and $\langle\cdot,\cdot\rangle_X$ denotes the dual pairing of X and X'.

Remark 2.1. a) The local bilinearform $b_i(\cdot,\cdot) := a_i(\cdot,\cdot) + \lambda^{-1}\langle\cdot,\cdot\rangle_{L_2(\partial\Omega_i)}$ is elliptic on $H^1_{\Gamma_i}(\Omega_i) \times H^1_{\Gamma_i}(\Omega_i)$, even if a_i is only coercive, see (5) Theorem 7.1. Thus, the local problems (2.4) have unique solutions. For instance $A = -\Delta$ on a domain with

$\operatorname{meas}_{d-1} \Gamma_i = 0$ induces only a coercive bilinearform.

b) The minimal regularity assumption on g_i in the low-dimensional inner product $\langle g_i, v_i \rangle_{L_2(\partial\Omega_i)}$ from (2.4) is $g_i \in (H_{00}^{1/2}(\partial\Omega \setminus \Gamma_i))'$, since in this setting v_i vanishes on Γ_i. Here the inner product would be interpreted as a dual pairing. The methods presented in this paper will generate sequences $g_i^{(k)}$ of Robin boundary values in $L_2(\partial\Omega_i \setminus \Gamma_i)$ and we will assume that the unknown Robin data $\tilde{g}_i$ corresponding to the restriction of the global solution to Ω_i also lies in this space. Thus only L_2 inner products on the boundaries $\partial\Omega_i \setminus \Gamma_i$ have to be treated numerically.

The need of handling boundary data on the interfaces leads to the notation g_{ij} for the restriction of $g_i \in L_2(\partial\Omega_i \setminus \Gamma_i)$ to γ_{ij}. We will avoid the cumbersome identification $g_i = (g_{ij})_{j \in N(i)} \in \Pi_{i \in N(i)} L_2(\gamma_{ij}) = L_2(\partial\Omega_i \setminus \Gamma_i)$.

Observe that the dependence of the solution $u_i = u_i(f, g)$ of (2.4) defines local affine linear operator $S_i^f g_i := u_i$, i.e.,

$$S_i^f g_i = S_i^f 0 + S_i^0 g_i. \tag{2.6}$$

The local spaces combine in a natural way to the following product spaces

$$\begin{aligned} \mathbb{H}_0^1 &:= \Pi_{i=1}^m H_{\Gamma_i}^1(\Omega_i), \\ \mathbb{H}^{1/2} &:= \Pi_{i=1}^m H_{00}^{1/2}(\partial\Omega_i \setminus \Gamma_i), \\ \mathbb{L}_2 &:= \Pi_{i=1}^m L_2(\partial\Omega_i \setminus \Gamma_i). \end{aligned}$$

The local definitions can be transferred to the global setting. For $\mathbf{v} = (v_i)_{i=1}^m$, $\mathbf{w} = (w_i)_{i=1}^m \in \mathbb{H}_0^1$ and $\mathbf{g} = (g_i)_{i=1}^m \in (\mathbb{H}^{1/2})'$ we have

$$a(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^m a_i(v_i, w_i), \quad \langle \mathbf{g}, \mathbf{w} \rangle := \sum_{i=1}^m \langle g_i, w_i \rangle_i, \quad S^f \mathbf{g} = (S_i^f g_i)_{i=1}^m,$$

with $\langle \cdot, \cdot \rangle_i$ denoting the inner product on $L_2(\partial\Omega_i \setminus \Gamma_i)$. As an abuse of notation $S^f \mathbf{g}$ will sometimes be considered as the element of $\mathbb{L}_2$ given by its trace. The meaning will always be clear from the context.

We define the energy norm

$$\|\mathbf{v}\|_a^2 = a(\mathbf{v}, \mathbf{v}),$$

which is indeed a norm if the local bilinearforms a_i are elliptic, otherwise it is not definite. The bilinearform b and norm $\|\cdot\|_b$ are defined analogous to a and $\|\cdot\|_a$, respectively.

Finally, we define an exchange operator $\overline{(\cdot)} : \mathbb{L}_2 \to \mathbb{L}_2$ by $\overline{g}_{ij} = g_{ji}$.

Remark 2.2. $\overline{(\cdot)}$ as an operator on $\mathbb{H}^{1/2}$ is continuous if and only if there are no internal crosspoints, that is for $i = 1, \ldots, m$ $\gamma_{ij} \cap \gamma_{ik} = \emptyset$ for $j \neq k,\ k, j \in N(i)$.

2.2. **Presentation of the methods.** It is now possible to define some domain decomposition methods. Each method will generate a sequence of Robin boundary values in $\mathbb{L}_2$, starting at an arbitrary $\mathbf{g}^{(0)} \in \mathbb{L}_2$. We begin with the well-known Lions' method, that reads in our notation

$$\begin{aligned} &\text{compute } S^f \mathbf{g}^{(k)}, \\ &\text{update } \mathbf{g}^{(k+1)} = 2\overline{S^f \mathbf{g}^{(k)} - \mathbf{g}^{(k)}}. \end{aligned} \tag{2.7}$$

Note that no conormal derivatives need to be computed and that $\mathbf{g}^{(k)} \in \mathbb{L}_2,\ k \in \mathbb{N}$. Lions' method is an alternating method. $g_i^{(k+1)}$ depends only on $S_j^f g_j^{(k)}$ and $g_{ji}^{(k)}$ for $j \in N(i)$, but no information of the solution $S_i^f g_i^{(k)}$ or the old boundary value $g_i^{(k)}$ on Ω_i is used for the update. To examine the update strategy a little closer, let $u = S^f \tilde{\mathbf{g}}$ be a classical solution of the global problem (2.1) with Robin boundary data $\tilde{\mathbf{g}} \in \mathbb{L}_2$. Denoting with u_i the restriction of u to Ω_i and the trace on the interfaces γ_{ij} with u_{ij} the solution fulfills

$$u_{ij} = u_{ji}, \quad \frac{\partial u_i}{\partial \nu_i} = -\frac{\partial u_j}{\partial \nu_j} \quad \text{on } \gamma_{ij}. \tag{2.8}$$

We observe

$$g_{ij} = u_{ij} + \lambda \frac{\partial u_i}{\partial \nu_i} = u_{ji} - \lambda \frac{\partial u_j}{\partial \nu_j} = 2u_{ji} - g_{ji}.$$

Roughly speaking, the update $2S_j^f g_j^{(k)} - g_j^{(k)}$ is the best guess for the boundary data $g_{ij}^{(k+1)}$ that depends only on data from the j-th domain.

It is imminent that including data from the i-th domain in $g_i^{(k+1)}$ may lead to faster convergence. This can be done by performing a Richardson iteration and using the update from Lions' method as a search direction, leading to

$$\begin{aligned} &\text{compute } S^f \mathbf{g}^{(k)}, \\ &\text{update } \mathbf{g}^{(k+1)} = \mathbf{g}^{(k)} + \delta^{(k)} \overline{2S^f \mathbf{g}^{(k)} - \mathbf{g}^{(k)}}. \end{aligned} \tag{2.9}$$

A strategy for the choice of $\delta^{(k)} \in \mathbb{R}$ is given by Guo and Hou in (3). There $\delta^{(k)}$ is chosen such that an interface bias functional $E^f \mathbf{g}^{(k)}$ is minimized in each step. This leads to a descent direction that is not a gradient direction in general.

If we just want to combine information from the current domain Ω_i and its neighbors, there is no need to hold on to the form of the update

in Lions' method. Going back to the situation in (2.8) we can combine the given data in a more direct way:

$$\begin{aligned} g_{ij} =& u_{ij} + \lambda \frac{\partial u_i}{\partial \nu_i} \\ =& (1-\mu)u_{ij} + \mu u_{ji} + (1-\eta)\lambda \frac{\partial u_i}{\partial \nu_i} - \eta\lambda \frac{\partial u_j}{\partial \nu_j} \\ =& (1-\mu)u_{ij} + \mu u_{ji} \\ &+ (1-\eta)(g_{ij} - u_{ij}) - \eta(g_{ji} - u_{ji}) \\ =& g_{ij} + \mu(u_{ji} - u_{ij}) \\ &\quad + \eta(u_{ij} - g_{ij} + u_{ji} - g_{ji}). \end{aligned}$$

This motivates the new method

$$\begin{aligned} &\text{compute } S^f \mathbf{g}^{(k)}, \\ &\text{update } \mathbf{g}^{(k+1)} = \mathbf{g}^{(k)} - \mu^{(k)}(S^f \mathbf{g}^{(k)} - \overline{S^f \mathbf{g}^{(k)}}) \\ &\qquad\qquad + \eta^{(k)}(S^f \mathbf{g}^{(k)} - \mathbf{g}^{(k)} + \overline{S^f \mathbf{g}^{(k)} - \mathbf{g}^{(k)}}). \end{aligned} \tag{2.10}$$

Similar to the choice of the parameter in (2.9), $\eta^{(k)}$ and $\mu^{(k)}$ can be chosen to minimize an interface bias functional in each iteration. Before the definition of the interface bias functional is given, note that this method is fully parallel and again no conormal derivatives need to be computed. The choice $\eta^{(k)} = \mu^{(k)}$ gives (2.9) and $\eta^{(k)} = \mu^{(k)} = 1$ reproduces Lions' method (2.7), so that this is truly a generalization. $\mu^{(k)}$ controls the update corresponding to the jumps in the Dirichlet data on the interfaces γ_{ij}, while $\eta^{(k)}$ can be interpreted as dealing with the matching of the Neumann data of the current solution.

Since the sequences $\mathbf{g}^{(k)}$ are all in $\mathbb{L}_2$, the following definition of the interface bias functional makes sense

$$E^f \mathbf{g} := \|S^f \mathbf{g} - \overline{S^f \mathbf{g}}\|^2_{\mathbb{L}_2} + \|S^f \mathbf{g} - \mathbf{g} + \overline{S^f \mathbf{g} - \mathbf{g}}\|^2_{\mathbb{L}_2}. \tag{2.11}$$

The first term measures jumps in the Dirichlet data given by the local solutions, while the second corresponds to the sums of the conormal derivatives along the interfaces, i.e., the error in the Neumann data. Both terms are measured in the same norm, what makes this functional easy to handle and of course the $\mathbb{L}_2$-norm is relatively easy to compute.

The idea is, that choosing the parameters $\delta^{(k)}$, $\eta^{(k)}$ and $\mu^{(k)}$ in (2.9) and (2.10), respectively, such that the interface bias $E^f \mathbf{g}^{(k)}$ is minimized in each step should also lead to a fast error reduction in $\mathbb{H}^1_0$. The choice of parameters and the relation of $E^f \mathbf{g}^{(k)}$ and the error $\|u - S^f \mathbf{g}^{(k)}\|_{\mathbb{H}^1_0}$, which is important if we want to use $E^f \mathbf{g}^{(k)}$ as a stopping criterion, is explained in detail in Section 4.

3. Proof of convergence

The proof of convergence for method (2.10) presented in this section is independent of the strategy used to choose the parameters of the method. We start with a technical lemma.

Lemma 3.1. *Let $a \in [0,1]$, $\mathbf{g}$, $\mathbf{h} \in \mathbb{L}_2$. Then*

a) $\|(1-a)\mathbf{g} + a\mathbf{h}\|^2_{\mathbb{L}_2} \leq (1-a)\|\mathbf{g}\|^2_{\mathbb{L}_2} + a\|\mathbf{h}\|^2_{\mathbb{L}_2}$,
b)

$$\begin{aligned}\|2S^f\mathbf{g} - \mathbf{g}\|^2_{\mathbb{L}_2} &= \|\mathbf{g}\|^2_{\mathbb{L}_2} + 4\langle S^f\mathbf{g} - \mathbf{g}, S^f\mathbf{g}\rangle \\ &= \|\mathbf{g}\|^2_{\mathbb{L}_2} - 4\lambda(\|S^f\mathbf{g}\|^2_a - \langle f, S^f\mathbf{g}\rangle_{\mathbb{H}^1_0}),\end{aligned}$$

c) $\langle \overline{\mathbf{g}}, \mathbf{h}\rangle = \langle \mathbf{g}, \overline{\mathbf{h}}\rangle$.

Proof. For the proof of a) we use $\langle \mathbf{g}, \mathbf{h}\rangle \leq 1/2(\|\mathbf{g}\|^2_{\mathbb{L}_2} + \|\mathbf{h}\|^2_{\mathbb{L}_2})$ in

$$\|(1-a)\mathbf{g} + a\mathbf{h}\|^2_{\mathbb{L}_2} = (1-a)^2\|\mathbf{g}\|^2_{\mathbb{L}_2} + 2(1-a)a\langle \mathbf{g}, \mathbf{h}\rangle + a^2\|\mathbf{h}\|^2_{\mathbb{L}_2}.$$

A simple computation yields the first equation in b), while the second is given by collecting the low-dimensional integrals in (2.4). Straightforward calculation using only $\gamma_{ij} = \gamma_{ji}$ shows c). □

Theorem 3.2. *Assume that the boundary value corresponding to the solution of the global problem fulfills $\tilde{\mathbf{g}} \in \mathbb{L}_2$ and that there are no internal crosspoints. Let a be elliptic and continuous with the constants α and β. Then there exists $C > 0$ that depends on α, β and the continuity constants of the trace and extension operators from $\mathbb{H}^1_0 \to \mathbb{H}^{1/2}$, $\mathbb{H}^{1/2} \to \mathbb{H}^1_0$, respectively, s.t. the two parameter method* (2.10) *converges to the global solution u, i.e.,*

$$S^f\mathbf{g}^{(k)} \to u \text{ in } \mathbb{H}^1_0,$$

if $\eta^{(k)}$, $\mu^{(k)} \in [0,1]$ and there exists a constant $c > 0$, s.t. the cone property

$$c \leq (\eta^{(k)} + \mu^{(k)}) - C|\eta^{(k)} - \mu^{(k)}|$$

holds for all k.

Proof. We use the Pythagorean Theorem and insert the definition of $\mathbf{g}^{(k+1)}$ to obtain

$$\begin{aligned}4\|\mathbf{g}^{(k+1)}-\tilde{\mathbf{g}}\,\|_{\mathbb{L}_2}^2 =&\|\mathbf{g}^{(k+1)}-\tilde{\mathbf{g}}+\overline{\mathbf{g}^{(k+1)}-\tilde{\mathbf{g}}}\|_{\mathbb{L}_2}^2\\ &+\|\mathbf{g}^{(k+1)}-\tilde{\mathbf{g}}-\overline{\mathbf{g}^{(k+1)}-\tilde{\mathbf{g}}}\|_{\mathbb{L}_2}^2\\ =&\|(1-\eta^{(k)})(\mathbf{g}^{(k)}-\tilde{\mathbf{g}}+\overline{\mathbf{g}^{(k)}-\tilde{\mathbf{g}}}\,)\\ &+\eta^{(k)}(2S^f\mathbf{g}^{(k)}-\mathbf{g}^{(k)}-\tilde{\mathbf{g}}+\overline{2S^f\mathbf{g}^{(k)}-\mathbf{g}^{(k)}-\tilde{\mathbf{g}}}\,)\|_{\mathbb{L}_2}^2\\ &+\|(1-\mu^{(k)})(\mathbf{g}^{(k)}-\tilde{\mathbf{g}}-\overline{\mathbf{g}^{(k)}-\tilde{\mathbf{g}}}\,)\\ &-\mu^{(k)}(2S^f\mathbf{g}^{(k)}-(\mathbf{g}^{(k)}-\tilde{\mathbf{g}})-\overline{2S^f\mathbf{g}^{(k)}-(\mathbf{g}^{(k)}-\tilde{\mathbf{g}})}\,)\|_{\mathbb{L}_2}^2.\end{aligned}$$

These are only two convex combinations. Using Lemma 3.1 *a*) we get

$$\begin{aligned}&4\|\mathbf{g}^{(k+1)}-\tilde{\mathbf{g}}\,\|_{\mathbb{L}_2}^2\\ &\le(1-\frac{\eta^{(k)}+\mu^{(k)}}{2})(\|\mathbf{g}^{(k)}-\tilde{\mathbf{g}}+\overline{\mathbf{g}^{(k)}-\tilde{\mathbf{g}}}\|_{\mathbb{L}_2}^2+\|\mathbf{g}^{(k)}-\tilde{\mathbf{g}}-\overline{\mathbf{g}^{(k)}-\tilde{\mathbf{g}}}\|_{\mathbb{L}_2}^2)\\ &\quad+\frac{\eta^{(k)}+\mu^{(k)}}{2}(\|2S^f\mathbf{g}^{(k)}-\mathbf{g}^{(k)}-\tilde{\mathbf{g}}+\overline{2S^f\mathbf{g}^{(k)}-\mathbf{g}^{(k)}-\tilde{\mathbf{g}}}\|_{\mathbb{L}_2}^2\\ &\qquad+\|2S^f\mathbf{g}^{(k)}-\mathbf{g}^{(k)}+\tilde{\mathbf{g}}-\overline{2S^f\mathbf{g}^{(k)}-\mathbf{g}^{(k)}+\tilde{\mathbf{g}}}\|_{\mathbb{L}_2}^2)\\ &\quad+\frac{\mu^{(k)}-\eta^{(k)}}{2}(\|\mathbf{g}^{(k)}-\tilde{\mathbf{g}}+\overline{\mathbf{g}^{(k)}-\tilde{\mathbf{g}}}\|_{\mathbb{L}_2}^2-\|\mathbf{g}^{(k)}-\tilde{\mathbf{g}}-\overline{\mathbf{g}^{(k)}-\tilde{\mathbf{g}}}\|_{\mathbb{L}_2}^2\\ &\qquad+\|2S^f\mathbf{g}^{(k)}-\mathbf{g}^{(k)}+\tilde{\mathbf{g}}-\overline{2S^f\mathbf{g}^{(k)}-\mathbf{g}^{(k)}+\tilde{\mathbf{g}}}\|_{\mathbb{L}_2}^2\\ &\qquad-\|2S^f\mathbf{g}^{(k)}-\mathbf{g}^{(k)}-\tilde{\mathbf{g}}+\overline{2S^f\mathbf{g}^{(k)}-\mathbf{g}^{(k)}-\tilde{\mathbf{g}}}\|_{\mathbb{L}_2}^2).\end{aligned}$$

For the first two parts of the right-hand side we use again the Pythagorean Theorem. Further, $\overline{\tilde{\mathbf{g}}}=2S^f\,\tilde{\mathbf{g}}-\tilde{\mathbf{g}}$ together with the affine linearity (2.6) of S^f and Lemma 3.1 *b*) yields

$$\begin{aligned}&\|2S^f\mathbf{g}^{(k)}-\mathbf{g}^{(k)}-\overline{\tilde{\mathbf{g}}}\|_{\mathbb{L}_2}^2=\|2S^0(\mathbf{g}^{(k)}-\tilde{\mathbf{g}})-(\mathbf{g}^{(k)}-\tilde{\mathbf{g}})\|_{\mathbb{L}_2}^2\\ &=\|\mathbf{g}^{(k)}-\tilde{\mathbf{g}}\,\|_{\mathbb{L}_2}^2+4\langle S^f\mathbf{g}^{(k)}-u-(\mathbf{g}^{(k)}-\tilde{\mathbf{g}}),S^f\mathbf{g}^{(k)}-u\rangle.\end{aligned}$$

A lengthy but straightforward computation using again $\overline{\tilde{\mathbf{g}}}=2S^f\,\tilde{\mathbf{g}}-\tilde{\mathbf{g}}$, Lemma 3.1 *c*) and $\langle\tilde{\mathbf{g}}-u,u\rangle=\lambda(a(u,u)-\langle f,u\rangle_{\mathbb{H}_0^1})=0$ gives

$$\begin{aligned}&\|\mathbf{g}^{(k)}-\tilde{\mathbf{g}}+\overline{\mathbf{g}^{(k)}-\tilde{\mathbf{g}}}\|_{\mathbb{L}_2}^2+\|2S^f\mathbf{g}^{(k)}-\mathbf{g}^{(k)}+\tilde{\mathbf{g}}-\overline{2S^f\mathbf{g}^{(k)}-\mathbf{g}^{(k)}+\tilde{\mathbf{g}}}\|_{\mathbb{L}_2}^2\\ &-\|\mathbf{g}^{(k)}-\tilde{\mathbf{g}}-\overline{\mathbf{g}^{(k)}-\tilde{\mathbf{g}}}\|_{\mathbb{L}_2}^2-\|2S^f\mathbf{g}^{(k)}-\mathbf{g}^{(k)}-\tilde{\mathbf{g}}+\overline{2S^f\mathbf{g}^{(k)}-\mathbf{g}^{(k)}-\tilde{\mathbf{g}}}\|_{\mathbb{L}_2}^2\\ &=16\langle S^f\mathbf{g}^{(k)}-u-(\mathbf{g}^{(k)}-\tilde{\mathbf{g}}),u-\overline{S^f\mathbf{g}^{(k)}}\rangle.\end{aligned}$$

By now we have

$$\begin{aligned}
&\|\mathbf{g}^{(k+1)} - \tilde{\mathbf{g}}\,\|_{\mathbb{L}_2}^2 \\
&\le \|\mathbf{g}^{(k)} - \tilde{\mathbf{g}}\,\|_{\mathbb{L}_2}^2 \quad + 2(\eta^{(k)} + \mu^{(k)})\langle S^f\mathbf{g}^{(k)} - u - (\mathbf{g}^{(k)} - \tilde{\mathbf{g}}), S^f\mathbf{g}^{(k)} - u\rangle \\
&\qquad\qquad + 2(\eta^{(k)} - \mu^{(k)})\langle S^f\mathbf{g}^{(k)} - u - (\mathbf{g}^{(k)} - \tilde{\mathbf{g}}), \overline{S^f\mathbf{g}^{(k)}} - u\rangle.
\end{aligned}$$

Using $\overline{u} = u$ in $\mathbb{H}^{1/2}$, the continuity assumption for $\overline{(\cdot)}$, the existence of a continuous extension operator $\mathcal{E}$ and trace from $\mathbb{H}^{1/2}$ to $\mathbb{H}_0^1$ and $\mathbb{H}_0^1$ to $\mathbb{H}^{1/2}$, respectively, together with the ellipticity and continuity of a gives the estimate

$$\begin{aligned}
&\langle S^f\mathbf{g}^{(k)} - u - (\mathbf{g}^{(k)} - \tilde{\mathbf{g}}), u - \overline{S^f\mathbf{g}^{(k)}}\rangle \\
&\le \|S^f\mathbf{g}^{(k)} - u - (\mathbf{g}^{(k)} - \tilde{\mathbf{g}})\|_{(\mathbb{H}^{1/2})'}\|u - \overline{S^f\mathbf{g}^{(k)}}\|_{\mathbb{H}^{1/2}} \\
&\le C_1 \sup_{h\in\mathbb{H}^{1/2}} \frac{\langle S^f\mathbf{g}^{(k)} - u - (\mathbf{g}^{(k)} - \tilde{\mathbf{g}}), h\rangle}{\|h\|_{\mathbb{H}^{1/2}}}\|u - S^f\mathbf{g}^{(k)}\|_{\mathbb{H}^{1/2}} \\
&\le C_2\lambda \sup_{h\in\mathbb{H}^{1/2}} \frac{a(u - S^f\mathbf{g}^{(k)}, \mathcal{E}\,h)}{\|h\|_{\mathbb{H}^{1/2}}}\|u - S^f\mathbf{g}^{(k)}\|_{\mathbb{H}_0^1} \\
&\le C_3\|u - S^f\mathbf{g}^{(k)}\|_a^2.
\end{aligned}$$

We use $\langle S^f\mathbf{g}^{(k)} - u - (\mathbf{g}^{(k)} - \tilde{\mathbf{g}}), S^f\mathbf{g}^{(k)} - u\rangle = -\lambda\|S^f\mathbf{g}^{(k)} - u\|_a^2$ and sum over k to get

$$\begin{aligned}
\sum_{k=0}^{N} 2\lambda((\mu^{(k)} + \eta^{(k)}) - C_3|(\mu^{(k)} - \eta^{(k)})|)\,\|u - S^f\mathbf{g}^{(k)}\|_a^2 \\
\le \|\mathbf{g}^{(0)} - \tilde{\mathbf{g}}\,\|_{\mathbb{L}_2}^2 - \|\mathbf{g}^{(N+1)} - \tilde{\mathbf{g}}\,\|_{\mathbb{L}_2}^2.
\end{aligned}$$

The right-hand side is finite, and because of the cone property for $\eta^{(k)}$ and $\mu^{(k)}$ the summands on the left-hand side are nonnegative. Thus the ellipticity of a gives the convergence result for $S^f\mathbf{g}^{(k)}$. □

Remark 3.3. If we restrict the choice of parameters to the case $\eta^{(k)} = \mu^{(k)}$, the continuity assumption on $\overline{(\cdot)}$ is not needed, and the above proof reproduces the convergence result from (3). Notice that even if there are crosspoints in the decomposition, the method remains well-defined and a numerical comparison reasonable.

4. Choice of parameters

In this section we consider the interface bias functional given by (2.11). We will point out its relation to the error of the current solution $u - S^f\mathbf{g}^{(k)}$ and then give formulas for the optimal parameters $\delta^{(k)}$, $\eta^{(k)}$ and $\mu^{(k)}$ in (2.9) and (2.10), respectively.

4.1. **Bounding the error with the interface bias functional.** We are interested in the error $u - S^f\mathbf{g}^{(k)}$ of the current solution. Using $u = S^f\,\tilde{\mathbf{g}}$ the affine linearity of S^f leads to

$$u - S^f\mathbf{g}^{(k)} = S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)})$$

and thus to the question of dependence of the solution of a Robin boundary value problem on its boundary data. This is essentially the same as the dependence of a Neumann boundary value problem on its boundary data.

Lemma 4.1. *There exist constants $0 < c_1$, c_2 s.t. for every $\mathbf{g} \in (\mathbb{H}^{1/2})'$ the following statement is true*

$$c_1\|\mathbf{g}\|_{(\mathbb{H}^{1/2})'} \leq \|S^0\mathbf{g}\|_{\mathbb{H}^1_0} \leq c_2\|\mathbf{g}\|_{(\mathbb{H}^{1/2})'}.$$

Proof. Because of the ellipticity and continuity of the bilinearforms b_i from Remark 2.1, the energy norm $\|\cdot\|_b$ induced by b is equivalent to $\|\cdot\|_{\mathbb{H}^1_0}$. By the definition of the local problems (2.4) and the existence of continuous extension and trace operators from $\mathbb{H}^{1/2}$ to $\mathbb{H}^1_0$ and $\mathbb{H}^1_0$ to $\mathbb{H}^{1/2}$, respectively, we have

$$\|S^0\mathbf{g}\|_b^2 = \langle \mathbf{g}, S^0\mathbf{g}\rangle \leq \|\mathbf{g}\|_{(\mathbb{H}^{1/2})'}\|S^0\mathbf{g}\|_{\mathbb{H}^{1/2}} \leq c_2\|\mathbf{g}\|_{(\mathbb{H}^{1/2})'}\|S^0\mathbf{g}\|_b$$

and

$$\begin{aligned}\|\mathbf{g}\|_{(\mathbb{H}^{1/2})'} &= \sup_{\mathbf{h}\in\mathbb{H}^{1/2}} \frac{\langle \mathbf{g}, \mathbf{h}\rangle}{\|\mathbf{h}\|_{\mathbb{H}^{1/2}}} = \sup_{\mathbf{h}\in\mathbb{H}^{1/2}} \frac{b(S^0\mathbf{g}, \mathcal{E}\,\mathbf{h})}{\|\mathbf{h}\|_{\mathbb{H}^{1/2}}} \\ &\leq C \sup_{\mathbf{h}\in\mathbb{H}^{1/2}} \frac{b(S^0\mathbf{g}, \mathcal{E}\,\mathbf{h})}{\|\mathcal{E}\,\mathbf{h}\|_{\mathbb{H}^1_0}} \leq C \sup_{\mathbf{v}\in\mathbb{H}^1_0} \frac{b(S^0\mathbf{g}, \mathbf{v})}{\|\mathbf{v}\|_{\mathbb{H}^1_0}} \leq c_1^{-1}\|S^0\mathbf{g}\|_b.\end{aligned}$$

□

Remark 4.2. Assuming a to be elliptic, we can use it instead of b in the last proof. Starting again with the global equation given by (2.4), i.e.,

$$a(S^0\mathbf{g}, \mathbf{v}) = \lambda^{-1}\langle \mathbf{g} - S^0\mathbf{g}, \mathbf{v}\rangle, \quad \text{for all } \mathbf{v} \in \mathbb{H}^1_0,$$

we can deduce the existence of constants $0 < c_1$, c_2 (possibly differing from those in Lemma 4.1) s.t.

$$c_1\|\mathbf{g} - S^0\mathbf{g}\|_{(\mathbb{H}^{1/2})'} \leq \|S^0\mathbf{g}\|_{\mathbb{H}^1_0} \leq c_2\|\mathbf{g} - S^0\mathbf{g}\|_{(\mathbb{H}^{1/2})'}, \quad \mathbf{g} \in (\mathbb{H}^{1/2})'.$$

We are now able to give an estimate for the $\mathbb{H}^1_0$-error of $S^f\mathbf{g}$.

Proposition 4.3. *Let a be elliptic. Then there exists a constant $C > 0$ depending on the ellipticity of a and continuity of the trace from $\mathbb{H}^1_0$ to*

$\mathbb{H}^{1/2}$, *s.t.*

$$\|u - S^f\mathbf{g}\|_{\mathbb{H}^1_0} \leq C(\|\mathbf{g}^{(k)} - S^f\mathbf{g}^{(k)} + \overline{\mathbf{g}^{(k)} - S^f\mathbf{g}^{(k)}}\|_{(\mathbb{H}^{1/2})'} + \|S^f\mathbf{g}^{(k)} - \overline{S^f\mathbf{g}^{(k)}}\|_{\mathbb{H}^{1/2}}).$$

Proof. We use Lemma 3.1 *c)* to get $2\langle \mathbf{g}, \mathbf{h}\rangle = \langle \mathbf{g} + \overline{\mathbf{g}}, \mathbf{h}\rangle + \langle \mathbf{g}, \mathbf{h} - \overline{\mathbf{h}}\rangle$ for arbitrary $\mathbf{g}, \mathbf{h} \in \mathbb{L}_2$. This, Remark 4.2 and the existence of a continuous trace operator from $\mathbb{H}^1_0$ to $\mathbb{H}^{1/2}$ gives

$$\begin{aligned}
&2\|S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)})\|_a^2 \\
&= 2\langle \tilde{\mathbf{g}} - \mathbf{g}^{(k)} - S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)}), S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)})\rangle \\
&= \langle \tilde{\mathbf{g}} - \mathbf{g}^{(k)} - S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)}) + \overline{\tilde{\mathbf{g}} - \mathbf{g}^{(k)} - S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)})}, S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)})\rangle \\
&\quad + \langle \tilde{\mathbf{g}} - \mathbf{g}^{(k)} - S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)}), S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)}) - \overline{S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)})}\rangle \\
&\leq \|\tilde{\mathbf{g}} - \mathbf{g}^{(k)} - S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)}) + \overline{\tilde{\mathbf{g}} - \mathbf{g}^{(k)} - S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)})}\|_{(\mathbb{H}^{1/2})'}\|S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)})\|_{\mathbb{H}^{1/2}} \\
&+ \|\tilde{\mathbf{g}} - \mathbf{g}^{(k)} - S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)})\|_{(\mathbb{H}^{1/2})'}\|S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)}) - \overline{S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)})}\|_{\mathbb{H}^{1/2}} \\
&\leq C\|S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)})\|_{\mathbb{H}^1_0}(\|\tilde{\mathbf{g}} - \mathbf{g}^{(k)} - S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)}) + \overline{\tilde{\mathbf{g}} - \mathbf{g}^{(k)} - S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)})}\|_{(\mathbb{H}^{1/2})'} \\
&\qquad + \|S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)}) - \overline{S^0(\tilde{\mathbf{g}} - \mathbf{g}^{(k)})}\|_{\mathbb{H}^{1/2}}).
\end{aligned}$$

Further the unknown solution is continuous, i.e., $S^f\tilde{\mathbf{g}} = \overline{S^f\tilde{\mathbf{g}}}$ in $\mathbb{H}^{1/2}$ and has matching Neumann data, i.e., $\tilde{\mathbf{g}} - S^f\tilde{\mathbf{g}} + \overline{\tilde{\mathbf{g}} - S^f\tilde{\mathbf{g}}} = 0$ in $(\mathbb{H}^{1/2})'$. For the latter the regularity assumption $\tilde{\mathbf{g}} \in \mathbb{L}_2$ is needed. This together with the ellipticity of a completes the proof. □

Remark 4.4.

a) The natural norms for measuring Neumann and Dirichlet data for a problem in $\mathbb{H}^1_0$ are $\|\cdot\|_{(\mathbb{H}^{1/2})'}$ and $\|\cdot\|_{\mathbb{H}^{1/2}}$, respectively. The use of $\|\cdot\|_{\mathbb{L}_2}$ in the interface bias functional $E^f\mathbf{g}^{(k)}$ (2.11) instead of these natural norms, like in the error bound from Proposition 4.3, may lead to a non optimal descent direction. The Neumann data may be over-weighted, while the Dirichlet data may make a to small contribution to the interface bias. Furthermore this shows that using $E^f\mathbf{g}^{(k)}$ as a stopping criterion, as proposed in (3), may lead to unsatisfying results.

b) The regularity assumption $\tilde{\mathbf{g}} \in \mathbb{L}_2$ is more than needed to guarantee the matching of the Neumann data of the looked for solution. Indeed we only need

$$\langle (\tilde{g}_{ij} + u_{ij} + \tilde{g}_{ji} + u_{ji})_{j\in N(i)}, v_i\rangle_i \tag{4.1}$$

to make sense for arbitrary $v_i \in H^1_{\Gamma_i}(\Omega_i)$ in order for the local problems to be well defined.

If this is the case, we take $h \in H^{1/2}_{00}(\gamma_{ij})$ and the Extension $v \in H^1_0(\Omega)$ with $v_{ij} = v_{ji} = h$ and $v_{kl} = 0, (i,j) \neq (k,l) \neq (j,l)$. Then we have

$$\langle \tilde{\mathbf{g}} + S^f\,\tilde{\mathbf{g}} + \overline{\tilde{\mathbf{g}} + S^f\,\tilde{\mathbf{g}}}, v\rangle = 2\langle \tilde{\mathbf{g}} + S^f\,\tilde{\mathbf{g}}, v\rangle = 2(a(S^f\,\tilde{\mathbf{g}}, v) - \langle f, v\rangle) = 0,$$

yielding the desired matching of the Neumann data by a density argument.

There are two ways to get (4.1). The first is to assume a decomposition without internal crosspoints. Here the restriction of $v \in H^1_{\Gamma_i}(\Omega_i)$ to $\gamma_{ij}, j \in N(i)$ is an element of $H^{1/2}_{00}(\gamma_{ij})$ giving $g_{ij} \in (H^{1/2}_{00}(\gamma_{ij}))'$ since $\langle g_i, \mathcal{E}\, v_{ij}\rangle_i$ is well defined with $\mathcal{E}$ being the extension by 0 of $v_{ij} \in H^{1/2}_{00}(\gamma_{ij})$ to $\partial\Omega_i \setminus \Gamma_i$. Thus (4.1) is fulfilled. The second way is to assume more regularity for the local boundary values. $g_i \in H^{-1/2+\epsilon}(\partial\Omega_i \setminus \Gamma_i)$ is enough for (4.1) to be well defined, since $v_{ij} \in H^{1/2}(\gamma_{ij}) \subset H^{1/2-\epsilon}(\gamma_{ij})$ can always be extended by 0 to Γ_i and Γ_j.

The comparison of $E^f\mathbf{g}^{(k)}$ and $\|u - S^f\mathbf{g}^{(k)}\|_{\mathbb{H}^1_0}$ is completed by the following lemma.

Lemma 4.5. *Assume $S^f\mathbf{g}^{(k)} \to u$ in $\mathbb{H}^1_0$ with $0 < \eta_{\min} < \eta^{(k)} < \eta_{\max} < 1$ for all $k > N_0$. Then $E^f\mathbf{g}^{(k)} \to 0$.*

Proof. Convergence of $\|S^f\mathbf{g}^{(k)} - \overline{S^f\mathbf{g}^{(k)}}\|_{\mathbb{L}_2}$ is trivial because of $\overline{(\cdot)}$ not changing the $\mathbb{L}_2$-norm and the continuity of the trace operator. For the other term in (2.11) we recursively apply

$$\begin{aligned}\|S^f\mathbf{g}^{(k+1)} - \mathbf{g}^{(k+1)} &+ \overline{S^f\mathbf{g}^{(k+1)} - \mathbf{g}^{(k+1)}}\|_{\mathbb{L}_2}\\ &\leq \|S^f\mathbf{g}^{(k+1)} - S^f\mathbf{g}^{(k)} + \overline{S^f\mathbf{g}^{(k+1)} - S^f\mathbf{g}^{(k)}}\|_{\mathbb{L}_2}\\ &\quad + |1 - 2\eta^{(k)}|\|S^f\mathbf{g}^{(k)} - \mathbf{g}^{(k)} + \overline{S^f\mathbf{g}^{(k)} - \mathbf{g}^{(k)}}\|_{\mathbb{L}_2},\end{aligned}$$

to get

$$\begin{aligned}&\|S^f\mathbf{g}^{(k+N)} - \mathbf{g}^{(k+N)} + \overline{S^f\mathbf{g}^{(k+N)} - \mathbf{g}^{(k+N)}}\|_{\mathbb{L}_2}\\ &\leq \sum_{j=0}^{k-1} C^{k-1-j}\|S^f\mathbf{g}^{(N+1+j)} - S^f\mathbf{g}^{(N+j)} + \overline{S^f\mathbf{g}^{(N+1+j)} - S^f\mathbf{g}^{(N+j)}}\|_{\mathbb{L}_2}\\ &+ C^k\|S^f\mathbf{g}^{(N)} - \mathbf{g}^{(N)} + \overline{S^f\mathbf{g}^{(N)} - \mathbf{g}^{(N)}}\|_{\mathbb{L}_2},\end{aligned}$$

where $C = \sup_{j>N_0}|1 - 2\eta^{(j)}| < 1$ and $N > N_0$. The last term is bounded by $C^k E^f\mathbf{g}^{(N)}$ and each summand by $C^{k-1-j}2\epsilon$ if N is large enough to ensure $\|S^f\mathbf{g}^{(j)} - \overline{S^f\mathbf{g}^{(j)}}\|_{\mathbb{L}_2} < \epsilon$ for all $j > N$. The convergence of the geometric series gives $\lim_{k\to\infty}\|S^f\mathbf{g}^{(k+N)} - \mathbf{g}^{(k+N)} + \overline{S^f\mathbf{g}^{(k+N)} - \mathbf{g}^{(k+N)}}\|_{\mathbb{L}_2} \leq 2\epsilon/(1-C)$ for arbitrary small ϵ. □

4.2. **Optimal parameters.** In this subsection we insert the update $\mathbf{g}^{(k+1)}$ from (2.10) into the interface bias functional (2.11) and interpret the expression as a quadratic function in $\eta^{(k)}$ and $\mu^{(k)}$. Minimizing it is a straightforward task.

We begin by giving a more compact notation for $E^f\mathbf{g}$. Using the fact that $\mathbf{g}+\overline{\mathbf{g}}$ and $\mathbf{h}-\overline{\mathbf{h}}$ are $\mathbb{L}_2$-orthogonal for arbitrary $\mathbf{g}$, $\mathbf{h}\in\mathbb{L}_2$ we can write the interface bias functional shorter

$$\begin{aligned} E^f\mathbf{g} &= \|S^f\mathbf{g}-\overline{S^f\mathbf{g}}+S^f\mathbf{g}-\mathbf{g}+\overline{S^f\mathbf{g}-\mathbf{g}}\|^2_{\mathbb{L}_2} \\ &= \|2S^f\mathbf{g}-\mathbf{g}-\overline{\mathbf{g}}\|^2_{\mathbb{L}_2}. \end{aligned} \tag{4.2}$$

Lemma 4.6. a) $E^f\mathbf{g}=0$ *iff* $\mathbf{g}=\tilde{\mathbf{g}}$.

b) *Let a be elliptic. Choosing $\eta^{(k)}=\mu^{(k)}=1$ like in Lions' method (2.7) reduces the interface bias if not already $\mathbf{g}^{(k)}=\tilde{\mathbf{g}}$. We have for $\mathbf{g}\in\mathbb{L}_2$*

$$E^f(\overline{2S^f\mathbf{g}-\mathbf{g}}) = \|\overline{2S^f\mathbf{g}-\mathbf{g}}-\mathbf{g}\|^2_{\mathbb{L}_2} - 4\lambda\|S^0(\overline{2S^f\mathbf{g}-\mathbf{g}}-\mathbf{g})\|^2_a.$$

Proof. Let $\mathbf{g}=\tilde{\mathbf{g}}$. Following Remark 4.4 *b*) and using $\tilde{\mathbf{g}}\in\mathbb{L}_2$ we have matching Dirichlet and Neumann boundary data for $S^f\,\tilde{\mathbf{g}}$ and thus $E^f\,\tilde{\mathbf{g}}=$ follows directly from the definition (2.11).

On the other hand, $E^f\mathbf{g}=0$ implies $E^0(\tilde{\mathbf{g}}-\mathbf{g})=0$. For instance the first term in (2.11) can be written as

$$\begin{aligned} 0 &= \|\mathbf{g}-S^f\mathbf{g}+\overline{\mathbf{g}-S^f\mathbf{g}}\|_{\mathbb{L}_2} \\ &= \|\mathbf{g}-S^f\mathbf{g}+\overline{\mathbf{g}-S^f\mathbf{g}}-(\tilde{\mathbf{g}}-S^f\,\tilde{\mathbf{g}}+\overline{\tilde{\mathbf{g}}-S^f\,\tilde{\mathbf{g}}})\|_{\mathbb{L}_2} \\ &= \|\mathbf{g}-\tilde{\mathbf{g}}-S^0(\mathbf{g}-\tilde{\mathbf{g}})+\overline{\mathbf{g}-\tilde{\mathbf{g}}-S^0(\mathbf{g}-\tilde{\mathbf{g}})}\|_{\mathbb{L}_2}, \end{aligned}$$

and similar for the second term. Proposition 4.3 gives $u-S^f\mathbf{g}=S^0(\tilde{\mathbf{g}}-\mathbf{g})=0$ in $\mathbb{H}^1_0$ and thus because the local boundary value problems admit unique solutions $\mathbf{g}=\tilde{\mathbf{g}}$. For the proof of *b*) we use (4.2) and Lemma 3.1 *b*) to get

$$\begin{aligned} E^f(\overline{2S^f\mathbf{g}-\mathbf{g}}) &= \|2S^f(\overline{2S^f\mathbf{g}-\mathbf{g}})-\overline{2S^f\mathbf{g}-\mathbf{g}}-(2S^f\mathbf{g}-\mathbf{g})\|^2_{\mathbb{L}_2} \\ &= \|2S^0(\overline{2S^f\mathbf{g}-\mathbf{g}}-\mathbf{g})-(\overline{2S^f\mathbf{g}-\mathbf{g}}-\mathbf{g})\|^2_{\mathbb{L}_2} \\ &= \|\overline{2S^f\mathbf{g}-\mathbf{g}}-\mathbf{g}\|^2_{\mathbb{L}_2} - 4\lambda\|S^0(\overline{2S^f\mathbf{g}-\mathbf{g}}-\mathbf{g})\|^2_a. \end{aligned}$$

The update from Lions' method reduces the interface bias functional, if not $\|S^0(\overline{2S^f\mathbf{g}-\mathbf{g}}-\mathbf{g})\|_a=0$. Using the ellipticity of a and uniqueness of the local solutions, this only happens for $\overline{2S^f\mathbf{g}-\mathbf{g}}-\mathbf{g}=0$. Inserting $2S^f\mathbf{g}-\mathbf{g}=\overline{\mathbf{g}}$ into (4.2) immediately gives $E^f\mathbf{g}=0$ and thus with *a*) $\mathbf{g}=\tilde{\mathbf{g}}$. □

With $\mathbf{g}^{(k+1)}$ defined by the update from (2.10) we want to identify $\eta_{\text{opt}}, \mu_{\text{opt}}$ solving

$$\underset{\eta^{(k)},\mu^{(k)}}{\operatorname{argmin}} E^f\mathbf{g}^{(k+1)} = \underset{\eta^{(k)},\mu^{(k)}}{\operatorname{argmin}} \|C + \eta^{(k)}D + \mu^{(k)}E\|^2_{\mathbb{L}_2} \tag{4.3}$$

where

$$\begin{aligned} C &:= 2S^f\mathbf{g}^{(k)} - \mathbf{g}^{(k)} - \overline{\mathbf{g}^{(k)}}, \\ D &:= 2(S^0(S^f\mathbf{g}^{(k)} - \mathbf{g}^{(k)} + \overline{S^f\mathbf{g}^{(k)} - \mathbf{g}^{(k)}}) \\ &\quad - (S^f\mathbf{g}^{(k)} - \mathbf{g}^{(k)} + \overline{S^f\mathbf{g}^{(k)} - \mathbf{g}^{(k)}})), \\ E &:= -2S^0(S^f\mathbf{g}^{(k)} - \overline{S^f\mathbf{g}^{(k)}}). \end{aligned}$$

$E^f\mathbf{g}^{(k+1)}$ is a quadratic function in $\eta^{(k)}$ and $\mu^{(k)}$. The set $\{(\eta,\mu)|(E^f\mathbf{g}^{(k+1)})(\eta,\mu) = c\}$ is an ellipse. Observe, that since the update from Lions' method reduces the interface bias functional, $(1,1)$ lies in the interior of the ellipse defined by $c = E^f\mathbf{g}^{(k)}$ as of course does the solution $(\eta_{\text{opt}}, \mu_{\text{opt}})$ of (4.3). Using the fact that $(0,0)$ lies on the ellipse, a geometric argument yields $\eta_{\text{opt}} + \mu_{\text{opt}} > 0$.

Depending on C, D and E the optimal parameters η_{opt} and μ_{opt} are given by the following differentiation of cases.

Theorem 4.7. *Let a be elliptic, $\tilde{\mathbf{g}}$ be the Robin data corresponding to the unknown solution of (2.1) and $\mathbf{g}^{(k+1)}$ be defined by the update in (2.10). If not already $\tilde{\mathbf{g}} = \mathbf{g}^{(k)}$ there exist $(\eta^{(k)}, \mu^{(k)}) \neq (0,0)$ solving (4.3). Depending on D and E the following cases can be distinguished:*

a) $D = E = 0$. In this case we have $\tilde{\mathbf{g}} = \mathbf{g}^{(k)}$.

b) $D = 0$, $E \neq 0$. $E^f\mathbf{g}^{(k+1)}$ does not depend on $\eta^{(k)}$. $E^f\mathbf{g}^{(k+1)}$ is minimized by $\mu_{\text{opt}} = -\frac{\langle C,E,\rangle}{\|E\|^2_{\mathbb{L}_2}} > \frac{1}{2}$. $E^f\mathbf{g}^{(k+1)}$ as a function in $\mu^{(k)}$ is strictly decreasing in $(0,\mu_{\text{opt}}]$ and strictly increasing in $[\mu_{\text{opt}}, 2\mu_{\text{opt}})$ and suffices $E^f\mathbf{g}^{(k+1)} < E^f\mathbf{g}^{(k)}$.

c) $E = 0$, $D \neq 0$. $E^f\mathbf{g}^{(k+1)}$ does not depend on $\mu^{(k)}$. $E^f\mathbf{g}^{(k+1)}$ is minimized by $\eta_{\text{opt}} = -\frac{\langle C,D,\rangle}{\|D\|^2_{\mathbb{L}_2}} > \frac{1}{2}$. $E^f\mathbf{g}^{(k+1)}$ as a function in $\eta^{(k)}$ is strictly decreasing in $(0,\eta_{\text{opt}}]$ and strictly increasing in $[\eta_{\text{opt}}, 2\eta_{\text{opt}})$ and suffices $E^f\mathbf{g}^{(k+1)} < E^f\mathbf{g}^{(k)}$.

d) $D \neq 0 \neq E$, $D = \kappa E$, $\kappa \in \mathbb{R}$. $E^f\mathbf{g}^{(k+1)} = \|C + (\kappa\eta^{(k)} + \mu^{(k)})E\|^2_{\mathbb{L}_2}$ is minimized for all $\eta^{(k)}$, $\mu^{(k)}$ on the line $\kappa\eta^{(k)} + \mu^{(k)} = \frac{-\langle C,E\rangle}{\|E\|^2_{\mathbb{L}_2}}$. The case $\kappa = -1$ would imply $C = D = 0$ and thus a). For every other value of κ the choice $\eta^{(k)} = \mu^{(k)}$ leads to $\eta_{\text{opt}} = \mu_{\text{opt}} = \frac{-\langle C,E\rangle}{(1+\kappa)\|E\|^2_{\mathbb{L}_2}} > \frac{1}{2}$. Here $E^f\mathbf{g}^{(k+1)}$ is strictly

decreasing in $(0, \eta_{\text{opt}}]$ *and strictly increasing in* $[\eta_{\text{opt}}, 2\eta_{\text{opt}})$ *with* $E^f\mathbf{g}^{(k+1)} < E^f\mathbf{g}^{(k)}$.

e) In every other case (4.3) *attains its minimum at*

$$\eta_{\text{opt}} = \frac{-\langle C, D\rangle \|E\|^2_{\mathbb{L}_2} + \langle C, E\rangle\langle D, E\rangle}{\|D\|^2_{\mathbb{L}_2}\|E\|^2_{\mathbb{L}_2} - (\langle D, E\rangle)^2},$$
$$\mu_{\text{opt}} = \frac{-\langle C, E\rangle \|D\|^2_{\mathbb{L}_2} + \langle C, D\rangle\langle D, E\rangle}{\|D\|^2_{\mathbb{L}_2}\|E\|^2_{\mathbb{L}_2} - (\langle D, E\rangle)^2}$$

and $\eta_{\text{opt}} + \mu_{\text{opt}} > 0$.

Proof. Let $D = 0$. Then for all $v \in \mathbb{H}^1_0$ we have by the definition of a and the local problems (2.4)

$$\begin{aligned} &a(S^0(S^f\mathbf{g}^{(k)} - \tilde{\mathbf{g}} + \overline{S^f\mathbf{g}^{(k)} - \tilde{\mathbf{g}}}), v) \\ &= \lambda^{-1}\langle S^f\mathbf{g}^{(k)} - \tilde{\mathbf{g}} + \overline{S^f\mathbf{g}^{(k)} - \tilde{\mathbf{g}}} - S^0(S^f\mathbf{g}^{(k)} - \tilde{\mathbf{g}} + \overline{S^f\mathbf{g}^{(k)} - \tilde{\mathbf{g}}}), v\rangle \\ &= \langle -\frac{1}{2}D, v\rangle = 0, \end{aligned}$$

what implies $S^0(S^f\mathbf{g}^{(k)} - \tilde{\mathbf{g}} + \overline{S^f\mathbf{g}^{(k)} - \tilde{\mathbf{g}}}) = 0$ and thus $S^f\mathbf{g}^{(k)} - \mathbf{g}^{(k)} + \overline{S^f\mathbf{g}^{(k)} - \mathbf{g}^{(k)}} = 0$ in $\mathbb{L}_2$. On the other hand, $E = 0$ gives $S^f\mathbf{g}^{(k)} - \overline{S^f\mathbf{g}^{(k)}} = 0$, because of the uniqueness of the local solutions. Looking at (2.11) $D = E = 0$ implies $E^f\mathbf{g}^{(k)} = 0$ thus with Lemma 4.6 *a)* $\mathbf{g}^{(k)} = \tilde{\mathbf{g}}$.

b), *c)* and *d)* share one idea. $E^f\mathbf{g}^{(k+1)}$ degenerates to a quadratic function in one variable. For instance in *b)* we have $E^f\mathbf{g}^{(k+1)}(2\mu^{(k)}_{\text{opt}}) = E^f\mathbf{g}^{(k+1)}(0) = E^f\mathbf{g}^{(k)}$. From Lemma 4.6 *b)* we know that $E^f\mathbf{g}^{(k+1)}(1) < E^f\mathbf{g}^{(k)}$ which gives $1 < 2\mu^{(k)}_{\text{opt}}$.

In *d)* $\kappa = -1$ is a special case. Then for $\eta^{(k)} = \mu^{(k)} = 1$ we have $E^f\mathbf{g}^{(k+1)}(1, 1) = E^f\mathbf{g}^{(k)}$ and thus again with Lemma 4.6 $\mathbf{g}^{(k)} = \tilde{\mathbf{g}}$.

All other results can be derived by straightforward calculation using

$$\begin{aligned} E^f\mathbf{g}^{(k+1)} = \|C\|^2_{\mathbb{L}_2} &+ (\eta^{(k)})^2\|D\|^2_{\mathbb{L}_2} + (\mu^{(k)})^2\|E\|^2_{\mathbb{L}_2} \\ &+ 2\eta^{(k)}\langle C, D\rangle + 2\mu^{(k)}\langle C, E\rangle + 2\eta^{(k)}\mu^{(k)}\langle D, E\rangle. \end{aligned}$$

□

Remark 4.8. For the one parameter generalization (2.9) a similar discussion for $\mathbf{g}^{(k)} \neq \tilde{\mathbf{g}}$ leads to the choice

$$\delta^{(k)}_{\text{opt}} := -\frac{\langle C, D + E\rangle}{\|D + E\|^2_{\mathbb{L}_2}}. \tag{4.4}$$

5. Numerical experiments

In this section numerical results for the methods of Subsection 2.2 together with the choice of parameters from Subsection 4.2 will be presented. The methods are compared with respect to the number of iterations and computation time. In the examples the looked for solution u is known. This allows estimating the usefulness of the interface bias function $E^f\mathbf{g}^{(k)}$ as a stopping criterion by comparing its value with the $\mathbb{H}^1_0$-error $\|u - \tilde{S}^f\mathbf{g}^{(k)}\|_{\mathbb{H}^1_0}$. Additionally the computed parameters $\delta^{(k)}_{\text{opt}}$, $\eta^{(k)}_{\text{opt}}$ and $\mu^{(k)}_{\text{opt}}$ minimizing (2.11) can be compared with the parameters that had minimized the $\mathbb{H}^1_0$-error.

Biorthogonal tensor-wavelets, as designed in (6), were used for discretization. We limit ourselves to the non-adaptive case, i.e., fixing a certain wavelet level J and taking the span of all wavelets up to J, denoted by $\Psi_i^J = \{\psi_{i,\lambda} \in \Psi_i | |\lambda| \leq J\} = \{\psi_{i,\lambda} | \lambda \in \Delta_i^J\}$, $i = 1, \ldots, m$, as the local search spaces. Here Δ_i^J is the set of all wavelet-indices up to the level J for the domain i and $|\lambda|$ denotes the level of a wavelet-index. A general advantage of using wavelets is, that the condition numbers of the stiffness matrices $(a_i(\psi_{i,\lambda}, \psi_{i,\mu}))_{\lambda,\,\mu\in\Delta_i^J}$, $i = 1, \ldots, m$, are bounded independently from J.

Analog to the infinite dimensional case the local solutions $\tilde{u}_i \in \text{span}\{\Psi_i^J\}$ of

$$a_i(\tilde{u}_i, v_i) + \lambda^{-1}\langle \tilde{u}_i, v_i\rangle_i = \langle f, v_i\rangle_{L_2(\Omega_i)} + \lambda^{-1}\langle g_i, v_i\rangle_i \quad \text{for all } v_i \in \Psi_i^J,$$

define local affine linear operators $\tilde{S}_i^f g_i$ and global $\tilde{S}^f\mathbf{g}$. With this, the methods under consideration give the following algorithms.

Algorithm 1 (Lions' method)**.**

1. Given arbitrary $\mathbf{g}^{(0)} \in \mathbb{L}_2$, fix $\lambda \in \mathbb{R}$ and $\epsilon > 0$ and set $k = 0$.
2. For $i = 1, \ldots, m$ compute in parallel $\tilde{S}_i^f\mathbf{g}^{(k)}$.
3. If $E^f\mathbf{g}^{(k)} < \epsilon$ break,
   else $\mathbf{g}^{(k+1)} = \overline{2\tilde{S}^f\mathbf{g}^{(k)} - \mathbf{g}^{(k)}}$, $k = k + 1$ go to 2.

Algorithm 2 (Method by Guo and Hou)**.**

1. Given arbitrary $\mathbf{g}^{(0)} \in \mathbb{L}_2$, fix $\lambda \in \mathbb{R}$ and $\epsilon > 0$ and set $k = 0$.
2. Compute in parallel $\tilde{S}^f\mathbf{g}^{(0)}$.
3. Compute in parallel $X = \tilde{S}^0(\overline{2\tilde{S}^f\mathbf{g}^{(k)} - \mathbf{g}^{(k)}} - \mathbf{g}^{(k)})$.
4. If $E^f\mathbf{g}^{(k)} < \epsilon$ break,
   else choose $\delta^{(k)}$ as in Remark 4.8 and update

$$\mathbf{g}^{(k+1)} = (1 - \delta^{(k)})\mathbf{g}^{(k)} + \delta^{(k)}\overline{2\tilde{S}^f\mathbf{g}^{(k)} - \mathbf{g}^{(k)}},$$
$$\tilde{S}^f\mathbf{g}^{(k+1)} = \tilde{S}^f\mathbf{g}^{(k)} + \delta^{(k)}X,\ k = k + 1 \text{ go to } 3.$$

Algorithm 3 (2 parameter generalization)**.**

1. Given arbitrary $\mathbf{g}^{(0)} \in \mathbb{L}_2$, fix $\lambda \in \mathbb{R}$ and $\epsilon > 0$ and set $k = 0$.
2. For $i = 1, \ldots, m$ compute in parallel $\tilde{S}_i^f \mathbf{g}^{(0)}$
3. compute in parallel

$$X = \tilde{S}^0(\tilde{S}^f \mathbf{g}^{(k)} - \mathbf{g}^{(k)} + \overline{\tilde{S}^f \mathbf{g}^{(k)} - \mathbf{g}^{(k)}}),\ Y = \tilde{S}^0(\tilde{S}^f \mathbf{g}^{(k)} - \overline{\tilde{S}^f \mathbf{g}^{(k)}})$$

4. If $E^f \mathbf{g}^{(k)} < \epsilon$ break,
   else choose $\eta^{(k)}$, $\mu^{(k)}$ as in Theorem 4.7 and update

$$\begin{aligned} \mathbf{g}^{(k+1)} &= \mathbf{g}^{(k)} + \eta^{(k)}(\tilde{S}^f \mathbf{g}^{(k)} - \mathbf{g}^{(k)} + \overline{\tilde{S}^f \mathbf{g}^{(k)} - \mathbf{g}^{(k)}}) \\ &\quad - \mu^{(k)}(\tilde{S}^f \mathbf{g}^{(k)} - \overline{\tilde{S}^f \mathbf{g}^{(k)}}), \\ \tilde{S}^f \mathbf{g}^{(k+1)} &= \tilde{S}^f \mathbf{g}^{(k)} + \eta^{(k)} X - \mu^{(k)} Y,\ k = k+1,\ \text{go to 3.} \end{aligned}$$

Observe that in every algorithm only one type of subproblem has to be solved per domain and that the stiffness matrix corresponding to the problem on a domain is the same in each algorithm. Because of this we may assume equal setup times for each algorithm and compare only the iteration times. The computation of $\delta^{(k)}$, $\eta^{(k)}$ and $\mu^{(k)}$ involves only lower-dimensional integrals, so that computational cost per iteration is dominated by solving the local Galerkin problems. Thus, we can expect the new method 3 to be roughly twice as expensive per iteration as the others. Keep in mind, that the two subproblems per domain in Algorithm 3 can be solved independently.

As a model problem we consider the Poisson equation $-\Delta u = f$ on Ω with homogeneous Dirichlet boundary conditions on $\partial\Omega$ and set $\mathbf{g}^{(0)} = \mathbf{0}$. The coupling parameter λ of the Robin data is set to 1. We use spline wavelets of order $d = 2$ with $\tilde{d} = 2$ vanishing moments and fix the maximal level $J = 5$. This leads to 63, 64 or 65 degrees of freedom per dimension, depending whether the basis fulfills homogeneous Dirichlet boundary conditions on no, one or both sides. That means 4032 (4095, 4096, 4160, 4225) degrees of freedom on a square domain with one neighbor (2 neighbors on opposing sides, 2 neighbors, 3 and 4 neighbors).

5.1. **The two domain case.** For the first two examples we consider the following decomposition into 2 domains.

$$\Omega = (0,2) \times (0,1), \quad \Omega_1 = (0,1) \times (0,1), \quad \Omega_2 = (1,2) \times (0,1)$$

1a) $u(x,y) = (x-2)\sin(x)y\cos(y\pi/2)$,

1b) $u(x,y) = x(x-2)y(y-1)\exp(-5(x-3/4)^2 - 7(y-1/4)^2)$.

The first example was already studied in (2) and (3).
Because of the symmetry of the domain decomposition, X in Algorithm

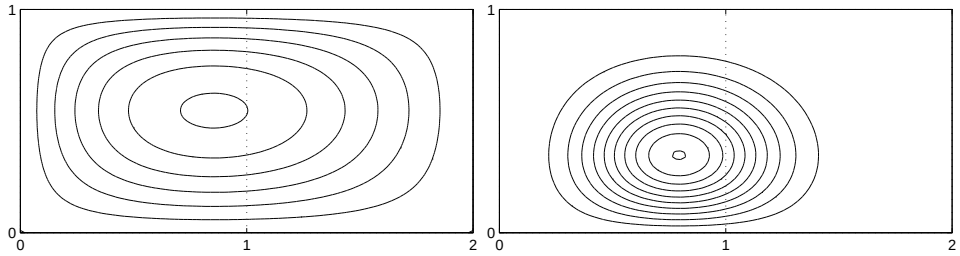

FIGURE 5.1. Solutions of Example 1a and 1b

3 will be symmetric, while Y will be antisymmetric corresponding to the interface $\{1\} \times [0,1]$. Thus, we can expect $\langle D, E \rangle = 0$ in Theorem 4.7. There are no internal crosspoints, and $\overline{(\cdot)} : \mathbb{H}^{1/2} \to \mathbb{H}^{1/2}$ is continuous with continuity constant 1.

For each of the three algorithms the $\mathbb{H}^1_0$-error of the computed solution $\tilde{S}^f \mathbf{g}^{(k)} \in \Pi_{i=1}^m \Psi_i^J$ and the looked for solution $u \in H^1_0(\Omega)$ is presented in Table 5.1. Values of the interface bias functional are given in Table 5.2. In Figure 5.2 the values of the interface bias functional and the $\mathbb{H}^1_0$-error are plotted against the time needed for the iterations. The computed parameters $\delta^{(k)}_{\text{opt}}$, $\mu^{(k)}_{\text{opt}}$ and $\eta^{(k)}_{\text{opt}}$ as well as the parameters that had minimized the $\mathbb{H}^1_0$-error are given in Table 5.3.

Looking at the $\mathbb{H}^1_0$-error, it is clear that Algorithm 3 outperforms the other two per iteration and also leads to faster convergence in time. The interface bias is also reduced faster. This proves that utilizing the jumps in the Dirichlet data and the matching of the Neumann data of the current solution separately for the update of the Robin boundary values $\mathbf{g}^{(k)}$ gives Algorithm 3 a qualitative advantage over the other algorithms. The strategy of minimizing $E^f \mathbf{g}^{(k)}$ in each iteration step works, but keeping Remark 4.4 *a*) in mind, the possible problem of not using the natural norms remains. For instance in 1b Algorithms 1 and 2 reduce the $\mathbb{H}^1_0$-error equally fast, while the iterates computed by Algorithm 2 have lower values of $E^f \mathbf{g}^{(k)}$. Also note, that in every iteration the parameters $\mu^{(k)}_{\text{opt}}$ and $\delta^{(k)}_{\text{opt}}$ are smaller than the parameters that had minimized the $\mathbb{H}^1_0$-error.

5.2. **The L-shaped domain.** On the domain decomposition

$$\Omega = (-1,1)^2 \setminus [0,1)^2,$$
$$\Omega_1 = (-1,0) \times (0,1), \quad \Omega_2 = (-1,-1) \times (-1,0), \quad \Omega_3 = (0,1) \times (-1,0),$$

TABLE 5.1. $\mathbb{H}_0^1$-error $\|u - \tilde{S}^f \mathbf{g}^{(k)}\|_{\mathbb{H}_0^1}$ per iteration

k		0	1	2	3	4	5	6
1a	A1	0.2749	0.1474	0.0744	0.0405	0.0205	0.0117	0.0061
	A2	0.2749	0.0645	0.0146	0.0100	0.0027	0.0021	0.0009
	A3	0.2749	0.0048	0.0012	0.0004	0.0002	0.0002	0.0002
1b	A1	0.2839	0.1606	0.0894	0.0554	0.0345	0.0237	0.0161
	A2	0.2839	0.1833	0.0687	0.0504	0.0251	0.0198	0.0111
	A3	0.2839	0.0431	0.0056	0.0022	0.0013	0.0008	0.0005

TABLE 5.2. $\log_{10}(E^f \mathbf{g}^{(k)})$ per iteration

k		0	1	2	3	4	5
1a	A1	-0.6645	-1.2310	-1.7937	-2.3484	-2.8879	-3.3994
	A2	-0.6645	-2.8807	-3.7021	-4.5157	-5.2754	-5.9936
	A3	-0.6645	-4.3751	-6.6346	-7.8566	-8.6099	-9.2416
1b	A1	-1.2135	-1.7142	-2.1728	-2.5815	-2.9420	-3.2640
	A2	-1.2135	-1.9990	-2.6412	-3.2222	-3.7225	-4.1449
	A3	-1.2135	-3.3823	-5.3228	-6.3142	-6.9341	-7.4610

TABLE 5.3. Parameters minimizing $E^f \mathbf{g}^{(k)}$ (opt) and the $\mathbb{H}_0^1$-error ($\mathbb{H}_0^1$)

k		0	1	2	3	4
1a	$\delta_{\text{opt}}^{(k)}$	0.6594	1.6330	0.6196	1.5913	0.6137
	$\delta_{\mathbb{H}_0^1}^{(k)}$	0.6684	2.0449	1.1523	2.0801	1.2871
	$\eta_{\text{opt}}^{(k)}$	0.6576	0.5740	0.5689	0.6118	0.5641
	$\mu_{\text{opt}}^{(k)}$	2.0254	2.2755	3.0689	2.6346	2.9774
	$\eta_{\mathbb{H}_0^1}^{(k)}$	0.6573	0.6047	-0.5052	-5.7913	-94.3044
	$\mu_{\mathbb{H}_0^1}^{(k)}$	2.0827	2.6410	3.7361	4.1171	4.9741
1b	$\delta_{\text{opt}}^{(k)}$	0.7012	1.4623	0.6804	1.4595	0.6955
	$\delta_{\mathbb{H}_0^1}^{(k)}$	1.1707	2.1090	1.5877	2.4697	2.1873
	$\eta_{\text{opt}}^{(k)}$	0.6358	0.5958	0.6237	0.5736	0.6073
	$\mu_{\text{opt}}^{(k)}$	2.0576	3.2780	2.9410	2.7540	3.0591
	$\eta_{\mathbb{H}_0^1}^{(k)}$	0.6444	0.6190	0.5956	5.9119	-20.8946
	$\mu_{\mathbb{H}_0^1}^{(k)}$	2.1463	3.6250	3.6731	4.3875	6.0431

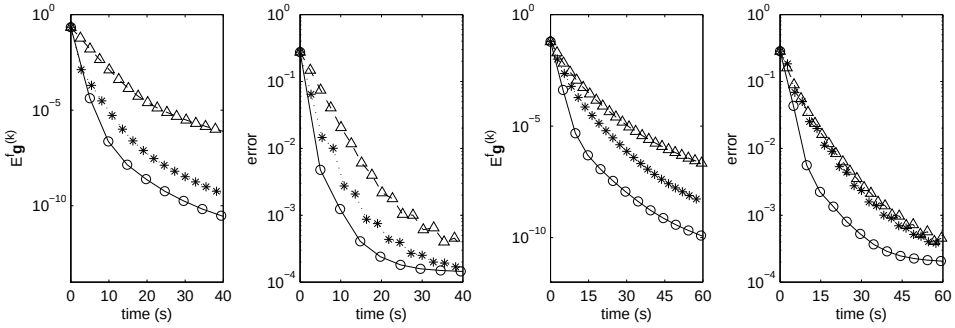

FIGURE 5.2. $E^f\mathbf{g}^{(k)}$ and $\mathbb{H}_0^1$-error for 1a and 1b. A1 ($\triangle$), A2 (*), A3 ($\bigcirc$)

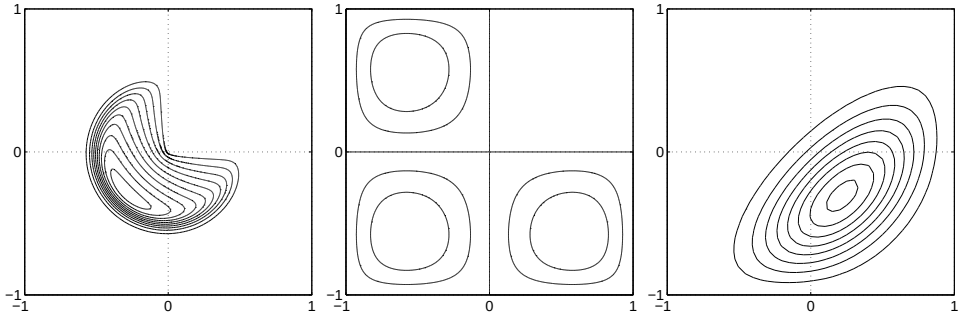

FIGURE 5.3. Solutions of Example 2a, 2b and 3

the algorithms are tested with

$$\text{2a) } u(r,\theta) = \chi(r) r^{2/3} \sin(2\theta/3),$$
$$\text{2b) } u(x,y) = (x-1)x(x+1)(y-1)y(y+1),$$

where 2a is in polar and 2b in Cartesian coordinates and

$$\chi(r) := \frac{w(1-r)}{w(r)+w(1-r)}, \quad w(t) := \begin{cases} \frac{1}{\exp(t^{-2})}, & t > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Because of the non-convex corner at $\mathbf{0}$, a solution u of the model problem will in general only have a Sobolev regularity of $s < \frac{5}{3}$, even if we assume $f \in L_2(\Omega)$. Still, there is a decomposition $u = u_R + u_S$ with $u_R \in H_0^2(\Omega)$ and u_S being a multiple of the explicitly known singularity function corresponding to $\mathbf{0}$ that is given by 2a. For more details on this decomposition we refer to (1). Because of its Sobolev regularity this singularity function can be seen as a worst case example. 2b is tested to analyze the algorithms in case of a smoother solution.

As in the two domain case Table 5.4, 5.5, 5.6 and Figures 5.4 and 5.5 present the results for Examples 2a and 2b.

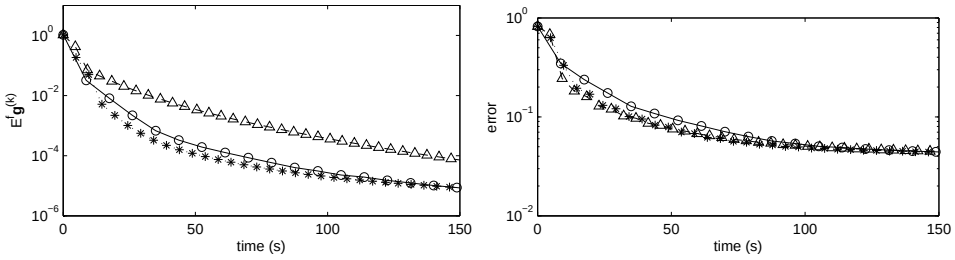

FIGURE 5.4. $E^f\mathbf{g}^{(k)}$ and $\mathbb{H}_0^1$-error for 2a. A1 ($\triangle$), A2 (*), A3 ($\bigcirc$)

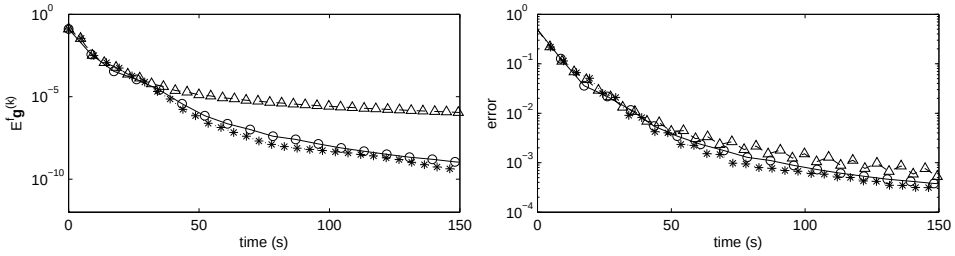

FIGURE 5.5. $E^f\mathbf{g}^{(k)}$ and $\mathbb{H}_0^1$-error for 2b. A1 ($\triangle$), A2 (*), A3 ($\bigcirc$)

In both examples Algorithm 3 leads, as before, to the fastest reduction of the $\mathbb{H}_0^1$-error and the interface bias functional per iteration. Algorithms 1 and 2 behave comparable in the reduction of the $\mathbb{H}_0^1$-error. Again the values of the interface bias functional are generally higher for Lions' method than for the other algorithms.

Comparing the methods in time, Lions' method and the one parameter generalization perform slightly better in reducing the $\mathbb{H}_0^1$-error than Algorithm 3 in 2a, although all algorithms converge to the best solution in the ansatz-space in about the same time. In 2b all methods perform comparatively with Lions' method showing an oscillatory behavior in the last iterations. Again $\delta_{\text{opt}}^{(k)}$ and $\mu_{\text{opt}}^{(k)}$ are underestimated.

5.3. **A decomposition into four domains.** The domain $\Omega = (-1,1)^2$ is symmetrically decomposed into 4 squares. We study the exponential function

$$\begin{aligned} 3)\ u(x,y) = &\exp(-1/2(x+y+1/5)^2 - 4(x-y-3/5)^2) \\ &\cdot (x-1)(x+1)(y-1)(y+1). \end{aligned}$$

All methods have in common, that information is only exchanged along the boundaries between adjacent domains, what may benefit the algorithms with cheaper - and thus more - iteration steps. Therefore a

TABLE 5.4. $\mathbb{H}_0^1$-error $\|u - \tilde{S}^f \mathbf{g}^{(k)}\|_{\mathbb{H}_0^1}$ per iteration

k		0	1	2	3	4	5	6
2a	A1	0.8212	0.6814	0.2445	0.1822	0.1601	0.1284	0.1198
	A2	0.8212	0.6308	0.3319	0.1948	0.1692	0.1309	0.1208
	A3	0.8212	0.3462	0.2382	0.1742	0.1277	0.1081	0.0921
2b	A1	0.4889	0.2196	0.1129	0.0687	0.0479	0.0291	0.0215
	A2	0.4889	0.2176	0.1141	0.0661	0.0508	0.0251	0.0209
	A3	0.4889	0.1270	0.0359	0.0218	0.0116	0.0055	0.0034

TABLE 5.5. $\log_{10}(E^f \mathbf{g}^{(k)})$ per iteration

k		0	1	2	3	4	5
2a	A1	0.0190	-0.3720	-1.1440	-1.3466	-1.5205	-1.6927
	A2	0.0190	-0.7341	-1.3030	-2.2878	-2.6571	-2.9955
	A3	0.0190	-1.4943	-2.0887	-2.6591	-3.1675	-3.4828
2b	A1	-0.8924	-1.4609	-2.4886	-2.9289	-3.2162	-3.6323
	A2	-0.8924	-1.4610	-2.5040	-2.9345	-3.2559	-3.7369
	A3	-0.8924	-2.4349	-3.4589	-3.9489	-4.5838	-5.4231

TABLE 5.6. Parameters minimizing $E^f \mathbf{g}^{(k)}$ (opt) and the $\mathbb{H}_0^1$-error ($\mathbb{H}_0^1$)

k		0	1	2	3	4
2a	$\delta_{\text{opt}}^{(k)}$	0.6543	1.0680	0.8910	0.6837	1.3689
	$\delta_{\mathbb{H}_0^1}^{(k)}$	0.6709	1.5172	1.3354	2.9738	3.8574
	$\eta_{\text{opt}}^{(k)}$	0.5457	0.6920	1.2001	0.7941	1.1283
	$\mu_{\text{opt}}^{(k)}$	1.5239	0.9806	1.0834	1.7505	1.3333
	$\eta_{\mathbb{H}_0^1}^{(k)}$	0.4665	1.2271	1.3655	6.3379	2.2508
	$\mu_{\mathbb{H}_0^1}^{(k)}$	1.9127	2.1799	2.9163	3.8629	5.5649
2b	$\delta_{\text{opt}}^{(k)}$	1.0090	0.9353	1.1211	0.7623	1.3869
	$\delta_{\mathbb{H}_0^1}^{(k)}$	1.5225	1.0408	2.4230	2.3977	2.6023
	$\eta_{\text{opt}}^{(k)}$	-0.0744	0.1467	4.1902	0.8044	0.7737
	$\mu_{\text{opt}}^{(k)}$	1.3199	2.1177	1.0756	1.3854	2.0024
	$\eta_{\mathbb{H}_0^1}^{(k)}$	0.3279	-0.6554	5.1651	1.0982	0.8492
	$\mu_{\mathbb{H}_0^1}^{(k)}$	1.5584	2.5703	2.1642	2.7240	3.1494

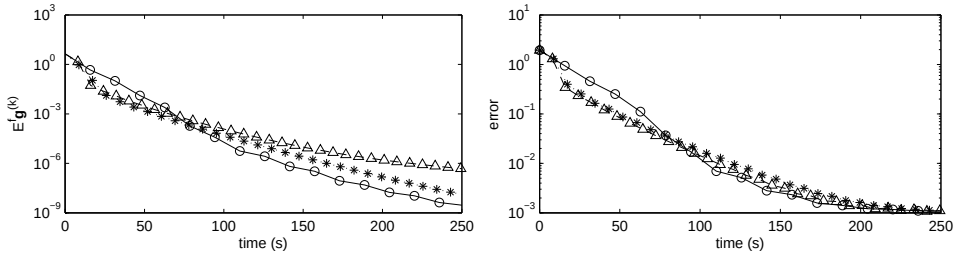

FIGURE 5.6. $E^f\mathbf{g}^{(k)}$ and $\mathbb{H}_0^1$-error for Example 3. A1 ($\triangle$), A2 (*), A3 ($\bigcirc$)

TABLE 5.7. $\mathbb{H}_0^1$-error $\|u - \tilde{S}^f\mathbf{g}^{(k)}\|_{\mathbb{H}_0^1}$ per iteration

k		0	1	2	3	4	5	6
3	A1	1.9406	1.3006	0.3431	0.2345	0.1708	0.1206	0.0894
	A2	1.9406	1.3023	0.3986	0.2535	0.1635	0.1252	0.0868
	A3	1.9406	0.9414	0.4593	0.2528	0.1116	0.0368	0.0171

TABLE 5.8. $\log_{10}(E^f\mathbf{g}^{(k)})$ per iteration

k		0	1	2	3	4	5
3	A1	0.6381	0.1664	-1.2728	-1.6275	-1.9130	-2.2183
	A2	0.6381	-0.0331	-0.9699	-1.8885	-2.2619	-2.5881
	A3	0.6381	-0.3347	-1.0017	-1.8893	-2.6080	-3.7267

solution u is studied that is smooth and localized at the bottom right domain, see Figure 5.3.

Looking at Table 5.7, Algorithm 3 again shows the best reduction of the $\mathbb{H}_0^1$-error measured in iterations. Note that the error after two iterations is actually smaller for the other two algorithms. Despite the different efficiency in the first iteration steps, the methods convergence in about the same time, see Figure 5.6. As in the other examples the graphs of $E^f\mathbf{g}^{(k)}$ and the $\mathbb{H}_0^1$-error behave similar for Algorithms 2 and 3, while Lions' method leads to a slower reduction of the interface bias functional. For the computed parameters $\delta_{\text{opt}}^{(k)} < \delta_{\mathbb{H}_0^1}^{(k)}$ and $\mu_{\text{opt}}^{(k)} < \mu_{\mathbb{H}_0^1}^{(k)}$ can again be observed.

TABLE 5.9. Parameters minimizing $E^f \mathbf{g}^{(k)}$ (opt) and the $\mathbb{H}_0^1$-error ($\mathbb{H}_0^1$)

k		0	1	2	3	4
3	$\delta_{\text{opt}}^{(k)}$	0.7156	1.2817	0.7824	1.1951	0.8376
	$\delta_{\mathbb{H}_0^1}^{(k)}$	0.8604	1.4055	1.0758	2.7146	2.7039
	$\eta_{\text{opt}}^{(k)}$	0.5429	0.9641	1.0018	0.7774	0.7510
	$\mu_{\text{opt}}^{(k)}$	1.0834	0.9732	0.9030	1.9102	2.5660
	$\eta_{\mathbb{H}_0^1}^{(k)}$	0.4539	0.9668	0.8878	0.2295	0.5798
	$\mu_{\mathbb{H}_0^1}^{(k)}$	1.4284	1.6292	1.7534	3.2578	3.5176

6. Conclusion

The proposed two parameter generalization (2.10) of Lions' method (2.7) utilizes two ideas. The first is to solve two subproblems per iteration step and subdomain, one corresponding to the jumps in the Dirichlet data and the other to the matching of the Neumann data of the current local solutions. They can be solved independently, opening up a new possibility for parallelization. These to kinds of error do not need to be equal in size. Also, the Robin boundary values used in the local subproblems consist of Dirichlet and Neumann data. Because of this the two parameter approach is natural.

The second idea is to combine the two kinds of local solutions such that the interface bias functional (2.11) is minimized in each step. Local L_2-norms are used to measure jumps in the Dirichlet data and the matching of the Neumann data, which would naturally be measured in local $H^{1/2}$ and $(H^{1/2})'$ norms.

Numerical examples show that the new method leads to a stronger reduction of the error in the product Sobolev norm per iteration, compared to Lions' method (2.7) and the one parameter generalization by Guo and Hou (2.9). Of course, solving two subproblems instead of one also means roughly twice as high computational cost per iteration. Still, looking at the error reduction in time, the new method shows a comparable performance on the L-shaped domain and a symmetrical decomposition of a square into four domains, while it clearly outperforms the other methods in the two domain case. This shows that the information gained by solving two subproblems per iteration step instead of only one is worth the higher iteration cost.

Comparing the error reduction in the product Sobolev norm with the value of the interface bias functional, Lions' method seems to reduce

the first quantity faster than the latter, while both behave similarly for the one and two parameter generalization. This suggests that using the interface bias functional as a stopping criterion for the new method works. In the numerical experiments the parameter corresponding to the jumps in the Dirichlet data was underestimated in every case, which suggests that the strategy of choosing the parameters may be improved. Since it is independent from the first idea of the new method and its proof of convergence, this poses no problem.

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Reports

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